beyond the buttons with numbers and the three buttons below that could be taken as numbers and beyond, to 15 and 18.

We end by asking whether our account of a this incident extends in some small way the notion of ZPD into a new area that might resonate with experiences that colleagues have encountered, or whether current alternative learning theories might have useful things to say by way of explanation. We also want to consider whether there is anything from this episode that we can use as teachers. Mellony has already used the video clip to indicate to early years teachers that children, when they enter school, may know a lot more about mathematics than the teachers might think, and that their environments (even the TV room) are rich contexts for mathematical exploration and extension if parents choose to engage children in these contexts. How might we structure numeracy lessons to stimulate the emergence of such learning events in contexts where there is increasing prescription of what must be taught and when? Are there other insights that the incident might illuminate for teachers?

Notes

[1] Mellony's teacherish style relates to her identity as a mathematics educator. She taught mathematics for several years, is a mathematics teacher educator and runs a weekly mathematics club.

[2] Duckworth's paper tells of 7 year-old Kevin who, before being told the aim of an activity to put a set of different length drinking straws into order from smallest to biggest, says "I know what I am going to do" and proceeds to take the straws and order them by size himself. He was very proud of himself and Duckworth puts this down to the task having been self-set, as she calls it.

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Intercultural dialogue and the geography and history of thought

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First vignette: China and Italy

Mariolina (Bartolini Bussi) is talking at a conference with a Chinese colleague, Xuhua. She has just presented a report on fractions and is writing on the whiteboard with a felt pen. Suddenly Mariolina notices that Xuhua writes fractions in a strange order, first the denominator, then the fraction bar and eventually the numerator:

Mariolina: Why do you write fractions in this way?

Xuhua: What do you mean? How should I write them?

Mariolina: I mean the order. We write them in the reverse order (top-down): first the numerator then the fraction bar and last the denominator.

Xuhua: Very strange, indeed! How do you know how many pieces you wish, if you do not know in how many pieces you have cut the whole?

Second vignette: Italy and Burma (Myanmar)

Mariolina and Alessandro (Ramploud) are talking with two Burmese colleagues (Thein Lwin, a mathematician, and Ko Ta, a doctor and coordinator of a network of Monastic schools) who are visiting their department:

Mariolina: How do you write fractions in Burmese? For instance two thirds.

Thein Lwin: [*Is a bit surprised, writes 2/3 top-down*] Why?

Mariolina: I have read in Wikipedia that the Burmese order is the same as the Chinese one: bottom-up.

Thein Lwin: [*Shakes his head*] No, it's the same as yours!

Ko Ta: [*Smiles*] I am not a mathematician!

Ko Ta closes his eyes, takes a pencil and traces gestures in the air. Alessandro has the impression that Ko Ta is looking for a kind of motion memory of the gesture used when he was a child in a primary school. After some seconds, Ko Ta smiles and shows a bottom-up process: first 3, then the fraction bar and eventually 2.

Thein Lwin: [*Smiles and nods*] He's right. I agree!

These two vignettes tell us a simple story. Chinese and Burmese are in the same family of Sino-Tibetan languages. Hence, it is not surprising that their way of saying fractions (and the process of writing fractions) are similar. Yet in Chinese the traditional process of writing (order) and saying fractions is still the same as in the past, taught in the same way in textbooks, whilst in Burmese it seems that a "Western" habit is changing the tradition. It would be interesting to know whether this process depends on the effect of colonialism (that for decades designed the Burmese education system according to the British tradition) or on the effort to run after Western mathematics and mathematics education as a way to overcome the negative effects of military rule. This issue deserves further analysis; however, it helped the participants in the interaction to reflect on each other's own *un-thought*. Here we are quoting

Jullien (2006), the French philosopher and sinologist, who explains his decision to start to study Chinese and to move to Beijing as a way better to understand the European and Greek philosophy. To observe one's own culture from a distance helps to understand one's own un-thought. The *geography of thought* (Nisbett, 2003) allows us to become aware that our beliefs are relative and that they could have been different had we come from different parts of the world (Bartolini Bussi & Martignone, 2013; Bartolini Bussi *et al.,* 2013).

The history of thought

These stories also raise our curiosity to learn about the *history of thought.* All European languages now share the top-down writing process of fractions and the consequent naming order. What are the roots of this process? *Liber Abaci* (by Leonardo Fibonacci, who introduced the so-called Indo-Arabic notation to Europe), wrote:

> When above any number a line is drawn, and above that is written any other number, the superior number stands for the part or parts of the inferior number; the inferior is called the "denominatus" (denominator), the superior the "denominans" (numerator). Thus, if above the 2 a line is drawn, and above that unity [1] is written, this unity stands for one part of two parts of an integer, *i.e.* for a half, thus ½. (As quoted in Cajori, 1928, p. 269)

Hence we know that the order of describing fractions (and probably, we assume, also that of writing fractions) for Leonardo Fibonacci was (in line with the "Eastern" order):

> denominator → fraction bar → numerator.

Probably the reverse top-down order used later was an effect of the standard way of writing from the top to the bottom of the sheet. The final written products are the same!

Yet there is still the issue of ordinal numbers. Why is the denominator expressed in ordinal numbers? This is even more counter-intuitive. We have not yet found any satisfactory answer to this second question in the books on the history of mathematics or in conversations with historians. We guess that it is related to the importance (as it was already in ancient Egypt) of unit fractions that were used more often than other fractions, and, in some cases, instead of other fractions. There were rules (also studied by Leonardo Fibonacci) that allowed the writing of any fraction as the sum of unitary fractions and this writing helped to solve practical problems in a very effective way. For instance, to divide 5 pizzas among 8 children, one can say that each child has 5/8 of a pizza, but this requires cutting each pizza into 8 pieces and giving 5 pieces to each child. It is quite different from what somebody would do in everyday life! The sum

$$\frac{5}{8}=\frac{1}{2}+\frac{1}{8}$$

mirrors the more natural idea of cutting 4 pizzas in half (to give one half to each child) and then dividing the last one into 8 parts, to give a small piece more to each child. This solution is similar to the one found in ancient civilizations and in the Liber Abaci itself. The recourse to the sequence of unitary fractions in problem solving could have been so natural and frequent that they were considered a special genre of numbers, similar to the natural ones:

$$1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{5},\frac{1}{6}$$

and so on. In this sequence, the order corresponds to the wording of the denominators (at least from the third one). We know that the systematic approach to general fractions with any numerator is a recent idea. Even more recent is the idea of considering fractions in mathematics education as numbers to be represented on a number line, exactly like the whole numbers.

Implications for mathematics education

The case of fractions is just one example of the richness of taking a different perspective on our own un-thought about a mathematical process. Discovering that some issues that had been considered obvious are, on the contrary, the products of long and complex cultural processes prompts teachers to reflect on their beliefs and on the hidden choices made in their context. Although a direct transposition might be impossible, we know that Western languages and traditions are not always the best ones to hint at the genesis of some mathematical processes. In the case of fractions, some Eastern languages seem to be to be facilitators for the construction of meanings (see Siegler *et al.,* 2013).

> **Third vignette: Italy—interaction between an expert and a low achiever**
>
> Anna (Baccaglini-Frank) is working with a low achiever, L, using the software *Motion Math* [1] an app for the iPad, in which learners have to tilt the device to make a falling ball containing a fraction fall towards the right point on the number line [0,1] (for a video, see [2]).
> L seems to be confused by the task. Without an intuition about the position of the fraction on the line it is not easy at all to tilt the device quickly enough during the very short falling time. Anna tries to help him by reading the falling fraction. She is using the Western mode: two thirds, three fourths, and so on.
>
> *Anna:* [*suddenly changes the way of reading*] Let's name the fractions as Chinese do!
>
> *Anna:* [*½ falls*] Of two parts, take one!
>
> *Anna:* [*¾ falls*] Four parts, three!
>
> L is a bit surprised, starts to be less anxious and improves very quickly his performance. The improvement is more evident with unitary fractions (*e.g.,* 1/5).
>
> *L:* Oh yeah, I have to divide the segment into 5!
>
> The same happens with other low achievers.

Motion Math exploits both epistemological and cognitive analyses of fractions (Riconscente, 2013), emphasizing, on the one hand, the importance of using the number line to give coherence to the study of fractions and of whole numbers and, on the other hand, the neurological evidence of the mental number line (Zorzi *et al.*, 2002). Moreover, *Motion Math* exploits embodied learning and, in particular, the integrated perceptual-motor approach (Nemirovsky *et al.,* 2012) in the development of such a mental number line.

From her research on students with mathematics learning difficulties (Karagiannakis *et al.,* in press), and in particular when engaging in interventions with low achievers, Anna is learning to combine neuroscientific findings with the outcomes of the intercultural semiotic analysis discussed in our research group, to smooth the scarce transparency of the Italian wording.

This very short episode from a study in progress shows the synergy between intercultural dialogue, neuroscience and technology for defining effective teaching-learning situations. We hope that this synergy will be further and more deeply developed in the future, and applied in mathematics teacher education and development.

Notes

[1] motionmathgames.com
[2] www.youtube.com/watch?v=hmm0D90vcYI

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Zed: the structural link between mathematics and mathematics education

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Zoltan Dienes (1916-2014), known as Zed, passed away on 11 January 2014. Some of us see it as a culmination of an era. Mathematics education is prone to forgetting its origins within the realm of mathematics, and Zed's passing away serves as a reminder to those who have witnessed the weakening of these origins. My intersection and interest with Zed's work began in the mid-1990s when there was excessive focus on the social turn in mathematics education. Being trained in mathematics, it was difficult to stomach at that point in time the associated set of sociological problems that were being addressed by mathematics education. I initiated a correspondence with Dienes which led to my discovery of *Building Up Mathematics* (Dienes, 1960) and *Thinking in Structures* (Dienes & Jeeves, 1965). Both these books have been influential to a generation of mathematics educators who entered the field in the 1970s and they remain classics to this day.

Trained as a mathematician in England, Zoltan became interested in the psychology of learning in the 1950s and earned a second degree in psychology. The field of mathematics education, seen through its origins in mathematics, is often outlined in terms of the classical tradition of Felix Klein followed by Freudenthal's re-conception with an emphasis on the humanistic element of doing mathematics. While the approach of Klein, steeped in an essentialist philosophy, gave way to the pragmatic approach of Freudenthal, Zoltan's approach influenced by structuralism and cognitive psychology remains unique from the point of view of developing a theory of learning which has left a lasting impact on the field. Most importantly this theory was grounded in fieldwork with school children that experienced the multi-embodiment approach to a mathematical idea through manipulatives, games, stories and even dance, before they were encouraged to abstract the essence of the activity leading to mathematical generalizations. The six-stages of learning consisted of free play, games, commonalities, representation, symbolization and finally formalization. I have always considered his approach to mathematical learning (and teaching) as falling within psychological structuralism à la Wilhelm Wundt because of its nuanced and layered approach to encouraging abstractions, with formalizations only occurring at the very end. This, to me, was similar to the focus on introspection as the method used by structuralists to understand conscious experience.

Zed's use of his theory of learning was powerful (to put it mildly). He had grown up surrounded by mathematicians. His father was a mathematician by training and gave Zed a book he authored on Taylor series for his 16th birthday. Zed's PhD thesis generalized one of Baire's category theorems by using Brower's intuitionist approach. In other words Zed believed in constructive mathematics in which *reductio ad absurdum* was viewed as a logical trick and frowned upon. When I met him in 2006, he was pushing 91, with a mind keen and fertile to talk mathematics (Sriraman & Lesh, 2007). I complained about being unable to find multiple embodiments to facilitate the learning of ideas in analysis. Two months later he sent me a paper on "A child's path to the Bolzano-Weierstrass theorem" (Sriraman, 2008), which essentially contained a structured story which allowed one to discover this deep theorem!

Zed embodied the common ground between mathematics and mathematics education, in a life that was dedicated to exploring the beauty of mathematics by making it accessible to schoolchildren. Given the climate of the "math wars" in the US and similar debates elsewhere in the world, it seems ironic that his seminal work on *Building Up Mathematics* remains forgotten. This book would appeal to both mathematicians and mathematics educators because of its focus on the foundational structures of mathematics.