Mathematics in Context: A New Course

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Our new course turns upside down many of the basic methods of teaching in undergraduate mathematics. It is "top down" rather than "bottom up". It starts off with general discussions as to the nature and extent of mathematics. It proceeds through a decision by students as to what they want to write about It is assessed by two extended essays which are expected to communicate a viewpoint, and to discuss issues

Background

One motivation for introducing the course was the following basic assumption:

An education of undergraduates should teach not only how to do something but should also give some training and consideration to the following:

1 The context of particular techniques
2 Relationships with other disciplines.
3 The role of the subject in history and in society
4 The notion of value in the subject
5 Communication skills
6 The methodology of research.
7 The extent and quality of current research
8 Current issues

In part this assumption was inspired by the findings of the McLone Report [McL], published in the early 1970s but still not well enough known This was a report on a survey "to obtain information about the educational qualities required and sought after by industry in the employment of mathematicians, and to ascertain the type of mathematics of most use to a graduate in his career, and the relevance in this context, of present undergraduate courses" In conclusion it summarised the employers' rather jaundiced view of the mathematics graduate as follows: "A description of the employers' view of the average mathematics graduate might be summarized thus: Good at solving problems, not so good at formulating them, the graduate has a reasonable knowledge of mathematical literature and technique; he has some ingenuity and is capable of seeking out further knowledge. On the other hand the graduate is not particularly good at planning his work, nor at making a critical evaluation of it when completed; and in any event he has to keep his work to himself as he has apparently little idea of how to communicate it to others "

If we list some of the characteristics of standard mathematics courses, this is not surprising:

(i) Each course is given through lectures which cover fully the material of the course Very often, this material is written fully on the blackboard and the students copy this into their notebooks
(ii) The courses are aimed at giving a thorough grounding in a particular topic, and in acquiring a particular range c techniques
(iii) The students are assessed on their ability to reproduc these techniques with rigour and accuracy under examina tion conditions
(iv) While the course may contain some motivation and back ground, such context is reflected either not at all or to a ver small extent in the examination.
(v) There is little room for the evaluation either of techniqu or of particular results.

Furthermore students are usually not given a professional view of the place and role of mathematics, and hence the are likely to be unaware of the nature and extent of curre mathematical research, and its potential for applications, bot academic and industrial

In summary, students at both undergraduate and postgradu ate levels are starved of a global viewpoint

Planned structure, themes and mode of assessment

In the session 1987/8 a small working group was set up t look at the above criticisms of standard courses in mathema ics It recommended a new course, equivalent to 1/6th c the final (third) year of the degree (replacing one tradition three hour examination paper). This course was to consi of half of a "unit" on the *History of Mathematics*, and th other half on *Mathematics in Context* Each part was to b assessed by two long essays, to be handed in at the begir ning of the second week of the second and third terms W will be concerned only with the "unit" on *Mathematics i Context*, since this breaks new ground.

The general aims of the course were stated as:

(a) To present some global picture of mathematical activity rather than merely of some individual topics.
(b) To encourage the student to assess, to evaluate, and t write and communicate.
(c) To encourage participation in a joint project
(d) To encourage independence and creativity

The course was intended to discuss some or all of the fo lowing themes:

The impact of mathematics on science and society.
Mathematics and technological advance.
The extent and size of mathematical research activit
The employment of mathematicians.
The teaching of mathematics
The communication of mathematics.
The funding of mathematics

Other less specific subjects and issues were also to be cons

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dered in the course. For example, an important but not so well recognized issue is the contrast in mathematics between problem solving and the development of concepts and language. Mathematicians tend to rate the former the most highly and the solution of some classical problem tends to get the most prizes. That is, the training of mathematicians places an emphasis on rigour, technique and achievement, and has little emphasis on problem formulation or concept formulation. By contrast, a study of the history of mathematics shows that in the applications of mathematics it is concepts and language which are often more important than particular theorems. Grothendieck [Gr] has written of the problems of "bringing concepts out of the dark".

Getting Context off the ground

How did we prepare for this new venture? It might be thought that it would be necessary to take a year to prepare the course, with the collection of duplicated articles on the various aspects that were to be considered. On the other hand, students should perhaps experience the fact that the way in which mathematics is taught does not help with the difficulties of gathering information in real life when you want to do something specific. One of the considerations for a choice of topic is thus the availability of information. There is for this course no established body of knowledge (whoopee!), much of the material involves primary sources, and too much preparation could well rigidify the course and make the tutors less responsive to the students. We knew in some sense what we were aiming at, but were quite unclear whether these aims were realizable. The starting point was only our perception that there was, to us, a gap in the students' professional training. We were well aware that there was a large element of risk in the success of the course, and this led us to concentrate on conveying the overall aims.

Our initial investigations suggested, in fact, that a considerable amount of source material was available, for example the following:

American Mathematical Society publications and articles on the uses of mathematics, for example, [AMS1] and [AMS2].

Reports of Government commissions in various countries (for example, the ACARD report in the UK on software) [ACA].

Publications on the teaching of mathematics from the UK Department of Education and Science and other similar sources.

Local sources on the preparation of exhibitions and the popularization of mathematics, which formed a background to the later article [BP].

University Grants Committee Statistics, and reports from the Committee of Heads of University Departments of Mathematics, the London Mathematical Society, and the Institute of Mathematics and Its Applications, all easily available within the UK.

Reports of conferences on the applications of mathematics.

The McLone report on "The Training of Mathematicians" [McL] mentioned earlier.

Articles in popular journals such as the *Mathematical Intelligencer* and *Scientific American*.

Books on the nature and achievements of mathematics by for example Ian Stewart [St1] etc., Keith Devlin [Dev], Davis and Hersh [D&H1] and [D&H2], Lakatos [Lak], Martin Gardner [Gar1] etc., Morris Kline [K1] etc., Bell [B1] and [B2], Hadamard [H], and so on (see the reference list at the end of the article).

We decided to "play it by ear". To hand out some material, giving references and additional material later as needed. Our main initial effort, therefore, went on deciding on how the course should be "taught".

Methodology and style

Teaching was done by both of us in one discussion period of 1 to $1\frac{1}{2}$ hours per week over the two terms. The advantage of having two people is that they can put views which complement, or even disagree with, each other. This conveys to the students the fact that the arguments in this area are not cut and dried, and it also improves the atmosphere for discussion. The venue was a seminar room with comfortable chairs; this arrangement encouraged easy and informal discussion.

Discussion of the questions mentioned earlier is clearly not the same as a traditional mathematics course. A proper discussion of these issues and questions brings up issues and methodologies from politics, sociology, psychology, education, history, and so on, for which the previous education of the students usually gives no pointers and no expertise. The same problem arises for the staff. Thus there is a severe problem in giving the discussion and the resulting projects an appropriate level of rigour. On the other hand, many of these issues about mathematics can be discussed properly only by someone with a good background in mathematics. If they are not discussed in a mathematics course they will be discussed nowhere. It seems better to attempt the study, to make students aware of the problems, to help them to assess these problems themselves, and for us to be aware in making the final assessments of the limitations in the training of the students. It was also helpful to remind the students that, in the real world, issues and even questions are not as clear cut as they are in a mathematics proof, and that even in mathematics the question of what is interesting and important is not as susceptible of a final answer as is the question of correctness.

An interesting feature of the course is that the matters raised are appropriate for controversy. A sad feature of mathematics, even at the research level, is the lack of controversy or of discussion of aims and methodology. The concentration is usually on success. As we explained to students, part of the prescription for success is a proper analysis of failure (on the grounds that if you never fail at anything, then you have probably never set yourself sufficiently hard tasks). So failure is actually more interesting than success because it can suggest what needs to be done. We are interested in the processes involved in the development of mathematics, both generally and in the progress of an individual.

One real problem for the management of such a course is how to relate specific projects in the course to the discussion of general issues For example, if the topic under discussion is the applicability of mathematics, it is clearly not enough just to give an example of an application of mathematics Some kind of conclusion and judgment has got to be drawn, there has to be some analysis of why this particular example was chosen, and the discussion should consider the example in its role as evidence for, or lack of evidence for, the social importance of mathematics.

As we have said, the aim has been to concentrate on issues, and to raise questions on methodology, judgment, and values. The standpoint is the opposite of authoritarian, and is intended to encourage the exchange and development of viewpoints. The course has to be run by iconoclasts, who are prepared to listen, and to examine every point of view for its value The attitude has to be not "You are wrong", but "How would you give that point of view some rigorous and justifiable basis?", "Why do you say that?", or "Have you considered the following evidence?" What the leaders have to give to the course is the benefit of their knowledge, and the fact that they have considered and discussed most of the points at issue at some time or another Thus a lot of ground work has been unconsciously prepared over a period of years.

The first session

The end result of these deliberations was that the session 1988/89 started with eight bold students and two staff coming to the first session and wondering what was going to happen Both of us were nervous about the whole course; about our preparation, or lack of it; and as to how the students would react to such an apparently unstructured and open-ended course. What was actually going to happen?

We need not have worried! The course gave people an opportunity to discuss questions which had been bothering many for some time: for example, about the nature of mathematics and the purpose of the courses as a whole The discussion was vigorous and stimulating.

Our first session explained the aims of the course and why we had set it up The initial handouts for the course included the following:

> Copies of articles from the Notices of the American Mathematical Society on current progress in mathematics [AMS1 & 2]
>
> Copies of the transparencies and handouts for an Evening Discussion by R. Brown on "The Public Image of Mathematics" which was given to the British Mathematical Colloquium at St. Andrews in 1986.
>
> A list of Themes for the whole course

This last list was not intended to be "definitive" but a working document that would evolve with the course and with the students. It was intended to indicate some of the issues and topics that required discussion for a full view of *Mathematics in Context*, and some of the Themes which could form the general area for projects.

We explained that we were concerned that students were not given a professional's view of the place and role of mathematics, and that they were bound to be unaware of the nature and extent of current mathematical research.

We asked them if they thought mathematical research was a common activity in universities They agreed that they had deduced or suspected that mathematical research probably went on in Mathematics Departments, since they were aware that research was a general requirement for universities. However, they had no idea what that research consisted of, nor how much mathematics research generally went on. Because of this, the first task that we set the students was to go to the Library, find *Mathematical Reviews*, and using this as a source, to come to some conclusions on the quantity of mathematical research, its changes over time, and the balance of subjects Of course, it would be quite impossible to read the whole material, and so some kind of sampling, and some splitting up of the analysis among the group, would be desirable. We did not delimit this task, except to point out that a report was required for next week, nor did we suggest what conclusions might result. Our attitude was: "Here is a potential resource. How might it be used in the time available?"

The response to this exercise was excellent Next week the students reported that mathematics was clearly expanding, judging by the increase in the number of pages of *Mathematical Reviews* over the years. One student had done a survey of numbers of papers in Ballistics over the years, and had come to some interesting conclusions. All of the class were astonished at the extent of current mathematical research. We were asked why we could not have just given them the information on the numbers of pages in *Mathematical Reviews* over the years. This perhaps illustrates the way in which students have been conditioned to expect information to be given to them in handy parcels. We explained that there was no substitute for picking the volumes off the shelf: Never mind the quality, feel the weight!

Themes and projects as a focus

We used the list of Themes as a focus for the initial sessions and also for the discussions leading up to the students' choice of project subjects. The listed themes were not intended to limit the course, rather they indicated our conception of some of the potential diversity of the area. Many of the themes were hardly discussed, if at all. This might be viewed as resulting from over-ambition on our part but in fact we intended the list to be more a stimulus than a syllabus!

The list is given in two parts, since the second set was given out at the start of the second term to widen the possible areas for discussion

MATHEMATICS IN CONTEXT:
THEMES FOR DISCUSSION I
(Given out at the start of the course)

A. What is mathematics?
 1 Notation
 a) improvements in notation through the ages
  1) numbers 2) graphs 3) geometry 4) algebra
 b) current notations
 c) computers
 2 Argument and logic
 a) origins of logic

b) logic and computers
   1) The history of computers
   2) The next generation of computers
   3) Artificial intelligence

c) probability

3 Calculation

a) numbers   b) areas and volumes   c) speeds and derivatives
d) interest   e) estimation   f) computers

4 Concepts

a) groups   b) symmetry   c) number   d) function

5 Problem solving

a) methods   b) strategies

B The extent of new mathematics
1 Numbers of mathematicians in universities
2 Numbers of mathematics undergraduates
   a) in UK   b) in EEC   c) in world
3 Quantity of research output
4 Quality of research output

C The funding of mathematics
1 University teaching
2 Further education and higher education teaching
3 Research
   a) Total funding   b) Comparison with other science subjects

D The importance of mathematics for science
1 Physics   2 Chemistry   3 Biology   4 Engineering

E The importance of mathematics for technology
1 Engineering   2 Computer science   3 Physics

F The importance of mathematics for commerce and finance
1 Use of old established techniques
2 Use of recent mathematics
   a) Cryptography   b) Optimisation

G The teaching of mathematics
1 Mathematics as a core subject in schools
2 The relevance (or otherwise) of university mathematics
3 New methods of teaching mathematics in schools
a) GCSE (A new examination in secondary schools in the UK, which involves pupils in investigations in mathematics)
b) Investigations
c) Where should A-level go? (A-level = U K advanced level examination taken at age 18).
d) Skills v. understanding
4 The teaching of communication skills
5 The teaching of writing skills
6 The teaching of problem solving skills
7 The teaching of problem formulation skills

H. The public image of mathematics
1 Measures of the public impression of mathematics
2 Mathematics and numerical skills
3 Improving the public image of mathematics
4 Mathematical recreations

I The employment of mathematicians
1 Where do mathematicians get employment?
2 The McLone report
3 Mathematics and computer science
4 Mathematics and software

MATHEMATICS IN CONTEXT: THEMES FOR DISCUSSION II

(Given out at the start of the second half of the course).

A. Nature of mathematics
1 Notion of proof
a) Any subject has criteria of validity.
b) Lakatos: Proofs and refutations
c) Creativity versus rigour
d) Influence of computers — the proof of the four colour theorem
e) Large proofs — the classification of finite simple groups
f) Stochastic theorems — article in *Scientific American*
2. What is interesting or good mathematics?
3 Current view of mathematics as based on logic and set theory
a) Lawvere view
b) Current programming debate on functional programming
c) Constructive mathematics, many-valued logics, fuzzy mathematics
4 Undecidability and computability
   a) Limits on mathematical applications
5. Infinitesimals and non-standard analysis
6 Category theory and mathematics as the study of structure
7 Logic and computers — complexity

B. Applications of mathematics
1 The influence of applications on the development of mathematics
2 Catastrophe theory: a case study in theory and applications
3 Mathematics and biology:
   a) Zeeman's Presidential Address
   b) Thom
   c) Dawkins and the evolution controversies
   d) Knots and DNA
4 The importance of probability in applications
5 Topological ideas in mathematics and its applications: differential equations and physics
6 Operational research, optimisation, finite mathematics, mathematics and management

C. Mathematics and society
1 Provisions of teachers of mathematics
2 The funding of mathematics
a) University funding
   1) UK   2) USA   3) France

Essentially each session was organised as a discussion lead by us in order to focus on and formulate issues in mathematics, with a view to directing students to areas where they might carry out specific projects. Because of the broad nature of the course and of the issues, there was a danger that the course would become completely shallow, reminiscent of the apocryphal essay question: "Write a brief history of human thought and compare it with some other kind of thought." The intention was to avoid this danger by leading the discussion to suggestions for projects which had an available base of information and knowledge, or where something could be attempted with a reasonable standard of rigour.

The miniprojects, examples and overview

Part of our discussion with students was about the nature of the projects and what we expected The qualities we looked for were:

Understanding the overall aims of the area of study
Assessment of materials and facilities.
Formulation of specific objectives for the essay or project
Clarity of thought and exposition.
Evaluation of the difficulties involved in carrying out the project as formulated, given the limited time and the possible difficulties in finding the material and documentation.
Evaluation of the extent of achievement of the original objectives.
Quality of presentation.
Possibilities for further work

We emphasised that a project should have a clear statement of aims, some kind of conclusion, and some assessment of methodology and the extent to which it had been possible to achieve the aims within the time scale and other restrictions necessarily imposed. In this respect, a properly carried out analysis of failure is as interesting in its own way as success.

A full list of titles of projects is included in the section "Evaluation"; there we will mention some examples in more detail. The miniproject could take the form of an essay, but we encouraged students not to restrict attention to that medium For example, one student found the idea of an essay boring and decided that his first project would consist of the preparation of exhibition material together with an analysis of the difficulties involved in presenting his chosen subject (Fractals) to the general public He produced five A3 boards illustrating ideas basic to this area, such as selfsimilarity, iteration and dimension In addition he wrote two booklets, one a brochure to accompany the boards, the other an essay explaining why he had chosen that particular medium and the difficulties he had encountered. Another student produced material suitable for use in Master-classes, again with an analysis. Another looked at the success of integrating the mathematics syllabus in schools with those of other subjects The analysis in this case included reactions to her requests for information from various mathematics advisors One of the most striking pieces of work was an essay on mathematics and art; this attempted to define relations between the two activities, and the text was illustrated with well chosen reproductions

How the assessment worked

The assessment of the course was a part of the students' final examination, contributing one twelfth of the final mark. This relatively low proportion meant we had to remind the students not to spend too much time on miniprojects and neglect their other courses.

Here is our agreed procedure for marking. Each miniproject was double marked The two markings were independent. When both were completed, we discussed each case in turn, defending our view of the merit of the work being considered. Both mark sheets and the final mark were made available to the external examiner.

The following shows the format of the assessment form used for marking the projects. This formulates some of the qualities for which we were looking.

Mathematics in Context Assessment form

Name of student:
Project title:
Project number (1/2):
1 Conception of project
  (a) Aims
  (b) Method
  (c) Analysis
2. Relation to overall theme of course
3 Initiative in choice of project.
4. Initiative in carrying it out
5 Reading required.
6. Use of references
7. Originality.
8 Assessment of evidence
9. Presentation
10 Other comment

Final Mark:

Assessor: Date:

Evaluation

At the examiners meeting in June, 1989, the external examiners commented on the excellent quality of work submitted for this course and said that the course seemed to have been very successful We asked the students their opinions and have so far had written reactions from two students, both very favourable, as well as comments in the projects themselves which made it clear how much the work had been enjoyed A key reason for this enjoyment seems to be that the course gave the students an opportunity to work on something which they themselves had chosen

We have been surprised and delighted by the quality of the work produced, and the way in which students have responded with energy, enthusiasm, and independence. We particularly liked the way, noted earlier, in which two students not only wrote on a specific topic, but also analysed the methodology underlying their approach. Mathematics could be criticized as a profession for its lack of interest in the understanding and the teaching of methodology.

The aim of the sessions was to discuss themes with the intention of helping students to formulate specific projects to be undertaken for the essays Unfortunately notes were not taken of the discussion and so we do not have a proper record of the progress of the course It would have been a good idea if, each week, a student had been required to take notes and prepare a written record of the topics and of the discussion.

As we have said, the course had for us a large element of risk and we were not sure how it was going to go. Did we succeed in keeping students within the limits of "context"? This is a matter of opinion and it is impossible to be definitive about it Certainly the majority of the miniprojects came well within the general framework of the "context" heading. As an indication of this we include below a list of the titles of the various projects This can only give a vague picture of what was produced, but does at least give that:

Project Titles: 1988/9

(The following are listed in pairs: each student wrote two projects.)

DJA	(i)	The importance of mathematics for commerce and finance
	(ii)	The usefulness and aim of an education in mathematics
SLB	(i)	Changes in primary mathematics — its effects
	(ii)	Applications of statistics (or manipulation of statistics)
KC	(i)	Worksheets based on patterns and colourings Report based on worksheets
	(ii)	Maths and art
DCH	(i)	Popularization with puzzles
	(ii)	Thoughts on the funding of mathematics in British universities
SK	(i)	Maths in a topic
	(ii)	The contents of mathematics degree courses around the world
CWP	(i)	An Introduction to fractals and fractal geometry. To examine and aid the popularization of mathematics Exhibition on fractals
	(ii)	A study of basic epidemiology
FR	(i)	Mathematics across the curriculum
	(ii)	The employment of mathematicians
AS	(i)	A lesson in mathematics should be a voyage of discovery
	(ii)	Game theory: concepts and applications

The fact that we can now regard the course as successful and worth developing is due to the responsiveness of the students They surprised us by their clear decisions as to what projects they wanted to do, and their success in taking into consideration the guidelines and criteria which were discussed in the course Much of our detailed advice was concerned with cutting projects down to size, and with helping in focusing aims.

Conclusion and afterword

Two further points have arisen from discussion with students and others One is the view of students on the course that the notion of "Mathematics in Context" should be widespread throughout the undergraduate course. A specialised third year course for a small proportion of students is not providing an answer to the point, made at the beginning of this article, on the required characteristics of an education for undergraduates Another, made by Prof J Montesinos (Madrid), is that before context can be taught to the students it has to be taught to the staff! But that is another story.

Our aim was not to give final answers but to give a professional background and methodology to a discussion of questions of value and context. Is mathematics important? If so, for what reason and in what way? Should there in general be more, less, or the same amount of mathematics taught? On what basis does one begin to formulate answers to these questions? In the main we feel these questions were discussed adequately, given the time. We believe that a number of students feel they are starved of any discussion which will help them come to terms with the rather peculiar nature of the activity called mathematics. So we want to contrast the presentation of mathematics as a normal activity, related to discovery and our exploration of the world, and the overly cynical view that the purpose of a course is simply to find some justifiable way of arranging students in a linear order.

We like the view that one of the aims of education in mathematics is to prove to pupils and students that they are cleverer than they thought they were. To our surprise and delight, this particular course has done this in some cases, releasing independence and creativity which was unexpected and unseen

References

[ACA] Software: a vital key to UK competitiveness. London: H.M.S.O., 1986 (Cabinet Office. Advisory Council for Applied Research and Development)

[AMS1] Renewing U.S. mathematics: critical resource for the future *Notices Amer Math Soc* 31 (1984) 434-446

[AMS2] A M. Jaffe, Ordering the universe: the role of mathematics *Notices Amer. Math Soc* 31 (1984) 589-608

[B1] E.T. Bell, *The development of mathematics* N.Y. McGraw-Hill, 1945 (2nd ed.)

[B2] E T Bell, *Men of mathematics* London, Penguin, 1953

[BP] R. Brown and T Porter. *Making a mathematical exhibition* Proc ICMI Seminar 1989 (available also as Bangor Mathematics Preprint No 89.15)

[D&H1] P J Davis and R Hersh. *The mathematical experience* Brighton. Harvester 1984

[D&H2] P.J Davis and R Hersh, *Descartes dream the world according to mathematics* Brighton, Harvester, 1986

[Dev] K.H Devlin. *Mathematics — the new golden age* London Penguin, 1988

[Gar1] M Gardner, *Mathematical puzzles and diversions* Penguin, 1965

[Gar2] M Gardner, *Mathematical magic show* Harmondsworth. Viking 1984

[Gar3] M Gardner *Knotted doughnuts and other mathematical entertainments* N Y. W.H Freeman, 1986

[Gr] A. Grothendieck. Private communication (1983)

[H] J Hadamard, *The psychology of invention in the mathematical field* Princeton, U P. 1945

[K1] M Kline. *Mathematics and Western culture*. N Y., O U P, 1953

[K2] M. Kline, *Mathematical thought from ancient to modern times* N Y, O.U P, 1972

[K3] M Kline, *Why the professor can t teach: mathematics and the dilemma of universities*. N Y. St Martin's Press, 1977

[K4] M Kline, *Mathematics an introduction to its spirit and use: readings from Scientific American*, with introduction by Morris Kline San Francisco. W H Freeman. 1979

[Lak] I. Lakatos, *Proofs and refutations: the logic of mathematical discovery* Cambridge. C.U.P, 1976

[McL] R.R. McLone. *The training of mathematicians* Social Science Research Council 1974

[P&St] T Poston and I. Stewart. *Catastrophe theory and its applications* London, Pitman. 1978

[St1] I Stewart *Concepts of modern mathematics* Penguin. 1975

[St2] I. Stewart *The problems of mathematics*. Oxford O.U.P., 1987