The Multiplication Algorithm: an Integrated Approach

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The usual approach to the multiplication algorithm in the Netherlands consists of calculation devoid of understanding. For example 52×39 is written this way:

$$\begin{array}{r} 52 \\ \underline{39} \times \end{array}$$

Children in the third grade (9 or 10 years old) have to reproduce this pattern:

> *"Nine twos are eighteen.*
> *Put down the eight, carry one*
> *Nine fives are forty-five*
> *Forty-five and one makes forty-six.*
> *Write down forty-six.*
> *Then put a nought on the next line*
> *Three twos are six".*
> And so on, and so on.

An important difference between this approach and new approaches to teaching arithmetic, including that to the multiplication algorithm by Wiskobas*, is to aim at more understanding of what is going on in the algorithm.

To develop a better approach to multiplication we experimented with some new ideas, starting in a Grade 3 class in 1979. Our aim was to develop the algorithm in harmony with the children's natural way of thinking. This article is a summary of the research. A full report with a detailed description of the results and procedures is available (in Dutch). [1]

2. The pre-algorithmic stage

At the beginning of the third grade, children learn the algorithms for addition and subtraction. They also learn the multiplication facts.

Before we start teaching long multiplication, we tell a story:

> *"I know a family named Squirrel. There are eight members of this family, called Jumper, Quick-boy, and so on.*
> *Oh! I forgot grandfather. Of course, grandfather cannot look for beechnuts any more, because he is too old. You may ask what grandfather is usually doing. Well, his job is counting the beechnuts dragged along by the eight squirrels.*
> *One day each squirrel finds twenty-three beechnuts, i.e, Jumper found twenty-three, Quick-boy twenty-three, and so on, altogether eight times. How many beechnuts will grandfather have counted by the end of the day?"*

The formulation of the problem in the story doesn't contain any reference to a multiplication problem. So we expect a variety of solutions. We wonder how the children will solve 8×23.

There are indeed many different solutions. We show five of them below.

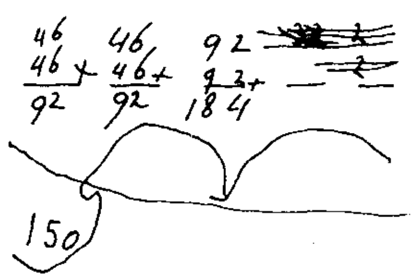

Figure 1

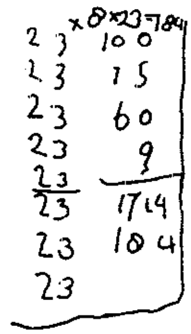

Figure 2

* "Wiskobas" refers to the approach developed by the former Dutch Institute IOWO

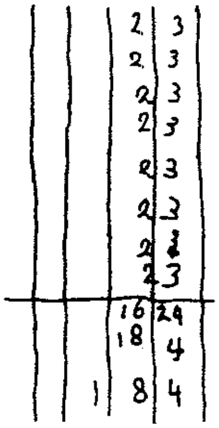

Figure 3

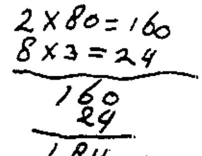

Figure 4

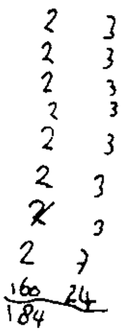

Figure 5

Some solutions, like those in Figures 2 and 3, use the algorithm which had been learnt for addition. But others use it in their own way, as in Figure 5 Doubling seems to be an important strategy as shown in Figure 1 Very surprising is the solution in Figure 4, which uses the distributive property of multiplication

Our conclusion after much experience is that children have few problems if they are allowed to attempt in their own ways multiplication problems which at a later stage they will solve algorithmically. The children use surprising methods, which they understand very well. A wide range of methods of solution occur; in some of them the multiplication facts are used, in others knowledge of the addition algorithm, and sometimes the properties of multiplication. Many of these solutions can provide the starting point for developing the multiplication algorithm

3. Place-value and repeated addition

"The sultan of a Middle East country liked to count his pieces of gold. He was a rich man. His wealth was stowed away in three treasure-chambers, each guarded by big moustached guards. The first treasure-chamber contained pieces of gold, the second treasure-chamber boxes with ten gold pieces in each box, and the third treasure-chamber cases – in each case you would find ten boxes of gold One day, the rich sultan owned

cases	*boxes*	*pieces of gold*
1	*5*	*9*

That day his wealth increased by 67 pieces of gold. What was the wealth of the sultan at the end of the day?"

Children who have already learned addition in an algorithmic way get the opportunity to deepen their understanding of place-value. This approach of packing up pieces of gold is at a higher level – mentally – than the change approach on the abacus, because the abacus contains a maximum of 20 beads on each bar, too few to represent 67 pieces of gold (= beads), for instance

The story of the sultan is important, because children realize what is going on when the sultan packs up his gold At later stages the children remember that particular story, preventing failures in place-value and change problems

Let us consider this situation: *the sultan receives 67 pieces of gold from a farmer who pays him rent. What happens?* Well, the first guard fills up an empty box with ten pieces of gold. Then another box with ten pieces, and so on. He describes his activities in this way:

Cases	Boxes	Gold
		67
	1	57
	2	47
	3	37
	4	27
	5	17
	6	7

Figure 6

So he finishes with six boxes, and seven pieces of gold are left. Each time the question can be asked: *How many pieces are left when the guard fills up three (four, ...) boxes?*

You are right, of course if you say that this level of exchanging is a low one Yet many children need it, would be my reply. There will be children with more knowledge and experience who find the solution directly, in this way:

Cases	Boxes	Gold
		67
	6	7

Figure 7

They know a lot about place-value!

The story continues: *That day eight farmers paid their rent, sixty-seven pieces each. The guards count and record:*

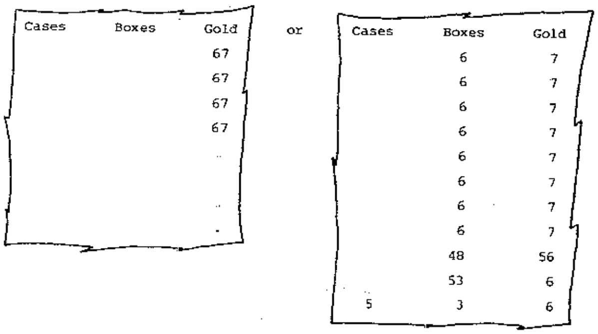

Figure 8

Again, you can observe many solutions. Some children add: $7 + 7 = 14, \ldots + 7 = 21, \ldots + 7 = 28$, and so on. Other children who remember a lot of the multiplication facts they have just learned, use them:

$$8 \times 7 = 56, 8 \times 6 = 48$$

Again, some make the exchange at once, others need "the long way" of working step-by-step.

We conclude that you can make a start on teaching the multiplication algorithm by using a repeated-addition strategy. However, important conditions for success are comprehension of place-value and some knowledge of multiplication facts.

4. The important factor 10

A few lessons later when cases, boxes and pieces of gold are given the abstract designation hundreds, tens and units, the problem to solve is 12×57.

How can it be done?
Use multiplication facts, please.
Most of the children calculate and record like this:

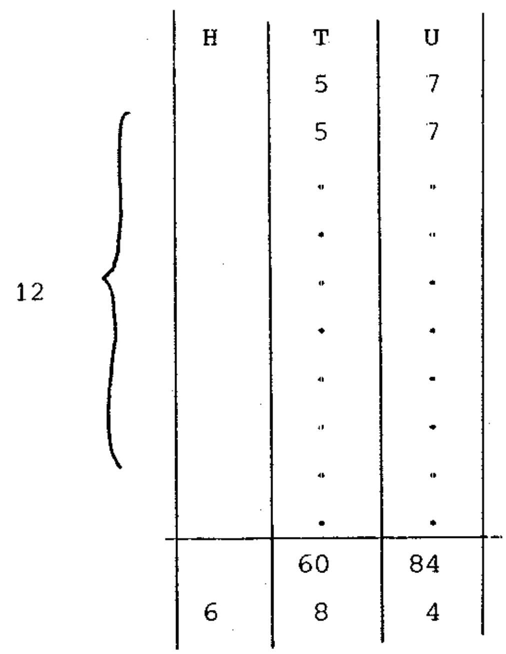

Figure 9

Of course, it is not necessary to record all the 57's. On the contrary, it is unwise to do that. Children have to shorten their solution if they can.

In the Netherlands nearly all children learn the multiplication facts up to 10×10. So 12×57 must be solved in some way such as 12×7 (units) $= 10 \times 7 + 2 \times 7 = 70 + 14 = 84$ (units).

Children might solve it this way: 10×7 (units) $= 70$, plus 7 is 77, plus 7 is 84. Multiplication of 10×5 (tens) in the same way. In both solutions the factor 10 plays an important role. So at this stage the important role of the factor 10 has to be emphasized. The multiplication problem 12×57 will be recorded in this way:

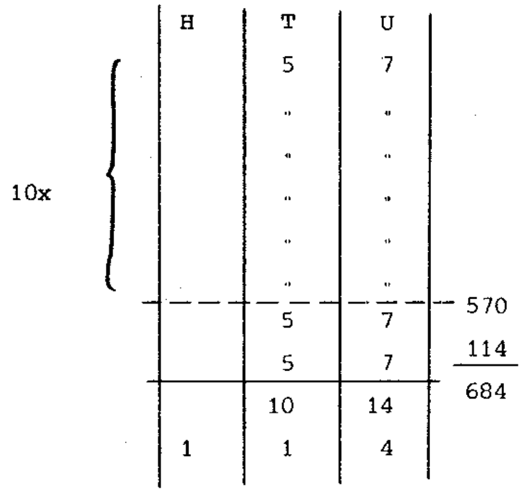

Figure 10

Many children have already discovered: "Ten times something is something with zero behind it"

Two partial products arise. In the last step they add the two partial products. For this procedure children need the multiplication facts $10 \times \ldots$
and $2 \times \ldots$
This is easy for them, because they have understood and mostly memorized these facts.

5. After fifteen lessons: 62×45.

In the algorithm programme and also in memorisation lessons of basic facts the children's skills increase in varying degree. One can notice three differences, which indicate the children's level:

- How are the exchanges done? In one step or more?
- Are they using multiplication facts instead of repeated addition?
- Do they shorten their schema or their recording?

There are many opportunities for the children to reach a higher level of development of the algorithm because there are many planned exercises.

After about fifteen lessons we gave the children a test, containing these multiplication problems:

part A	*part B*
7×43	62×45
12×49	8×416
84×8	17×68
60×35	59×89

The test takes thirty minutes. All children finished part A, nearly all of them finished 62×45 as well. We show a few interesting results:

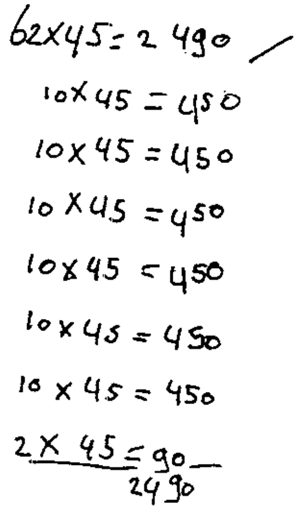

Figure 11

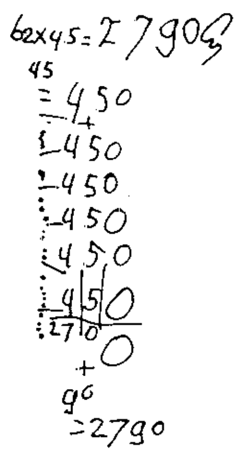

Figure 12

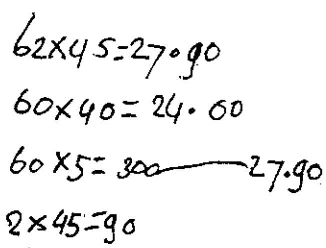

Figure 13

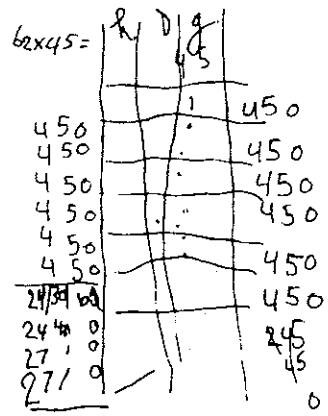

Figure 14

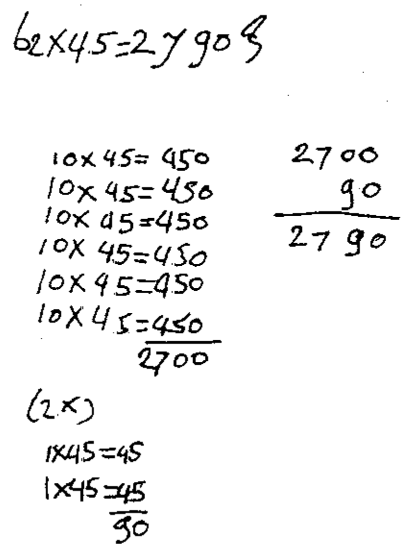

Figure 15

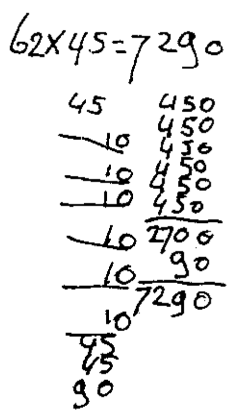

Figure 16

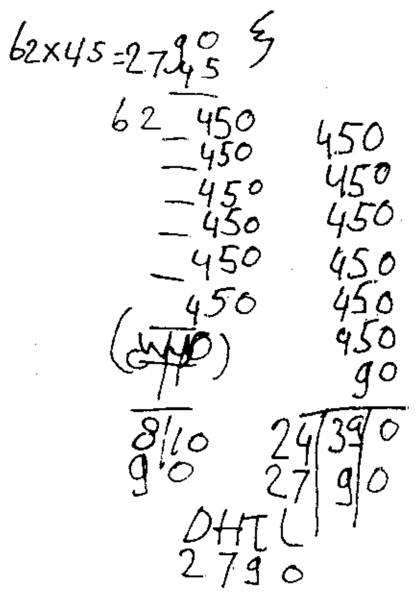

Figure 17

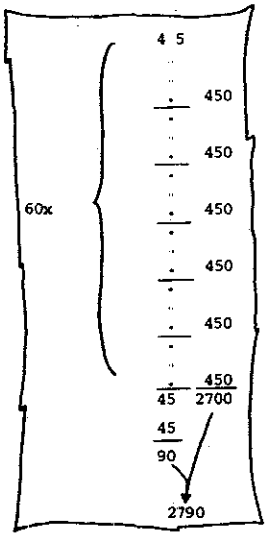

Figure 19

or

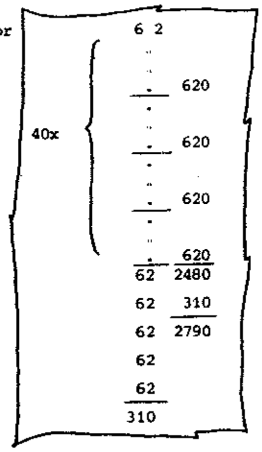

Figure 20

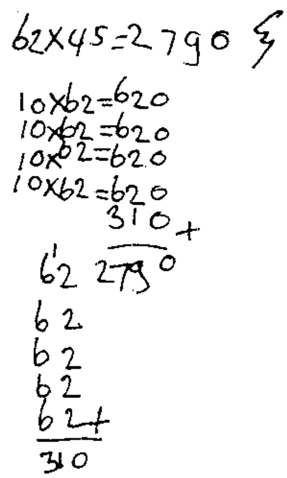

Figure 18

Figure 14 shows no solution The recording is at a low level and is based directly on the sultan story Some children solved 62 × 45, others did the commutative problem 45 × 62 In 62 × 45, 2 × 45 = 90 is easy (by doubling), as shown in Figure 12, and probably in Figures 11 and 13 The pupil in Figure 13 shortens the schema very well. He sets the fashion: we know the direction we shall use to develop the algorithm. Figure 17 and Figure 14 show that exchanging is not without difficulties.

After the test it seems clear that we can expect this level from our children:

6. Toward the end of the partial course

The end of the course can be established as the time when children are able to multiply in an algorithmic way. Ask yourself if this means that you are able to solve problems like this:

$$\begin{array}{r} 47958 \\ \underline{4963} \times \end{array}$$

My answer is no!

After about thirty lessons the children generally show the following type of solution:

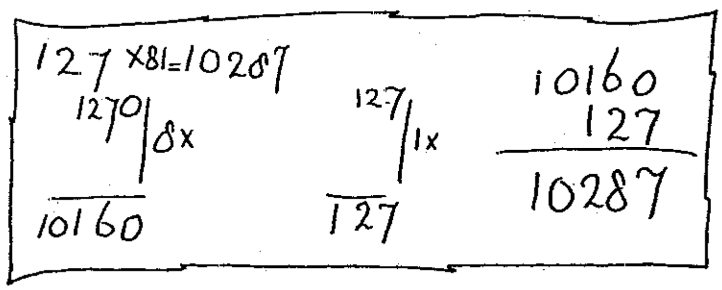

Figure 21

Note the way they shorten their schema. One remembers that 80 × 127 = (8 × 10) × 127 = 8 × (10 × 127) = 8 × 1270! This is quite clearly the algorithm everbody knows.

So there are different levels of algorithmisation The children are able to remember a later stage of algorithmisation if it is necessary for them. An algorithmic problem like 238 × 46 is usually solved in another way:

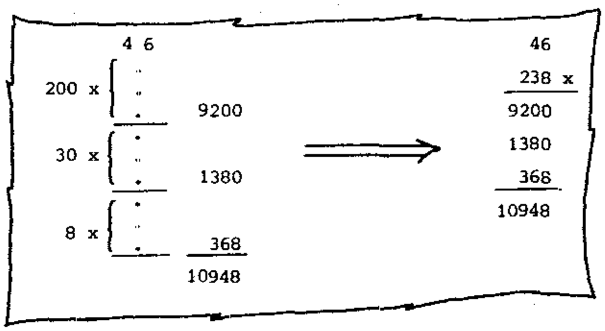

Figure 22

Or, by choice:

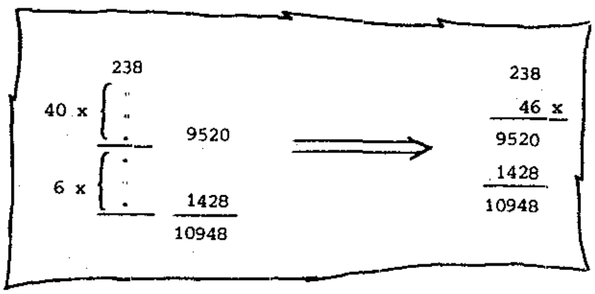

Figure 23

7. Principles

Learning the algorithm of multiplication in the way shown is characterized by an increasing understanding of the algorithm. When there is a need for a better understanding it is desirable that the children go back to a previous stage which they understand well Even at the end of the process it is possible to fall back to the natural stages of the beginning.

The shortening of the notation scheme used by the children is very important Shortening the schema and understanding the algorithm go hand in hand

A few of the principles mentioned are, in summary,

- the course starts with context problems, like those of the sultan stories
- the method of repeated addition is successively replaced by the use of multiplication facts
- the role of the factors 10 and 100 is important
- the relational use of flexible calculation, based on commutativity, distributivity, and the basic facts of multiplication, is characteristic of this course
- although not described in detail, the pupils' verbalisations of their solutions form an important factor in the development of higher student levels.
- the main goal of the course is to employ a process of increasing, progressive schematisation

Note

[1] *Cijferend vermenigvuldigen en delen volgens Wiskobas* by A Dekker, H ter Heege, A Treffers. Utrecht (OW OC). 1982

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