# FOREGROUNDING THE BACKGROUND: TWO USES OF COORDINATE SYSTEMS [1]

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A common way we model, communicate, and analyze phenomena is through graphical representations—graphs represented on coordinate systems. Despite the importance of graphical representations in mathematics, researchers have shown persistent difficulties students experience as they construct and interpret graphs. Particularly noteworthy among these difficulties is that students often treat graphs as literal representations of a situation, making figurative associations with the shape of the graph and some physical/visual characteristic of the phenomena being graphed (*e.g.,* Leinhardt, Zaslavsky & Stein, 1990; Oehrtman, Carlson & Thompson, 2008). While this identification of *what* students struggle with is useful, more accounts of *why* this might occur [2] and *when* this association could be productive are equally important. In this paper, we present a framework distinguishing two types of coordinate systems as such an account.

We contend students' graphing understandings can be better understood by carefully attending to students' understandings of underlying coordinate systems. To illustrate why foregrounding coordinate systems is important, we interpret how coordinate systems might be used to solve two tasks, one from a precalculus and the other from a calculus textbook (see Figure 1).

Although the two tasks may seem similar as each asks students to imagine and mathematize part of a rider's experience on a Ferris wheel, there are several key differences. The difference relevant to our discussion is that in Task A the implied coordinate system serves to describe the location of points *within* the Ferris Wheel context, whereas in Task B the implied coordinate system would be used to depict the relationship between two quantities extracted from the Ferris Wheel context *outside* the Ferris Wheel context.

In what follows, we argue that depending on the nature of quantities and how they are coordinated, coordinate systems can serve two distinct but related purposes. After characterizing important features of each type of coordinate system, we provide examples of how this distinction might inform research endeavors, provide alternative interpretations of findings from previous research, and close with educational implications for the learning of coordinate systems and graphical representations generally.

## Coordinate systems and frames of reference

We define a coordinate system broadly as a representational space in which quantities (Thompson, 2011) are coordinated to systematically organize some phenomenon. We use 'representational' to refer to mental constructs that can, but need not, be physically represented (*e.g.,* drawn on a piece of paper). The systematic quantitative organization of the phenomenon into a coordinate system is established based on some initial qualitative organizations (Piaget, Inhelder, and Szeminska, 1960), which we call frames of reference—mental structures through which an individual gauges the relative extents of various attributes of objects within a phenomenon. Like Joshua, Musgrave, Hatfield and Thompson (2015), we consider a frame of reference to involve a commitment to a reference point and directionality; however, we do not require a frame of reference involve a unit of measure. When multiple frames of reference are coordinated and a unit of measure has been adopted for each frame of reference, we consider an individual to have established a coordinate system.

Depending on where the organizing structure of the phenomenon is conceived of—within or outside of the phenomenon—we distinguish between spatial and quantitative coordinate systems. In either case, when an individual has established frames of reference, she can construct quantities by enacting or anticipating a measurement process for attributes within the phenomenon to quantitatively describe it. In other words, establishing frames of reference is a necessary but not sufficient condition for constructing coordinate systems.

We offer two points of clarification. First, we assume a coordinate system does not represent by itself; it must be created and interpreted by individuals. Therefore, we consider coordinate systems to be constructed by an individual in goal-directed activity, and we unpack two such goals a coordinate system could be constructed to achieve. Second, we assume coordinate systems can, but need not, contain graphs. We emphasize that we are distinguishing uses of *coordinate systems,* which is different from the distinctions others have made about *graphs* and graphing activity (*e.g.,* Moore & Thompson, 2015).

## Two uses of coordinate systems

Below, we elaborate on each type of coordinate system in relation to frames of reference and the meaning of points in the coordinate plane, which form the foundation of graphs.

### Spatial coordinate systems

*Spatial coordination* refers to an individual using a coordinate system in a representation of some physical or imagined space. A *spatial coordinate system* can be viewed as a coordinate system mentally overlaid onto the space being represented and objects within that space being tagged with coordinates [3]. In this case, the coordinate system quantitatively organizes the space in which the phenomenon is

**31. Entertainment** The Ferris Wheel first appeared at the 1893 Chicago Exposition. Its axle was 45 feet long. Spokes radiated from it that supported 36 wooden cars, which could hold 60 people each. The diameter of the wheel itself was 250 feet. Suppose the axle was located at the origin. Find the coordinates of the car located at the loading platform. Then find the location of the car at the 90° counterclockwise, 180°, and 270° counterclockwise rotation positions.

1. *Ferris Wheel Problem:* When you ride a Ferris wheel, your distance, $y(t)$, in feet from the ground, varies sinusoidally with time $t$, in seconds since the wheel started rotating. Suppose that the Ferris wheel has a diameter of 40 ft and that its axle is 25 ft above the ground (see Figure 3-8j). Three seconds after it starts, your seat is at its high point. The wheel makes 3 rev/min.

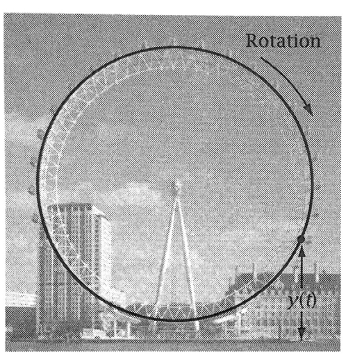


*Figure 3-8j*

a. Sketch the graph of function $y$. Figure out the particular equation for $y(t)$.

*Figure 1. Examples: Task A (above) from Holliday, Cuevas, McClure, Carter & Marks, 2001, p. 95 and Task B (below) from Foerster, 2005, p. 112.*

situated. An example of where an individual might use a spatial coordinate system is a GPS, which could involve the coordination of orthogonal distances.

Prior to constructing a spatial coordinate system an individual must establish frames of reference, which they impose onto the spatial phenomenon to gauge extents of various spatial attributes of objects within the space (*e.g.*, relative location or orientation of an object). Once such frames of reference are established, an individual can enact or imagine measurements of spatial attributes of the objects and tag those quantities' values *onto* the space in which the phenomenon occurs. Through this process, an individual constructs what we refer to as a spatial coordinate system.

To illustrate, consider the Ferris Wheel context in Task A (Figure 1). By constructing horizontal and vertical lines through the axle as frames of reference, an individual can organize relative positions of each car in reference to the axle. Specifically, the location of a car can be conceived of along a horizontal line through the axle with a realization that it simultaneously has a specific location along a vertical line through the axle (see Figure 2, left). Using the two orthogonal lines and their intersection as coordinated frames of reference, the individual can gauge and measure two quantities—the horizontal displacement (Figure 2, center left) and vertical displacement (Figure 2, center right) from the reference point to each car. Finally, by tagging the two orthogonal distances onto objects in the Ferris Wheel space, the individual constructs a spatial coordinate system, which could be used to quantitatively describe each car's location (Figure 2, right).

Figure 2 presents one possible spatial organization of the Ferris Wheel situation using orthogonal lines and a reference point as frames of reference. However, there are other ways individuals may conceive of frames of reference, and therefore other spatial coordinate systems individuals might construct. For example, an individual may create a frame of reference consisting of rays and a vertex point centered at a spectator on the ground to gauge the rider's location from the spectator's position (Figure 3). As a result, the individual may establish a different spatial coordinate system by tagging an angle measure of rotation and radial distance to organize each car's location.

Although in the above examples we described how spatial coordinate systems can be used to mathematize a single instance in time (*i.e.*, a snapshot), spatial coordinate systems can also be used when representing dynamic situations. An individual may think of overlaying a coordinate system over an imagined movie and tracking changes of a point over time within the space. For example, in the Ferris Wheel context, one can observe or imagine a rider's position changing over time within a spatial coordinate system. The contexts in which spatial coordinate systems are used are also not restricted to locating objects in perceptual or sensorimotor space. An individual can also use spatial coordinate systems to organize or mathematize abstract objects, such as geometric shapes in an imagined space [4].

### Quantitative coordinate systems

*Quantitative coordination* refers to an individual using a coordinate system to coordinate sets of quantities by

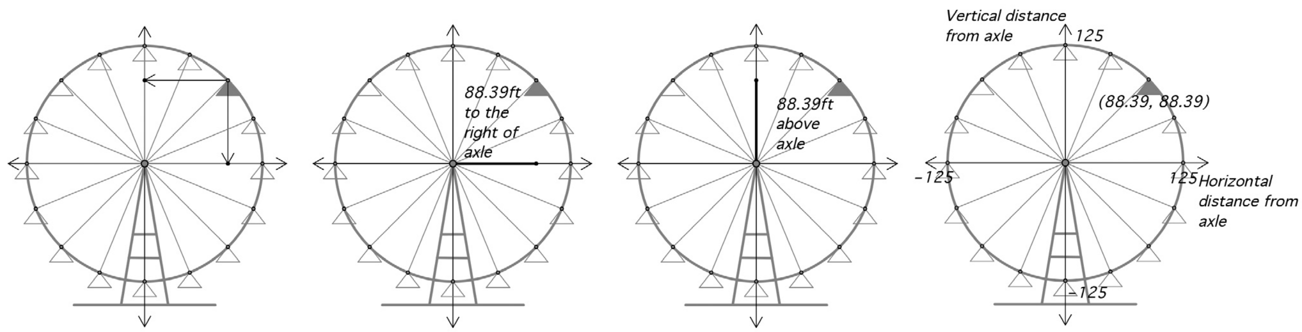


*Figure 2.* *Constructing a spatial coordinate system for Task A.*

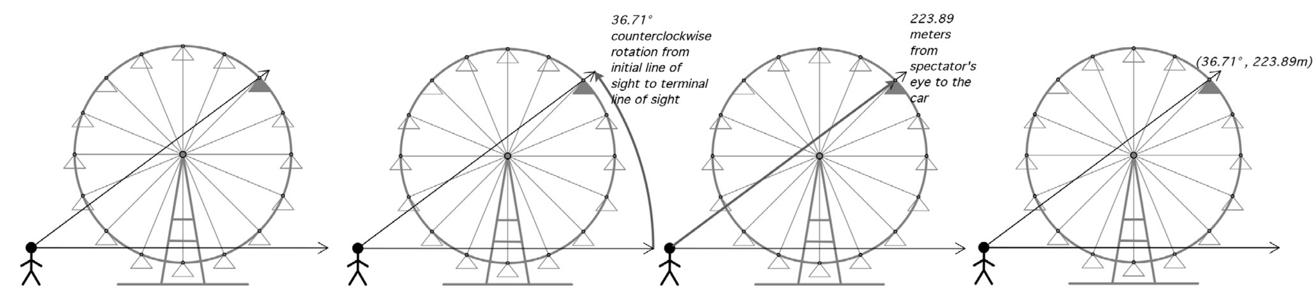


*Figure 3.* *Constructing a different spatial coordinate system for Task A.*

constructing a geometrical representation of the product of measure spaces. To establish a *quantitative coordinate system,* an individual must establish quantities within the given space/phenomenon, *disembed* (Steffe & Olive, 2010) these quantities (*i.e.,* extract them from the situation while maintaining an awareness of the quantities within the situation), and project them onto some *new space,* which is *different from the space* in which the quantities were originally conceived. This construction of a new space involves establishing frames of reference for each quantity and coordinating these frames of reference (Joshua *et al.,* 2015).

To illustrate, an individual could construct a quantitative coordinate system in the following way for Task B (Figure 1). First, the individual might conceive of two relevant quantities—elapsed time and vertical distance of the car in relation to the ground's location (Figure 4, left). Next, the individual can disembed the two magnitudes and overlay them onto a line, producing two drawn or imagined number lines (see Figure 4, center). By placing the two number lines orthogonally, the individual can produce a new product space of measures in which they could reason about the relationship between the two quantities (See Figure 4, right). The two-dimensional space made by the product of the two number lines above form a new space (a {time×vertical distance} plane), which is different from, but related to, the space containing the Ferris Wheel itself.

In both types of coordinate systems, points are multiplicative objects (Thompson, 2011), but the way in which the multiplicative objects are conceptualized differ. In a spatial coordinate system, situational quantities are coordinated to

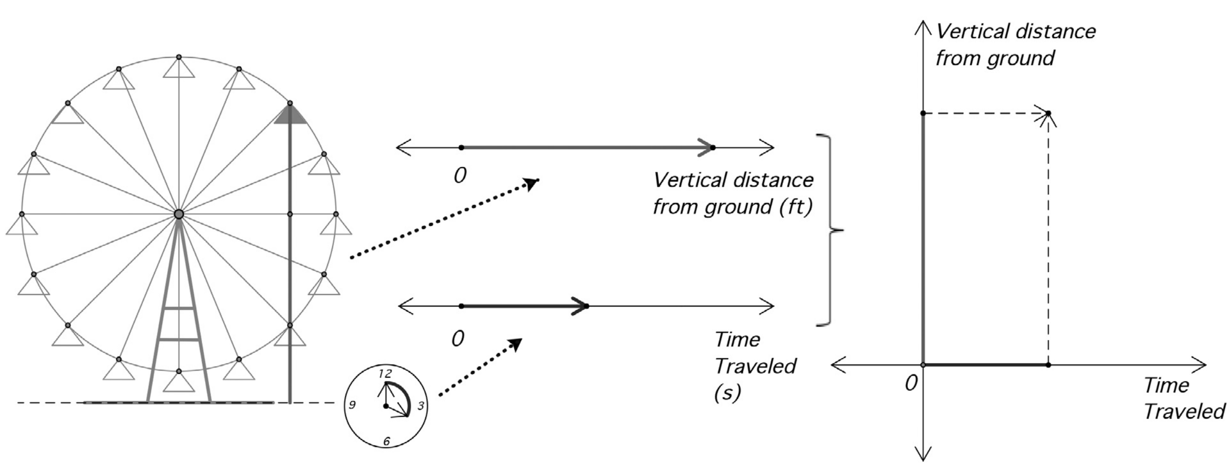


*Figure 4.* *Constructing a quantitative coordinate system for Task B.*

locate and/or describe a point in the same or an analogous space in which the phenomenon occurs. On the other hand, in a quantitative coordinate system, a point is constructed via coordinating quantities disembedded from the space in which the phenomenon occurs.

## A new lens to investigating students' graphing activity

The spatial/quantitative coordinate system distinction and the associated meanings for graphs can inform research on students' graphing activity.

### Considering graphs in each type of coordinate system

Graphs constructed upon each coordinate system type may entail different meanings due to the nature of the coordinate system. Graphs constructed on spatial coordinate systems can be viewed as a collection of objects or traces of an object within the space containing the original objects or phenomena. On the other hand, graphs within quantitative coordinate systems are not projections of physical objects or phenomena from the original space. Instead, graphs constructed on a quantitative coordinate system can be viewed as a collection of points representing relationships between disembedded magnitudes. To illustrate the different natures of graphs constructed in each coordinate system, consider Tasks C and D (Figure 5) extracted from a precalculus textbook and an algebra textbook, respectively.

Presumably, the authors of Task C intend for students to establish a spatial coordinate system. To gauge the relative locations of the two towers, frames of reference are suggested: a horizontal axis through the two towers, and vertical axis aligned north-south through the midpoint between the two towers. Once these frames of reference are established, vertical and horizontal distances can be measured (*e.g.,* how far Juana lives north of the horizontal axis). As a result, objects in Juana's neighborhood can be tagged with pairs of quantities describing distances from the midpoint between the two towers in signed easterly and northerly directions. In this case, an individual is coordinating the location of objects *within the space* in which the two towers are situated. The graph of the equation modeling the path of

**41. Communication** Radio waves emitted from two different radio towers interfere with each other's signal. The path of interference can be modeled by the equation $\frac{y^2}{12} - \frac{x^2}{16} = 1$, where the origin is the midpoint of the line segment between the two towers and the positive $y$-axis represents north. Juana lives on an east-west road 6 miles north of the $x$-axis and cannot receive the radio station at her house. At what coordinates might Juana live relative to the midpoint between the two towers?

**55. MULTI-STEP PROBLEM** A kernel of popcorn contains water that expands when the kernel is heated, causing it to pop. The equations below give the "popping volume" $y$ (in cubic centimeters per gram) of popcorn with moisture content $x$ (as a percent of the popcorn's weight). ▶ Source: *Cereal Chemistry*

$$\textbf{Hot-air popping: } y = -0.761x^2 + 21.4x - 94.8$$

$$\textbf{Hot-oil popping: } y = -0.652x^2 + 17.7x - 76.0$$

**c.** The moisture content of popcorn typically ranges from 8% to 18%. Graph the equations for hot-air and hot-oil popping on the interval $8 \le x \le 18$.

*Figure 5. Two textbook problems involving coordinate systems: Task C (above) from Holliday, et al. 2001, p. 135 and Task D (below) from Larson, Boswell, Kanold & Stiff, 2004, p. 255.*

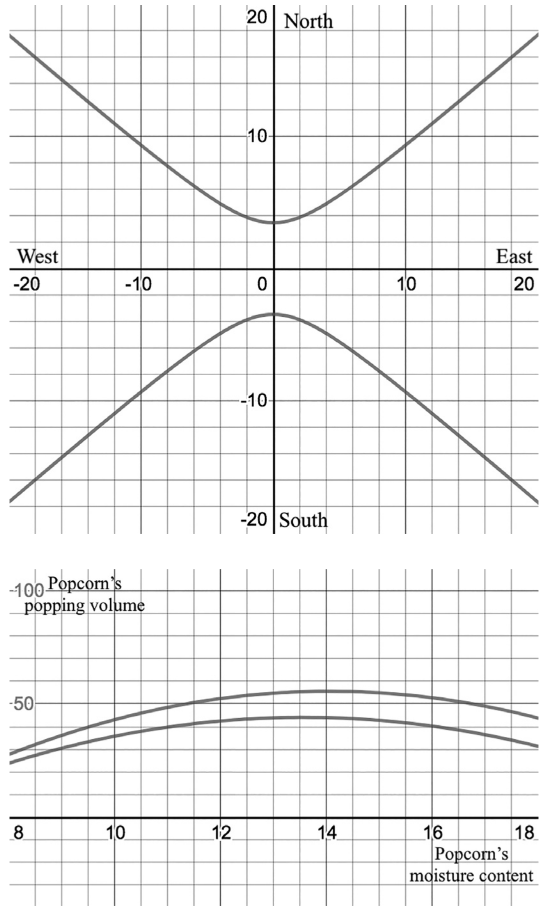


*Figure 6. Graph in Task C (above), Graph in Task D (below).*

interference as shown in Figure 6 (above) represents the literal path where the two radio tower signals interfere each other within this spatial coordinate system.

On the other hand, Task D involves producing a graph within a quantitative coordinate system. To complete the task, an individual must abstract two quantities—the popcorn moisture and popcorn volume when the kernel pops—from the kernel popping phenomenon. Once the individual disembeds both quantities from the situation, the individual can organize the quantities along two number lines. The coordinate system obtained by orthogonally arranging these number lines is a new product space in which the two measures could be coordinated simultaneously. Critically, this new space created by a coordinate system does not entail the spatial situation in which the popcorn pops. Hence, the graph modeling the phenomenon shown in Figure 6 (below) represents how the quantities change in tandem rather than literal trajectories of popcorn kernels popping or other literal motions from the popcorn situation.

### Connecting to research on students' graphing activity

As we have illustrated, different ways of reasoning are involved when creating and interpreting spatial and quantitative coordinate systems and associated graphs. Therefore, it is critical to attend to this distinction when investigating students' reasoning with graphical representations. Specifically, we propose that when a student constructs or is reasoning about a coordinate system it is important to attend to whether the student is explicitly aware of the purpose of the coordinate system. For example, when a student draws axes, a researcher or teacher might examine whether the student is (a) attempting to quantitatively organize, *i.e.,* tagging quantities *onto*, a particular phenomenon within the space it is occurring, (b) representing magnitudes of quantities abstracted from an outside situation, or (c) attempting to reproduce an image from their prior classroom instruction.

We noted earlier that researchers reported that students often make inappropriate figurative associations between the shape of the graph and some physical/visual characteristic of the phenomena being graphed (*e.g.,* Leinhardt, Zaslavsky & Stein, 1990; Oehrtman, Carlson & Thompson, 2008). One possible explanation for students appearing to treat graphs as literal representations of a situation is that students are interpreting what a teacher or researcher intends to be a quantitative coordinate system as a spatial coordinate system. In other words, students may have used a spatial coordinate system to organize the phenomenon rather than reasoning within a quantitative coordinate system with disembedded magnitudes as the researcher intended.

On alleviating students' difficulties with a task to draw a speed-time graph of a bicycle traveling a hill, Oehrtman, Carlson and Thompson (2008) remarked, "a student must ignore the fact that the picture [of the hill] looks like a graph […] then, while ignoring the shape of the hill in the picture, determine how to represent the result graphically" (p. 153). However, the boundary of the hill can be interpreted as a graph if the student is reasoning within a spatial coordinate system; in such a case, the student may be constrained to reasoning about points and spatial attributes in the graph as opposed to reasoning about the intended contextualized quantities (*e.g.,* speed) outside of the perceptually available situation. We also note, as illustrated in Tasks A (Figure 1 and C (Figure 5), that there are indeed contexts in which thinking about graphs in such a way is productive. By attending to students' meanings for coordinate systems, researchers might develop richer models for students' graphing activities.

## Educational implications

In addition to providing a new lens for research investigating students' graphing activity, our framework has educational implications. First, our framework provides a tool to analyze and reflect on descriptions of coordinate systems used in curricula. For example, students' thinking of a graph as a literal representation of a physical situation may be unsurprising when students are presented with graphs that seem to conflate the two uses of coordinate systems. In the task below (Figure 7), we see a quantitative coordinate system used for relating time and distance. Yet, the use of arrows and labels in the text is problematic from our perspective. The stoplight, school, and school zone are *not* represented physically on this graph as the arrows seem to indicate. These objects exist in a space different from the coordinate plane established by the quantities time and distance. We hypothesize that conflations between quantitative

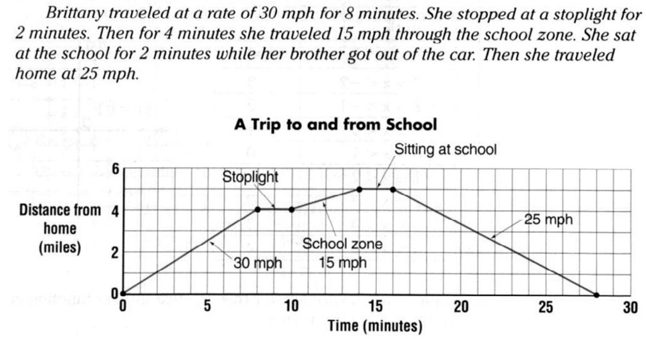


*Figure 7. Example from Holliday et al., 2001, p. 45.*

and spatial coordinate systems in tasks can lead to students' difficulties in graphing activity. By attending to the underlying coordinate systems more carefully, researchers, teachers, and curricula designers can support students in developing understandings compatible with their intentions. Additional research is needed to examine the prevalence of these conflations in curricula, as well as the relationships between these tasks and students' thinking.

Second, using our distinction of coordinate systems, we can reflect on the current way in which upper grade level or undergraduate students experience and learn graphs in their mathematics courses. From their analysis of graphs in STEM textbooks and journals, Paoletti *et al.* (2017) noted commonly used precalculus and calculus textbooks almost exclusively represented two decontextualized quantities typically represented as $x$ and $y$. We contend when presented with decontextualized graphs, students may interpret graphing tasks in their textbooks in terms of either type of coordinate system, which can lead to different interpretations. We suspect students may be unsure as to what the graphs constructed upon a decontextualized coordinate system are meant to represent, if anything, and are instead left with finding coping mechanisms to find the 'correct' answer. Further research is needed to investigate these hypotheses.

Looking forward, we intend for our framework to provide a new lens for extending empirical research examining students' construction of coordinate systems and graphing activities. By bringing attention to the coordinate systems on which students reason when engaging in graphing activity, we hope to provide an additional resource for consideration when researchers and educators work to understand why students have difficulty constructing or interpreting graphical representations. Ultimately, we hope researchers' and educators' awareness of two types of coordinate systems will result in their providing students with more opportunities to engage in powerful ways of reasoning with the mathematical concepts that are often represented through these two distinct coordinate system types.

## Notes

[1] An earlier version of this paper was presented at the 40th annual meeting of the *North American Chapter of the International Group for the Psychology of Mathematics Education.*

[2] Roth's (2002) semiotic model for graph reading is an example of such an account.

[3] We note that alternatively, one can think of overlaying the spatial situation onto a coordinate system.

[4] For an example of this and additional examples of both spatial and quantitative coordinate systems, refer to Lee, Hardison & Paoletti (2018).

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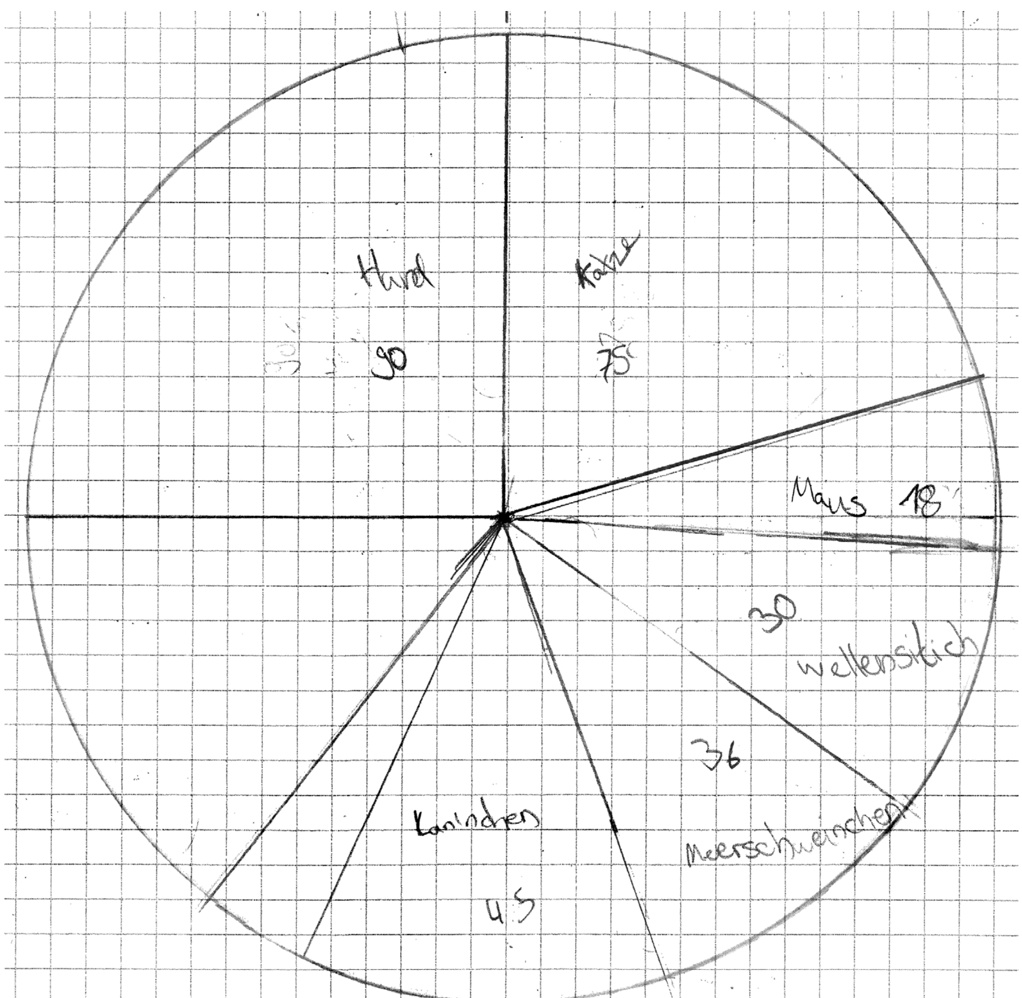


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