METAPHOR AND LEARNING MATHEMATICS

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Since the introduction of conceptual metaphor theory in the late 1970s (*e.g.,* Ortony, 1979; Lakoff & Johnson, 1980), the role of metaphor in learning has evolved into a major theme in the cognitive sciences. In tandem, discussions of the role of metaphor in mathematics have been prominent since Pimm (1987) helped to bring the matter into focus some decades ago. Among mathematics education researchers, it is the latter that has attracted more attention, and so the considerable progress that has been made on understanding the role of metaphor in the learning of mathematics has been more focused on the mathematics than the learning. That observation serves as the backdrop of this discussion.

Another part of that backdrop is a multi-year review of more than 2500 discourses on learning in education, each examined for orienting foci and implicit metaphors. That evolving study is presented as a website (Davis & Francis, 2019), and we draw heavily on its "mapping" of discourses in our thinking. In particular, this writing represents an attempt to engage simultaneously with two of its overlapping regions—namely, "Embodiment Discourses" and "Embeddedness Discourses", along with the webs of figurative association invoked across them.

Prompted by that analysis, we are intrigued by the suggestions that, (i) when it comes to making sense of learning, it is more appropriate to think in terms of clusters of metaphors than two or three prominent ones, and (ii) there are two especially useful clusters for making sense of the *learning* of mathematics. One addresses the subjective, largely unstructured level of individual sense-making. Unfortunately, invoked alone, there seems to be a tendency for its descriptions to fade into prescriptions for learners' experiences in the hands of educators. In our experience, that tendency is mitigated when that cluster is deliberately associated with the other, which is concentrated in the intersubjective realm of social action and cultural accord. This second cluster of metaphors usually operates more implicitly and is much less volatile. With regard to mathematics, it is mostly manifest in definitions, conjectures, and inferences, and with regard to schooling, it is most commonly invoked around matters of norms, structures of engagement, habits of interpretation, and so on.

Before going much further, it should be noted that, for the most part, "we" is used in self-reference in this writing. In some spots, however, "we" can be read to refer more broadly to a partnership of researchers, resource developers and teachers who have been working together for nearly a decade. Some of the products of that shared work are available online, in the form of an interactive website where the elaborated "we" present an emergent model of mathematics teaching (Math Minds, 2020).

Recursive, raveled difference

As Sfard (2015) has observed, "Whereas everybody seems to agree that learning must be defined as some kind of change, there is no consensus with regard to what it is that changes" (p. 129). That is, it seems that the sole point of agreement across perspectives on learning is that something is different after learning has happened.

On the matter of "difference", our own review of embodiment and embeddedness discourses on learning reveals another emphasis. Across these two clusters, learning is understood not just in terms of change—that is, after-the-fact difference—but as in-the-moment acts of perceiving, amplifying, and utilizing difference. Phrased differently, our starting points in this analysis are the coupled realizations that (i) human learning is enabled by acts of differentiating and (ii) formal domains of human knowledge are means of organizing distinct sets of noticed differences. With nods toward both these elements, Bateson (1979/2002) commented:

> Science is a *way of perceiving* and making what we may call "sense" of our percepts. But perception operates only upon *difference*. All receipt of information is necessarily the receipt of news of difference, and all perception of difference is limited by threshold. Differences that are too slight or too slowly presented are not perceivable. They are not food for perception. (p. 27)

As we develop, the final two sentences of this passage have a particular relevance for teaching mathematics, which we understand principally as managing the strength and pace of variations.

But teaching mathematics is also about well-sequenced juxtapositions of variations. That is, learning mathematics is not just about noticing; it is about associating those noticings in more and more sophisticated and specific ways. That, of course, is not news. But, with regard to learning mathematics, it has only been over the last half century that researchers have come to terms with the cognitive dynamics that are most important in enabling these associations among differences. In this regard, and to the surprise of many, analogy plays a more significant role than deduction—with metaphor figuring especially prominently, as Pimm (1987) and others (*e.g.*, Lakoff & Núñez, 2000) have illustrated.

Partially in consequence, learning mathematics is better characterized as an elaborative phenomenon than an accumulative process. Mathematical knowing is tiered, involving occasional leaps of reorganization as a weight of experience and interpretation compels new, more expansive ways of thinking. Embracing this core element of embodiment discourses on learning, Pirie and Kieren (1994) proposed a visual metaphor of varied and nested modes of mathematizing. Like them, we are interested in *organizing understanding,* and we share a view of understanding as unfolding in hierarchical and recursive levels. With a primary interest in what such a view might mean for teaching mathematics, we frame our analyses in terms of categories of difference *(i.e.,* what and how teachers might vary), and we focus our attention on the boundaries between those categories. That is, we are interested in the transitions that bring new levels of apparent mathematical objects into being (see Figure 1).

It is at the boundaries between levels of variation, we argue, that mathematics happens—and, so, those are the sites to which pedagogy must prompt attention. The particulars of these boundaries are identified through a process of raveling mathematical ideas and intentionally prompting to key transitions.

The choice of the word "raveling" in the preceding sentence is deliberate. A contranym, raveling can be used to describe both the weaving and the untangling of a rope or web—and we mean to summon both meanings simultaneously when we use the word. That is, for an educator, raveling a mathematical concept involves both deconstructing and reconstructing its web of associations. By engaging in deliberate and disciplined processes of raveling, we have found that we can greatly increase the likelihood of prompting learners (i) to make mathematically significant critical discernments (and thus be ready to act upon them at higher levels) and (ii) to develop a shared understanding that allows learners to proceed *together*. Ultimately, then, movement between levels allows *intentional* and *accelerated* movement between the implicit-idiosyncratic-unstructured (main emphases of embodiment discourses on learning) and the explicit-intersubjective-structured (main emphases of embeddedness discourses on learning).

Because we are stressing the importance of particular content and of prompting to that content, we are compelled to consider (i) how we might organize (or ravel) mathematical content and (ii) the nature of prompting. If we first take care to ravel the layers of metaphor and logic that comprise a particular mathematical space of relevance, we have found that principles of variation pedagogy can powerfully prompt awareness not just to the *things themselves* but to the various transformations that comprise them. Here, we draw from Marton's (2015) theory of variation, as well as from articulations of a variation *pedagogy* common in China (Gu, Huang & Marton, 2004; Lai & Murray, 2012).

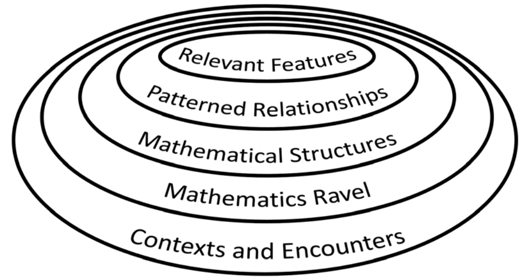

Figure 1. What happens at the boundaries of categories of difference?

Marton (2015) highlighted that the noticing of a specific aspect of an object of learning is greatly enabled by encountering variations of that aspect. This "Principle of Difference" is especially relevant to the progression we describe here. Taken seriously, this principle has the potential to uproot much of how we think about teaching and learning. Rephrasing, it emphasizes that human perception is geared toward that which changes, which means that if we want to draw attention to something, we should vary it against a constant background. This strategy may seem simple enough, but often when wanting to highlight something, there is a tendency to offer multiple examples before adequately highlighting what those examples are *examples of.* Learners can *generalize* from a single example in ways that allow them to see similarity to other instances (see Mason & Pimm, 1984), but similarity cannot be perceived if features are not yet discerned. At each level of our progression, it is what we do with *difference* that generates awareness of particular types of features, which can then be acted upon at the next level.

With that animating principle in mind, we have found it helpful to begin by distinguishing among *types* of variation and to configure them into recursive *levels*. Drawing on others' insights, in Table 1, we have loosely mapped our levels onto what is distinguished as "conceptual variation" and "procedural variation" in Chinese variation pedagogy (Gu, Huang & Marton, 2004).

Although these levels have much in common with other recursive structures of mathematical knowing, such as Van Hiele's levels or the Pirie-Kieren model mentioned earlier, we are especially interested in the role of prompting to differences that allow movement between levels.

Level 1 begins with what Hewitt (1999) would call the "arbitrary" bounding of a space of relevance. Marton (2015)

Level	Tool of Variation	Category of Difference	Type of Variation (Chinese Variation Pedagogy)	Arbitrary/ Necessary
1	Examples	Relevant Features	Conceptual	Arbitrary
2	Relevant Features	Patterned Relationships	Procedural Type 1	Necessary
3	Patterned Relationships	Mathematical Structures	Procedural Type 2	
4	Mathematical Structures	Mathematics Ravel	Procedural Type 3a	
5	Mathematics Ravel	Contexts and Encounters	Procedural Type 3b	

Table 1. Levels of variation.

used the notions of *separation* through contrast of essential details (against a constant background) and *generalization* through contrast of inessential details to achieve this purpose, while in Chinese variation pedagogy, conceptual variation of examples, non-examples, and non-standard examples is used to separate (and name) relevant distinctions (Gu, Huang & Marton, 2004). Even at Level 1, the "objects" under consideration are in fact *distinctions* that allow learners to separate aspects of experience from a previously undifferentiated whole. Consideration of what this or that distinction means for the articulation of boundaries belongs in the realm of metaphor: already we are asking, "Is this a difference that makes a difference?" Notably, what is deemed relevant may be first identified by a knowledgeable *other*—an important aspect of raveling that allows the identification and creation of a *shared* space of relevance. To work successfully as a collective pedagogic space, relevance must emerge from pre-conceptual awareness accessible to a particular group of learners.

While Level 1 uses difference to *bound a space of relevance* with details that have the potential to interact and possibly form dependency relationships (Watson, 2017), Level 2 prompts attention to perturbations of Level 1 distinctions and to examinations of what might be "necessary" (Hewitt, 1999) within the system. Of particular interest are ways Level 1 distinctions might be varied in order to observe their influence on others within the bounded space. Here, we might ask, "What would happen if …?" and consider the nature of correlations and dependencies that emerge in that space. This action too is enabled by metaphor, in the sense that awareness of potential implications must occur metaphorically before such implications can be judged logically. Again, we are concerned with whether the differences make a difference. If accepted, the features that arise through correlations and dependencies at Levels 2 and 3 become tools of analysis that can be used at Level 4, when we consider under what perturbations the relationships themselves remain stable. At Level 4, relationships become the tool, as we compare and consider whether the relational structure in one situation also applies to another, and whether explanatory structure might therefore be similarly applicable. Such intentional use of metaphor is an inherently analogical move, enabled by a careful ravel that makes possible *perception* of similarity in the first place. Level 5 *integrates* the transforms of varied progressions across diverse encounters with the emergent concept. Here, relevance is often defined by a problem space in which different conceptual progressions may be brought to bear on the same problem.

Chinese variation pedagogy differentiates procedural variation into three types (Lai & Murray, 2012): (i) varied problem conditions, (ii) varied strategies, and (iii) varied applications. Interpreting through the lens of difference, we find resonance with the levels we have just described. Varied problem conditions may be seen in terms of the perturbations of features at Level 2, and varied strategies can emerge in the varied relationships of Level 3 (though not all comparisons of strategies are meaningful in this way); varied applications are sometimes described in terms of the transfer of explanatory structure that we describe at Level 4 and, at other times, in terms of application to different contexts, which is our Level 5.

The distinctions made explicit through a focus on difference have had a profound effect on the way we think about both mathematical raveling and the way we prompt to essential noticings. At each stage, we consider differences between differences (and their logical and metaphorical implications) brought to awareness at the previous level. If such awareness has not been prompted, all we can offer are empty symbols to *stand in the place of* those differences instead of *describing* the hierarchy of *transformations*. No further growth is possible.

The tension between mathematical symbols, which can seem to imply *things*, and the mathematics that they point to has long been an important theme in Pimm's (1995) work: "Images get objectified, partly through being impressed into objects, partly through being named, and as a result are rendered far more static. With stasis comes particularity, as they can no longer partake of the dynamic which links them" (p. 36). How, then, might we retain the dynamism of images—and other symbolic representations, so as not to lose intended meaning to the stasis of that-which-has-been-named (or drawn)? Doing so is central to the series of levels that we propose.

Raveling difference

An example of a move through our progression may be helpful to illustrate our points. We have recently been involved in curricular discussions surrounding Euler's formula. The topic is often introduced by offering a variety of three-dimensional shapes (perhaps sorted into prisms, pyramids, and others) and asking learners to count the faces, vertices, and edges of each, and then to look for patterns among those numbers. In our experience, the culminating question of "What do you notice?" is often met with confusion, as learners struggle to make sense of what the teacher is looking for. This is a prompt to what Marton rejects as "induction" and seems also to be an instance of what Hewitt (1992) called "train-spotting". Not only does it require learners to find similarities among un-discerned relationships, it prompts them only to number patterns instead of geometric transformations.

If we shift attention to varied transformations that generate the polyhedron under examination, a very different experience may be prompted. At Level 1, we might highlight distinctions between face, vertex, and edge, perhaps including contrasts of one-, two-, and three-dimensional spaces.

At Level 2, using these features to examine patterns and relationships, we might then highlight a distinction between two- and three-dimensional space by stretching a polygon into a prism (or a pyramid), perhaps starting with a triangle and then extending to other shapes. What *happens* to the faces, vertices, and edges as we do so—and what does that mean for their numbers? Note that this prompt is very different from asking learners to find *patterns* in identified features of geometric objects. We are asking learners to *look for similarity among discerned relationships.* As well, we are questioning which properties hold under transformation—not asking learners *to look for relationships among multiple variables* presented as residing *within* static objects. Once something is discerned, it can be seen again; however, humans do not *discern through similarity.*

At Level 3, as we begin to use the patterned relationships to explore mathematical structures, we might consider ways of transforming our triangular prism into a prism with a four-sided base and four rectangular faces, instead of three. There are a number of ways this transformation might be accomplished, each offering different insights into the geometry of the solids and the relations between faces, vertices, and edges. For example, we might imagine: (i) "splitting" one of the edges between rectangular faces in such a manner that a new face emerges between them, like opening the flaps of a box (Figure 2); (ii) "slicing off" one of the edges between rectangular faces such that a new face appears on the cross-sectional cut (Figure 3); or (iii) "creasing" a rectangular face, transforming one face into two (Figure 4).

What happens to the vertices and edges in each of these face-generating moves? Can this process be continued such that the square becomes a pentagon, the pentagon becomes a hexagon, and so on? Note that the images used to prompt such moves might be different from the standard array of (regular) triangular to octagonal prisms often typically offered in the pattern-spotting task described earlier. A near-triangular prism with a slight crease in one face or with a mildly truncated edge may not immediately present itself as a quadrangular-based prism, but it might be imagined as an intermediate form between a triangle and a square, even though its properties (as defined by number of faces, vertices, and edges) qualify it as equivalent to a rectangular prism. As Pimm (1995) aptly put it, "The aim of using such diagrams is to see *through* the particularity of the diagram to grasp the generality of what the drawer is attempting to focus attention on" (p. 57). Here, generality is far from the emptiness sometimes associated with increasing levels of abstraction. It is filled, rather, with a bounded space of possibility that can be perceived in an instant. Clearly, dynamic geometry software could play a role in such contexts, although perhaps one or more intermediate static images would do more to prompt learners' attentions toward the gaps, where relevant difference becomes perceptible.

At Level 4, we could compare how the Level 3 actions work with prisms to how they work with pyramids—perhaps with the assistance of algebraic symbols whose juxtaposition makes it easier to compare numbers of faces, edges, and vertices. We might also start the cycle anew by questioning whether other solids (*e.g.,* Platonic or Archimedean solids) conform to (or might be transformed to fit with) a now-identified *relationship*.

At Level 5, we might combine insights developed in this progression with already-developed understandings of area to consider surface area. Similarly, we might reconsider earlier work with nets in conjunction with new understandings of relationships between faces, edges, and vertices. In such cases, there is little *new* to be discerned. Rather, we are combining familiar discernments in new ways. Significantly, discernments from each of the levels may remain relevant, depending on the nature of the contexts that announce their relevance.

Throughout this progression, we have emphasized the centrality of *difference* to perception of intended relationships. Although it may seem that this emphasis already implies the importance of *making distinctions,* it is of course possible to offer differences with the mere assumption that others will make intended distinctions and attend to the implications of those distinctions—implications that might

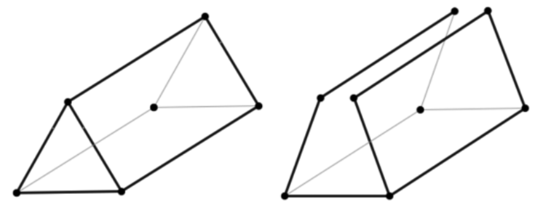

Figure 2. Transformation 1: Splitting an edge.

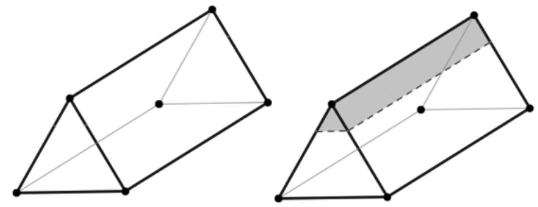

Figure 3. Transformation 2: Slicing off an edge.

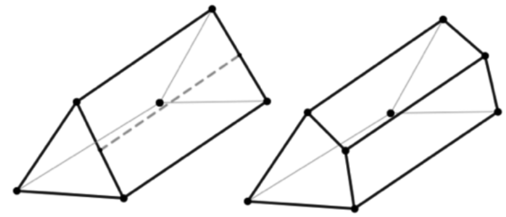

Figure 4. Transformation 3: Creasing a face.

prompt us to examine schemes of definition and classification, to consider dependency relationships, and to evaluate metaphorical substrates. We have thus found it helpful to present a dual conception of prompting: (i) offering salient differences and (ii) inviting learners to make relevant distinctions and associations (Preciado-Babb, Metz, Davis & Sabbaghan, 2020). We refer to such distinctions as "critical discernments"—itself a noun phrase that implies the verb that generated it: critical discernments carry and extend a history.

The construct of critical discernment has afforded us a way around the frequently noted tension between procedural fluency and conceptual understanding. Here we follow Pimm (1995), who dedicated considerable attention to describing tensions between, on the one hand, methods, tools, and calculating devices and, on the other, the meaning embedded there. He emphasized an important distinction between using tools (such as Dienes blocks) to *teach* and using tools to *calculate*—noting that, at times, even alleged pedagogical use of such tools is reduced to mere calculation. We have heard many teachers defend their focus on algorithms with the claim that they *are* teaching the meaning underlying the steps.

To assist in making sense of the distinctions at hand, in the first two columns of Table 2, we have used the example of long division to contrast procedural steps and conceptual steps. However, we have also seen learners (and, often, their parents) struggle to make sense of conceptual steps, at which point they sometimes plead for the simplicity of procedures that might allow them to "just do the math". A focus on critical discernments is different from a focus on either procedural or conceptual steps, and so the third column of

Table 2 offers a further contrast, listing critical discernments that emerge from *raveled difference.* Discerning critical discernments requires attention to a longer-term raveling of meaning that weaves and elaborates discernments that have much broader significance than their immediate relevance to a particular algorithm. For instance, whereas "bring down the next digit of the dividend" is relevant only in a very narrow context, "combining remainders" has much broader relevance.

Although we began with a geometric example to illustrate levels of raveled difference, then, the same principles apply to differences—and to differences between differences—in arithmetic spaces. Both geometrical and arithmetic relationships can, of course, be described algebraically. As Pimm (1995) noted,

> What is written in an algebraic demonstration are not the actions but the *results* of actions: sequences of equations show the results of transformations, not the transformations themselves. Hence, the algebra takes place *between* the successive written statements and is not the statements themselves. (p. 89)

We have found it helpful to attend to this notion of difference already at Level 1. Offering transitional prompts that recursively highlight the "space between" can do much to help learners see intended relationships at what might be considered increasing levels of abstraction in increasingly interconnected contexts.

While the idea of recursive transformations is not new, setting it alongside Marton's Principle of Difference as a means of intentionally prompting attention to successive *levels* of difference—in the context of a mathematical ravel carefully designed to interact with common (though never fully predictable) habits of perception—is proving to be a powerful pedagogical design principle. This principle allows

Procedural Steps	Conceptual Steps	Critical Discernments
1. Figure out how many times the divisor fits into the digit with the highest place value in the divisor. Write the number of times the divisor fits above the corresponding digit in the dividend (ignore remainder). This is the first digit in the quotient.	1. Divide each place value one at a time. Starting with the highest place value, divide it into the number of groups specified by the divisor. Write the quotient above the digit you're dividing.	CD1: Division can be thought of in partitive or quotitive terms. Here, we will focus on partitive division.
2. Multiply the first digit in the quotient by the divisor. Write the product beneath the first digit of the dividend.	2. Multiply that answer by the divisor to find out how many of that place value have now been placed. Write the total beneath the corresponding place value in the dividend.	CD2: When dividing, you can break the dividend into parts, divide each part into the number of groups specified by the divisor, then combine them (distributive property).
3. Subtract.	3. Subtract the number you got in Step 2 from the digit in the dividend that you were working with. The difference is the remainder that's left to be divided.	CD3: If you break the dividend into parts that divide evenly by the divisor, only the final group (if any) will have a remainder. If you don't, the remainders from each group will also need to be combined. The combined remainders may be large enough to further divide.
4. Bring down the next digit of the dividend. Write it beside the number you got when subtracting. This will form a two-digit number.	4. Combine the difference in Step 3 with the digit in the next place value. You can do this by simply bringing the next digit down.	CD4: The standard algorithm for long division uses a special case of this strategy whereby the parts being divided are specified by the digits in each place value.
5. Divide the two-digit number by the divisor. Write the answer you get beside the one you got in Step 1.	5. Unless Step 1 had a remainder of zero, you will now have a two-digit number.	
6. Multiply the second digit of the dividend by the divisor. Write the product beneath the two-digit number you divided in Step 4. Subtract.	6. Divide the two-digit number you found at the end of Step 3 by the divisor. Write the quotient beside your answer in Step 1.	
7. Repeat Steps 2-4 until all digits have been divided (Divide, Multiply, Subtract, Bring Down, Repeat).	7. Repeat Steps 2-4 until you have divided each place value. If there is still a remainder when you're finished, you can just state how many are left over.	

Table 2. Using long division to contrast procedural steps, conceptual steps, and critical discernments.

us—or perhaps *teaches* us—to "speak metonymically" (Pimm, 1995, p. 58), which is to say, it helps us to *maintain the presence of the dynamic* so that we do not get lost in static images and symbols. This is one of the great powers of mathematics—or perhaps, the *language of mathematics* (Pimm, 1987)—one way we can learn "to *mean* like a mathematician" (p. 207).

Implications

We opened this article by talking about discourses on learning in education, a survey of which underscores that making sense of current perspectives is not a matter of appreciating two or three prominent notions, but of recognizing that multiple intersecting flocks of metaphor are at play in discussions of learning mathematics. As a tool, then, metaphor can be deployed to reveal the complementarities and discontinuities of current thought, as opposed to being allowed to lie hidden in implicit associations, rendering debate unproductive if not impossible.

We also noted that an analysis of the metaphoric substrates of contemporary learning theories reveals that learning mathematics involves metaphor in at least two ways. Metaphor operates on the level of the active body, somewhat haphazardly, where it is used to sort and associate idiosyncratic noticings. Metaphor also operates on the level of the collective, much more systematically, to highlight specific notions and to privilege particular interpretations. For the most part, school mathematics has paid little attention to bridging these two categories of sense-making.

Developing habits and structures that lend themselves to deep mathematical knowing requires the elaboration and integration of many leveled progressions, as we have attempted to exemplify here. Carefully raveled mathematics curricula and resources can support students in moving from grade to grade and from teacher to teacher, without having to make leaps of inference that necessarily result not only in loss of meaning, but in a cascading loss of subsequent acts of meaning-making. It is impossible to make sense of differences among differences if the original differences have not been offered. It is precisely at the moment of attempting to do so that teachers often feel pressured to offer—and learners may feel pressured to accept—a stand-in (but empty) symbol that may appear identical to something meaningful, but that no longer describes a meaningful difference. The earlier that such critical discernments are missed, the fewer chances for subsequent sense-making that are afforded.

As we signaled at the start of this writing, this discussion is fueled by more than speculation. For nearly a decade, we have been involved in an extended partnership that includes school districts and a resource developer, through which we have accumulated extensive and compelling evidence that a pedagogy developed around fine-grained ravels and well-structured prompts can, in fact, contribute to impactful mathematics learning experiences—across grades and demographics. That said, we have also learned that responsibility for raveling mathematical content cannot be borne by individuals. On the contrary, it is only manageable if distributed across multiple teams of experts. Mathematics teaching, that is, must be understood as a partnership with a well-designed resource.

Generating such a resource requires sophisticated insight into children's pre-conceptual understanding of mathematics as well as nuanced understandings of formal mathematics. That is, in our experience, raveling is difficult and time-consuming work that can only be accomplished with the collaboration of mathematicians, logicians, learning specialists, and teachers. Although we have not yet found any single resource for teaching mathematics that adequately contrasts and sequences relevant differences, it is certainly the case that some are vastly better than others.

Not all students, or even teachers, have sufficient background to attend to the large gaps in many resources. With critical discernments carefully named and prompted through difference, teachers themselves can be supported in making those discernments, which enables them to then prompt to those differences, attend to the manner in which their students engage with critical differences, and respond in ways that allow for the continued elaboration of understanding. Even with a strong resource, such engagement requires an artful, attentive, and responsive teacher who is capable of harnessing the power of difference in order to prompt attention in ways that allow diverse learners to make collective sense of the complexities of mathematics.

References

Bateson, G. (1979/2002) *Mind and Nature: A Necessary Unity.* Hampton Press.

Davis, B. & Francis, K. (2019) *Discourses on Learning in Education.* https://learningdiscourses.com

Gu, L., Huang, R. & Marton, F. (2004) Teaching with variation: a Chinese way of promoting effective mathematics learning. In Fan, L., Wong, N.-Y., Cai, J. & Li S. (Eds.) *How Chinese Learn Mathematics: Perspectives from Insiders,* 309–347. World Scientific.

Hewitt, D. (1992) Train-spotters' paradise. *Mathematics Teaching* **140**, 6–8.

Hewitt, D. (1999) Arbitrary and necessary part 1: a way of viewing the mathematics curriculum. *For the Learning of Mathematics* **19**(3), 2–9.

Lai, M.Y. & Murray, S. (2012) Teaching with procedural variation: a Chinese way of promoting deep understanding of mathematics. *International Journal for Mathematics Teaching and Learning* **13**. Online at http://www.cimt.org.uk/journal/lai.pdf

Lakoff, G. & Johnson, M. (1980) *Metaphors We Live by.* University of Chicago Press.

Lakoff, G. & Núñez, R.E. (2000) *Where Mathematics Comes From: How the Embodied Mind Brings Mathematics into Being.* Basic Books.

Marton, F. (2015) *Necessary Conditions of Learning.* Routledge.

Mason, J. & Pimm, D. (1984) Generic examples: seeing the general in the particular. *Educational Studies in Mathematics* **15**(3), 277–289.

Math Minds (2020) *Math Minds Online Course.* https://www.structuringinquiry.com

Ortony, A. (Ed.) (1979) *Metaphor and Thought.* Cambridge University Press.

Pimm, D. (1987) *Speaking Mathematically: Communication in Mathematics Classrooms*. Routledge & Kegan Paul.

Pimm, D. (1995) *Symbols and Meanings in School Mathematics.* Routledge.

Pirie, S. & Kieren, T. (1994) Growth in mathematical understanding: how can we characterise it and how can we represent it? *Educational Studies in Mathematics* **26**(2–3), 165–190.

Preciado-Babb, A.P., Metz, M., Davis, B. & Sabbaghan, S. (2020) Transcending contemporary obsessions: the development of a model for teacher professional development. In Linares, S. & Chapman, O. (Eds.) *International Handbook of Mathematics Teacher Education,* 361–390. Sense.

Sfard, A. (2015) Learning, commognition, and mathematics. In Scott, D. & Hargreaves, E. (Eds.) *The Sage Handbook of Learning,* 129–138. SAGE.

Watson, A. (2017) Pedagogy of variations: synthesis of various notions of variation pedagogy. In Huang, R. & Li, Y. (Eds.) *Teaching and Learning Mathematics through Variation: Confucian Heritage Meets Western Theories,* 85–103. Sense.