Communications

Empty set, variable and some other mathematical concepts as figures of speech

EVGENII M. VOROB'EV

This short communication analyzes three concepts: the empty set, variable, and indefinite integral. We turn our attention to them, first, because they are between the basic concepts of set theory and calculus. Second, either they are undefined or their definitions contain significant gaps. Third, the terms 'empty set' and 'indefinite integral' are *catachreses, i.e.,* word combinations consisting of words mutually denying each other. Catachreses often cause students difficulty because they perceive them as meaningless and learn them only by memorization. It is desirable that any mathematical term has something to say to learners and does not lead them to a dead end. In this sense, 'antidifferentiation', 'antiderivative' and 'sieve of Eratosthenes' are good terms. The first two are actually equivalent to their definitions as the reverse of differentiating and the outcome of this action. The third metaphorical term concisely expresses the essence of the method.

We use here *conceptual metaphor* and *metonymy*, as introduced in Lakoff and Johnson (1980), to reveal the meaning of the catachreses 'empty set' and 'indefinite integral', and to make a more precise definition of 'variable'. The important role of metaphor and metonymy for teaching mathematics has been widely recognized (see, *e.g.,* English, 1997). Various studies have indicated that the language problem is one of the major factors contributing towards the poor performance of many students in mathematics (Barton & Neville-Barton, 2003).

Empty, one-element sets, and the container metaphor

According to Cantor's definition: "A set is a gathering together into a whole of definite, distinct objects of our perception or of our thought, which are called elements of the set." [1]. There is nothing to gather in the empty set, and it is senseless to gather into a whole a single element; therefore, the empty and singleton sets are not sets under Cantor's definition and need to be defined separately.

Mathematicians, following Cantor and Zermelo, apply curly braces to designate a set. For example, $\{\,\}$ is an empty set, $\{a\}$ a one-element set, $\{a, b, c\}$ a three-element set and so on. This suggests that mathematicians unconsciously use *the container metaphor* (Lakoff & Núñez, 2000) when writing about sets. In fact, we can treat braces as the walls of a one-dimensional container. It is also usual to speak of sets as *containing* their elements. If the set is metaphorically a container, then an empty container or container with one object inside is not difficult to imagine. Moreover, we can treat the symbols $\{\,\}$ and $\{x\}$ as definitions of the empty and one-element sets. From the point of view of semiotics, $\{\,\}$ and $\{x\}$ are iconic signs since they coincide with their concepts.

Euler had the idea of using circles for visual representation of sets and set-theoretic operations (Figure 1).

Euler diagrams, while representing containers, like braces, illustrate well the fragmentary nature of metaphor, namely, the fact that metaphor does not fully explain the concept, but highlights only some of its aspect. If it were not so, then metaphorical understanding could replace more formal mathematical understanding. In the case of Euler diagrams, the intersection of sets is not, in general, a circle. Disjoint sets not only generate no circle, but also no bounded figure that can represent the empty set.

Variable

Variables play an important role in at least three fields of mathematics: calculus, algebra, and computer science (see Usiskin, 1999 for a more detailed discussion). In the first two fields, variable is a synonym for a function argument or for an indeterminate in a polynomial. If a polynomial occurs in an equation, then the variable becomes an unknown. From a computer point of view, the name of a variable is the address of some specific memory register. We combine the first (argument/indeterminate) and the third (memory register) approaches here.

Modern definitions of 'variable' use the terms 'represent' or 'stand for', as in the definition: "A variable is a symbol that stands for all or for some of the elements of a class of numbers" [2]. Understood literally, this definition means that any character, under all circumstances, is a variable. Clearly, this is not the case; therefore, it would be desirable to make precise when and which symbols really stand for something. Observe that, while letters of the end of the alphabet, *x, y, z, etc.,* denote variables and those at the beginning of the alphabet, *a, b, c, etc.,* denote constants, nothing forbids the symbol x in a certain context to denote a constant (as in $x + 2 = 4$) and the symbol a to denote a variable (as in the polynomial $a^2 + 1$). Therefore, the definition of a variable is not complete unless it specifies a context.

Computer Science prefers another approach to the definition of a variable, in which the emphasis is shifted to symbols that are the names of function arguments. Here, the central concept is a *placeholder*. There are different versions of the definition of 'placeholder' depending on the

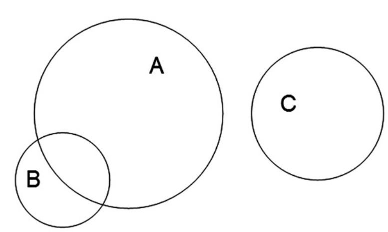

Figure 1. The Euler diagram for three sets.

specific area of application. In our opinion, it is fruitful to consider the term 'Placeholder[]', which is found in *Mathematica*. In *Mathematica*, the *Placeholder*[] command creates a *place* in *Mathematica* expressions to be filled. The command *Placeholder*[*x*] creates a place named *x*. Figure 2 shows how created places, arguments of the function *f*, look in *Mathematica*.

a) f[□, □]   b) f[*x*, *y*]

Figure 2. Placeholders (a) and named placeholders (b) in Mathematica.

To give meaning to the notion of a variable, we suggest again the *container* metaphor. Namely, in the traditional notation *f*(*x*, *y*) for the function of two variables, the places that the characters *x*, *y* occupy should be understood as containers. They can contain various elements of the sets *X*, *Y*, which are projections onto the *x*- and *y*-axes, projections of the domain of definition $D(f)$ of the function *f*. We can treat the letters *x* and *y* as the names of these containers.

Any element of the set *X* can occupy the container *x*, therefore, elements of the set *X* receive the symbolic name *x* of the container. In this sense, the name *x* also stands for the elements of the set *X*, or, which is the same, the symbol *x* represents the elements of the set *X*. This name transferring mechanism is called *metonymic extension. Metonymy* means transfer of a name. It is a figure of speech consisting of the regular or occasional transfer of a name from one class of objects or a single object to another class or a separate object.

For example, the White House refers to the White House staff or to any member of the staff. If we meet the phrase, 'The White House stated', it is clear that the president or his spokesperson made this statement in the White House. You can also say that the name of the White House replaces or represents any of its inhabitants. This example belongs to the class of metonymic extensions, characterized by the *transfer of a name from the container to its contents.* Proceeding from the foregoing, we propose the following definition of a variable as a function argument, for brevity limiting ourselves to the case of functions of one argument.

> A variable is a name of the function argument, metonymically transferred to the elements of the domain of the function definition.

Given that the words 'replace' and 'represent' delineate metonymic extension, this definition can take the following form.

> A variable is a name of a function argument that replaces or represents the elements of the domain of the function definition.

Indefinite integral

The imperative: "compute the indefinite integral" is a semantic catachresis, since 'unknown' and 'indeterminate' are the chief synonyms of 'indefinite' [3]. It is impossible to calculate what is indeterminate. The term indefinite integral occurs because in the definite integral:

$$\int_a^b f(x)\,dx$$

the limits of integration *a* and *b* are set, while there are no such limits in the indefinite integral $\int f(x)dx$.

If the function $f(x)$ is continuous on an interval Δ, then the following expression is its antiderivative on this interval

$$F(x)=\int_a^x f(y)\,dy \quad \text{for } x, a \in \Delta$$

It is clear that this expression is useless for calculating definite integrals by the Newton-Leibniz formula. Therefore, the antiderivative is often calculated in the class of elementary functions. It turns out that in this case the antiderivative may depend on the interval Δ.

As an example, consider the function $f(x) = 1 / (2 + \sin x)$ defined on a real axis and the indefinite integral $\int 1 / (2 + \sin x)\, dx$. Let us start by calculating the elementary function that is an antiderivative of $f(x)$ on the interval $\Delta_1 = (-\pi, \pi)$. After the canonical substitution $t = \tan(x/2)$ mapping one-to-one Δ_1 onto R^1, we obtain the antiderivative of $f(x)$ on Δ_1 equal to

$$F_1(x)=\frac{2\arctan\left(\frac{1+2\tan\left(\frac{x}{2}\right)}{\sqrt{3}}\right)}{\sqrt{3}}$$

As another interval, consider the interval $\Delta_2 = (0, 2\pi)$. The substitution $t = -\cot(x/2)$ mapping one-to-one Δ_2 onto R^1 yields the antiderivative

$$F_2(x)=\frac{2\arctan\left(\frac{-1-2\cot\left(\frac{x}{2}\right)}{\sqrt{3}}\right)}{\sqrt{3}}$$

The antiderivative $F_1(x)$ has a discontinuity at the point $x = \pi$ (see Figure 3a). Therefore, $F_1(x)$ is not the antiderivative of the function $1 / (2 + \sin x)$ on Δ_2. If we calculate the definite integral of the function $f(x)$ over Δ_2 using the Newton-Leibnitz formula and this antiderivative, then we obtain zero, which is the wrong result. At the same time, the application of the antiderivative $F_2(x)$ yields the correct result $2\pi/\sqrt{3}$. Similarly, the function $F_2(x)$ is not an antiderivative on the interval Δ_1, since it is discontinuous at the point $x = 0$ (see Figure 3b).

This example demonstrates that the definitions of indefinite integral often met in textbooks are incomplete. If we want to calculate the antiderivative in the class of elementary functions, we must specify the interval in which we are going to calculate the antiderivative.

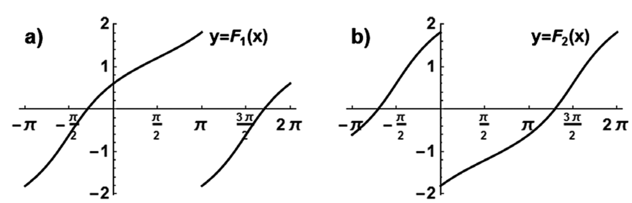

Figure 3. The graph of the function $F_1(x)$ (a), and the graph of $F_2(x)$ (b).

Conclusions

Accidentally or not, basic concepts of mathematical theory such as set, function, variable, infinity, *etc.*, do not have clear and transparent definitions. Their 'definitions' are actually *descriptions*, because they express these basic concepts using other fuzzily defined objects or processes. Therefore, it is difficult to overestimate the importance of the analysis of the meaning of mathematical texts by methods of cognitive linguistics. In this regard, it seems proper to analyze literally contradictory, and difficult to learn, mathematical concepts as figures of speech. We believe that it will help teachers of mathematics to offer a more complete and adequate exposition of the concepts under consideration to students.

Acknowledgments

The author is grateful to the reviewers of this communication for their helpful comments.

Notes

[1] This definition can be found in many sources. See, for example, https://en.wikipedia.org/wiki/Set_(mathematics)#Definition

[2] From the Cambridge English Dictionary, online at https://dictionary.cambridge.org/dictionary/english/variable.

[3] From the Oxford Living Dictionaries Thesaurus, online at https://en.oxforddictionaries.com/thesaurus/indefinite

References

Barton, B. & Neville-Barton, P. (2003) Language issues in undergraduate mathematics: a report of two studies. *New Zealand Journal of Mathematics* **32**, 19–28.

English, L. (Ed.) (1997) *Mathematical Reasoning: Analogies, Metaphors, and Images.* London: Lawrence Erlbaum Associates.

Lakoff, G. & Johnson, M. (1980) *Metaphors We Live By.* Chicago: The University of Chicago Press.

Lakoff, G. & Núñez, R. (2000) *Where Mathematics Comes From: How the Embodied Mind Brings Mathematics into Being.* New York: Basic Books.

Usiskin, Z. (1999) Conceptions of school algebra and uses of variables. In Moses, B. (Ed.) *Algebraic Thinking, Grades K-12: Reading from NCTM's School-Based Journals and Other Publications,* pp. 7–13. Reston VA: NCTM. (Reprinted from Coxford, A.F. & Shulte, A.P. (Eds.) *The Ideas of Algebra, K-12*, pp. 8–19. Reston, VA: NCTM.)

Introducing Problem Based Learning in Engineering Calculus

CONCEPCION VARELA, OLATZ GARCÍA, EUGENIO BRAVO, CAROLINA REBOLLAR, JAVIER BILBAO

The authors of this communication are professors of the Department of Applied Mathematics of the University of the Basque Country/Euskal Herriko Unibertsitatea (UPV/EHU), and we teach in the first years of several programs at the Engineering School of Bilbao. Each year we receive hundreds of new students and, every year, after the first weeks of class, we observe in many of them a disenchantment stemming from not being able to see immediately the practical application of the theoretical concepts that they are studying. Questions like: 'What type of discontinuity does this function have at this point?' or, 'Is this a differentiable function?', are not exactly attractive nor do they seem to lead to the 'construction' of anything practical. Nevertheless, they are basic concepts, essential in the training of our students. Here we present an experience with some first year students in the Engineering School of Bilbao that is intended to increase motivation and interest in the subject of mathematics taught in the field of Engineering.

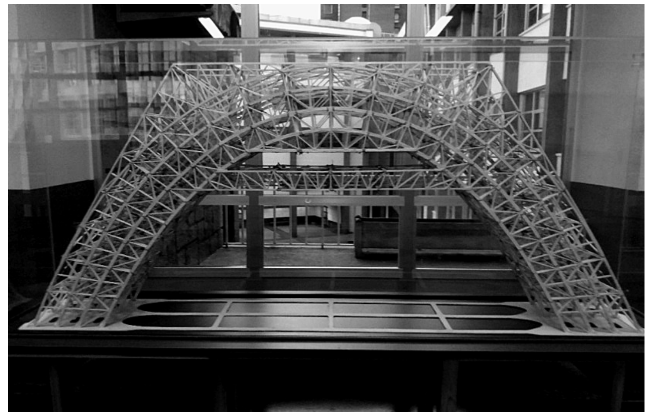

Figure 1. Bridge.

At the Engineering School of Bilbao, a bridge building contest using only ice cream sticks and glue has been held for many years. In the contest, two categories are evaluated and awarded: the aesthetics of the bridge and its resistance. In order to evaluate the latter, tiles are placed on the bridge, until it breaks or the builders decide not to risk anymore, because they actually value their work more than the prize they can get. These bridges are the result of hours of calculation in which the builders must apply their theoretical knowledge, followed by hours of handwork, gluing, with great care, thousands of sticks (Figure 1). In those two aspects lies, surely, the attractiveness of the contest. Because what encourages an 18-year-old to study an Engineering degree? It is almost certain that, although each person has his or her own motivation, most would include that 'I am good at technical subjects, physics, mathematics, drawing' and 'I would like to apply them to build things'.

Students who participate in the contest realize the importance of mathematics and the other subjects they are taking when they do the calculations for designing and building the bridge. In the same direction, the center is developing different projects such as 'Formula Student' or 'Moto Student' that are clearly motivating for the students, but with the restrictions that only students in the last years participate.

In the Engineering School of Bilbao there is enormous concern about the high percentage of students who drop out of studies in the first year due to, among other reasons, lack of motivation. It is in order to motivate the students that some of the teachers of the faculty suggested that we could find alternative approaches. For example, we looked for a way to teach real functions of one real variable without resorting to the traditional method of presenting theory, followed by applying it to exercises, which, were usually quite theoretical, too. We talked about the opportunities that the implementation of the EHEA (European Higher Education Area) has brought in relation to changes in teaching-learning methodologies. Among the ideas that we found when reviewing the literature related to that topic, we noted a focus on the student:

> The organization of the teachings will revolve around learning, using active methodologies, where the student is a basic element. (Trasobares & Gilaberte, 2007, p. 31; a similar idea is expressed in Ministry of Education, Culture and Sports, 2003) [1]

This is similar to what was already proposed in the 1960s, in Postman and Weingartner's book *Teaching as a Subversive Activity.* They removed the foundations of the teaching that was taught at that time, suggesting revolutionary ideas on education, including focusing the learning on the student and directing them to learn by asking questions. For the authors of the book, "the new education has as its purpose the development of a new kind of person, one who is an actively inquiring, flexible, creative, innovative, tolerant, liberal personality who can face uncertainty and ambiguity without disorientation" (Postman & Weingartner, 1969, p. 184).

This idea was also used at McMaster University when a group of professionals found that health problems of the population were not always treated adequately by health professionals, and decided to take a new approach to the way in which those professionals had to acquire the knowledge, skills and abilities necessary for the performance of their profession (Ribas, 2004). The success of this methodology caused it to expand to other universities.

Following these trends, in 2010 the UPV/EHU approved the implementation of its own educational model with the name *IKD model,* initials that correspond, in Basque, to Cooperative and Dynamic Learning. It is a cooperative, multilingual and inclusive model that emphasizes that students are the owners of their learning. With this premise, in its Strategic Plan 2018/2021 [2], and within the section dedicated to training, the UPV/EHU establishes as its objective "To deepen the development of the own educational model" (p. 15).

To achieve this objective, among the proposed actions are:

> Increase, and where appropriate consolidate, the use of innovative methodologies in teaching-learning processes. (p. 15)

> Promote the continuous orientation of the undergraduate students in order to facilitate greater use of university life and academic itineraries, as well as improve their performance. (p. 15)

> Intensify the development of transversal competences: leadership, critical thinking, multilingualism and multiculturalism, problem solving, digital competences. (p. 16)

Accepting the idea that, if we want students to have an active role in their learning, classical teaching methodologies (in which the focus is on the teacher) should be banished in favor of active methodologies in which the student is at the center. Supported by our own university environment, we asked ourselves a simple question:

> Are we capable of implementing this type of active methodology in the subjects we teach?

That question opened a wide field of discussion which we summarize in the following sections.

What methodology to choose?

The authors of this article have participated in various training programs in active teaching/learning methodologies, and we thought that, among them, Problem Based Learning (PBL) methodology seemed appropriate and a challenge at the same time.

Appropriate because, according to Barrows, its founder, the PBL methodology is "a learning method in which problems are used as a starting point for acquiring new knowledge" (Barrows & Tamblyn, 1980, p. ix). We also can define Problem Based Learning as a didactic method based on the use of problems as a starting point for the acquisition and integration of new knowledge. It is a strategy aimed at promoting problem-solving skills, which favors cooperative learning and critical thinking, through the resolution of real problems. And, at the same time, we understood that, using PBL, students improve the possibilities of learning when:

- prior knowledge is activated to incorporate new knowledge,
- there are application opportunities, and
- new knowledge is learned in the context in which it will be used afterwards.

This is a challenge because, in each of the PBL training courses in which we have participated, those who taught those courses considered that applying this methodology in such basic and theoretical subjects as ours was an added handicap. The reasons for thinking this way were fundamentally two: (a) It is very complicated to find real problems in which only the basic mathematical concepts that are studied in the first year are applied; (b) The content of many subjects is theoretical, also in their subsequent evaluation.

In which subject area to apply the methodology?

The authors of this article teach Calculus, Algebra, Differential Equations, and Numerical Analysis. We opted for Calculus because it is taught in the first year, students already bring certain knowledge about from pre-university courses, and the way it is structured, between Lectures (face-to-face classes with usually groups of more than 60 students) and Seminars (face-to-face classes with sub-groups, with about 20 students), offered the opportunity to apply PBL in only part of it (that is, in the Seminars). In this way, we made the question more precise:

> Are we capable of developing the study of Real Functions of One Real Variable based on the PBL methodology?

As 'simple' as that. There was no question among us that we wanted to make the attempt. We only needed to specify what we wanted the learning results to be, and 'which problem would be our starting point'.

The question of learning results was simple. Based on the existing teaching context for the study of real functions of one real variable the learning outcomes should include:

- Analysis of continuity, derivatives and differentials,
- The Riemann integral and its application to calculating areas in the plane and volumes of solids of revolution.

The second question of what problems to start with, was more difficult to solve. Having established the two learning outcomes we sought not one problem, but two.

For the analysis of continuity, derivatives and differentials, the problem that has been posed to the students as a starting point is:

> We are going to design a skateboard track. To do this, we must properly distribute structures through which we can slide or jump: parabolic tubes, inclined planes [...]
>
> Draw the profiles of some of the tracks you would like to design and try to define functions of one variable that fit (at least approximately) those profiles you have drawn. Start by using elementary functions that you can associate with small parts of the tracks. Then try to merge those parts to complete longer tracks.

For the part related to the Integral Calculus, this problem has been proposed:

> On the football field 'San Mames Barria' a fungus that is killing the grass has appeared. So that it does not spread, they have decided to change a part of the field. The piece that must be cut is the region that is limited by the curve shown in Figure 2. Before starting, and to be able to make the change as quickly as possible, they want to know how much new grass they should ask for. Can you think of a simple way to do that calculation?

The answer we are looking for is not the direct application of the integral, but the approximation to the integral by adding several rectangles that fit the curve (Figure 3).

In both cases, the students work on, in groups of 4 or 5 members, a series of tasks aimed at enabling them to deduce theoretical concepts as learning objectives. The material provided includes the estimated time to carry out the tasks.

We have designed 19 activities for the first learning outcome, and 24 for the second one. These are short activities that students have worked on, both outside and within the classroom. In total, we dedicate six 90 minute seminars (nine face-to-face hours) to the development of this work. Outside the classroom, we have estimated the time required to complete the proposed tasks to be five hours.

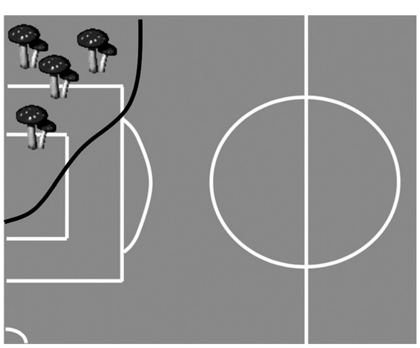

Figure 2. The area to be treated in the Football Field problem.

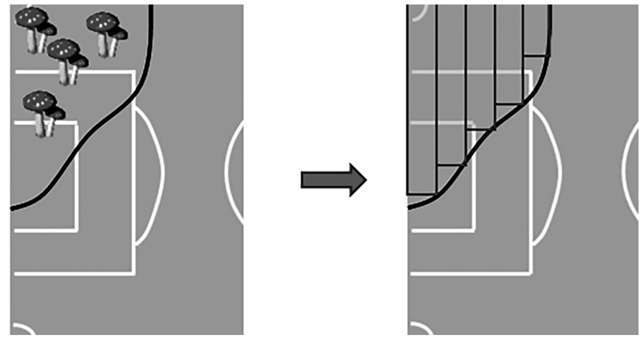

Figure 3. Adding rectangles to fit the curve.

For the preparation of these activities, the students are provided with written material in which the activities and the learning objectives are described, as well as explanatory videos related to some concepts. They also have tutorials available with the corresponding teacher.

Analysis of continuity, derivatives and differentials begins with an activity in which students must draw the tracks that they want to design, trying to associate them with the graphs of elementary functions and piecewise-defined functions. From there, we introduce the study of continuity and the types of discontinuity that can be defined. In order to introduce the concepts of derivative and differential, we ask students to start by analyzing how the height of two different tracks varies when the skater glides through them. It is important that the chosen tasks:

a) are quick to solve (due to the short time we have),

b) are attractive for students,

c) include concepts that can be applied in a few steps,

d) do not combine several concepts at the same time, at least in the first tasks.

For the Football Field problem, the activities are designed to calculate the area as the sum of the areas of infinite rectangles. What we are trying to do is to rigorously establish the conditions that must be verified, and to use a correct mathematical expression. In the subsequent activities, they must be able to deduce the basic properties of the definite integral and reach, as well, the concept of improper integral. For example, in one of the activities, they are asked to calculate

$$\int_{-1}^{1} \frac{dx}{x^2}$$

to state if they think the calculation and the obtained result is coherent with the properties they have seen for the Riemann definite integral, and try to explain it. The particular objective of this example is to see that, if it were resolved as a Riemann integral, we would get a negative result, when we know it must be positive. In view of this, they should observe that, actually, it is not a Riemann integral, since the function we integrate is not bounded in the integration interval.

Our initial objective, besides improving academic results, was stimulating students' interest in the subject they had to study. To assess our success, we designed a satisfaction survey with 24 questions that students should answer in the last session of the seminars. It gathers information about the student, the design of the activities and the methodology used. The statistical results obtained can be summarized as follows:

From the 102 students that have participated, 72 are male and 30 female. With respect to the questions related to the design of the activities, 77% worked on the activities before the face-to-face session; however, the time dedicated, between 15 minutes and one hour, is far from our estimate of the necessary time. In relation to multimedia resources, although 72% of the students consider that they are valid tools for understanding the concepts they have to work on, more than a quarter of the students admit not having watched any videos. And, finally, the opinion of the students on the methodology used is very positive. More than 83% think that the experience carried out in the classroom is appropriate or very adequate.

Based on these statistical results, especially the last one, we conclude that the experience has been clearly satisfactory. However, the students also had the opportunity to make suggestions for improvement or to simply give us their opinion on the material and the proposed way of working. Based on those answers, we divide our students into two groups: 'detractors' and 'defenders' of the PBL methodology. Here is a sample of the opinions of the group of 'detractors':

> The work has been hard. It would be better to have more typical classes so that you do not have to spend so many hours outside the classroom.
>
> The videos that you have to watch outside the classroom would be better to watch during the classes.
>
> We are not used to working as a group outside of classroom.

And here, a sample of the opinions of the 'defenders':

> The use of this methodology should be generalized.
>
> I would like to study more topics of the subject using this methodology, and not only the Calculus of one variable.
>
> It's good for us to work on our own. We learn more.

We cannot say that we were surprised by the results obtained. They show, in general, satisfaction with the methodology used and the work done, but the students still resist 'sharing' their work and doing part of it outside the classroom.

Completion of the degrees that we teach at the Engineering School of Bilbao implies great dedication and effort on the part of the students, work that is awarded with fast insertion in the labor market and with success of the graduates. However, we have to recognize that, in general, our students are used to working at their own pace, as evidenced in the surveys, and not as scheduled in the planning. Thus the biggest handicap we have to overcome is to ensure that students work properly, because this will lead to a considerable improvement in the results obtained.

In any case, we believe that the decision to start using the PBL methodology is the correct one, although we believe that the materials can be improved. In that sense, we have already developed a new version, including new tasks and eliminating others, with the aim that the path that students should follow from the initial problem to reach the final learning objectives is more compelling.

One of the main obstacles that we detected, when students were working on the activities in our first version, was that the timing that we had proposed was not adjusted to reality. Our students, in general, needed more time than we had planned for each of the activities. In some cases, the least, we chose to eliminate an activity. In other cases, keeping the activity as designed, we simply gave them more time. And in other cases, we decided to subdivide the activity into shorter ones, adding information (it is often enough to set out questions to the student), so the objective that was intended is more achievable. For example, in the study of the properties of the Riemann definite integral, we want them to understand and justify why, given a function *f* integrable in an interval [a, b], it is considered, by definition,

$$\int_b^a f(x)dx = -\int_a^b f(x)dx$$

In the first version, the activity was set up exactly as stated here. Students were not able to justify it. In the new version, it is suggested that they remember the way the integral was defined, the partitions that were made in the interval [a, b], how the term *dx* is reached, and analyze what the sign of this term is. Next, they are instructed to repeat the process to try to define

$$\int_b^a f(x)dx$$

The initial objective of this project is to develop a methodology for integrating mathematics subjects into the teaching-learning process in some of the degrees taught at the Engineering School of Bilbao, in order to improve student motivation and with the ultimate aim of reducing the drop-out rate of first-year students. If we focus on the Degree in Industrial Technology Engineering, which is the degree in which the highest percentage of students have participated, the drop-out rate in the first year has gone from 23.44% in 2016-17 to 19.01% in the academic year 2017-18 after applying the model in the first year mathematics courses. It is clear that the drop-out rate is still high, but we believe that the methodology presented has contributed to the reduction and it is in the way that must be followed to achieve our objectives.

Notes

[1] All translations from Spanish documents are our own.

[2] Online at https://www.ehu.eus/es/web/idazkaritza-nagusia/plan-estrategikoa-2018-2021

References

Ayala Calvo, J.C. (Ed.) (2007) *Conocimiento, Innovación y Emprendedores: Camino al Futuro.* Logrono: Universidad de La Rioja.

Barrows, H.S. & Tamblyn, R.M. (1980) *Problem-Based Learning: An Approach to Medical Education.* New York: Springer.

Ministry of Education, Culture & Sports (2003) *La Integracion del Sistema Universitario Espanol en el Espacio Europeo de Ensenanza Superior.* Madrid: Ministry of Education, Culture and Sports. Online at http://www.uhu.es/convergencia_europea/documentos/documentos-2007/docmarco_MEC_feb2003.pdf

Postman, N. & Weingartner, C. (1969) Teaching as a Subversive Activity. New York: Delacorte Press.

Ribas, A.F. (2004) Líneas maestras en el aprendizaje por problemas. *Revista interuniversitaria de formación del profesorado,* **18**(1) 79–95.

Trasobares, A.H. & Gilaberte, R.L. (2007) Aplicación del aprendizaje basado en problemas (PBL) bajo un enfoque multidisciplinar: una experiencia práctica. In Ayala Calvo, J.C. (Ed.) *Conocimiento, Innovación y Emprendedores: Camino al Futuro,* pp. 30–43. Logrono: Universidad de La Rioja.

How low can you go?

MELISSA TOLLEY NINK

I began teaching at Wingate University in the fall of 2013. Since then I have taught Calculus I eight times, and College Algebra six times. I began flipping Calculus in the fall of 2014 and have flipped ever since. I have flipped Algebra only once, in the fall of 2015. In this short communication I would like to explain why.

Currently, at Wingate Calculus is considered an intermediate mathematics class and is considered a tougher course that counts as a mathematics credit. Calculus is unique in that it meets every day of the week for a total of six contact hours. In contrast, Algebra is a remedial course and does not count towards the university mathematics requirement. Students who take this course are doing so to prepare for another higher level course. A typical student in Algebra struggled with mathematics in high school and needs to refresh before going on. Algebra meets three times a week for a total of three contact hours.

The goal of a flipped class is to minimize lectures in class while providing more resources for students outside of class. Each night, students are assigned between one and three videos for a total of about 30 minutes in Calculus and 20 minutes in Algebra. They are expected to have watched these videos and taken notes by the start of the next class. At the beginning of class, a short quiz is given. This quiz is open note, with the idea being that students who took notes will easily make 100%. After the quiz, we review for 10-20 minutes. After the review time, students are assigned daily work for the remainder of the class. They are encouraged to work together in pairs or groups, but not penalized for working alone. At the end of class this daily work is turned in and two to three problems graded. From other professors that I have talked to, it seems that most people follow this type of flipped-classroom setup, or one very similar.

After flipping Calculus the first time, I compared the final exam scores to those from the previous semester, since the final exam is very similar between semesters, and I found the average final exam score increased by 3.18 points. I also compared the final exam scores of the flipped Algebra course to the previous years class and found the average on the final exam *decreased* by 4.91 points. I compared the daily quizzes between classes and found the Calculus class did, on average, 9.5 points better than the Algebra class. So, overall, grades in the upper level course went up, while grades in the lower level course went down. Naturally, I wondered why.

In the classroom, Calculus students had higher attendance rates, more participation, and more completed work. The Algebra students like to do part of the daily work assignment, and then leave class early. Sometimes a Calculus student will do this, but after getting a bad grade they'll stop pretty quickly. The students in the Calculus class had, overall, higher quality of work with more motivation to do well. Even with the promise of a possible 100% on a quiz, students in the Algebra course still did not watch the videos. I could tell by comparing the number of views of the videos. My suspicion is that, because the material moves so quickly and is so complex, Calculus students learn very quickly that not watching the videos will be a major setback for them. Algebra students, on the other hand, skipped watching the videos more and would rely on the review in class to be enough.

I noticed more differences when comparing the two classes. First, student success in Calculus skyrocketed. We are able to cover more material more in depth than before. In Algebra, some students were very successful, but the average student did not thrive. In Algebra we were forced to move at a slower pace and scores were still lower. In Calculus, the grades are more centered with less variance, where Algebra had two extremes: either the students did great, or did well below what was normal in the unflipped class. In my office hours, Calculus students are constantly attending, but when I flipped Algebra I had one student stop by in the entire semester. The students course evaluations are also opposites. My evaluations for flipped Calculus continue to be great. The students love the structure of the course (with a few exceptions) and over the semester we really develop a nice relationship with each other. In Algebra the course evaluations went down when I flipped the course. Many students felt like the videos were excessive and a waste of time. A few even said they were distracted from their learning and that they need a lecture to succeed.

So what did I learn from this experience? Flipping is not a panacea. For students who struggle with and fear mathematics, flipping may be a bad choice. Other teaching forms serve them better. For motivated students, however, flipping can be very effective.