A Linguistic Approach to the Justification Problem in Mathematics Education

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'The limits of my language mean the limits of my world.' (Wittgenstein, *Tractatus Logico-Philosophicus*, 5.6)

1. Introduction

Mathematics is taught at all levels of the educational system world-wide. In school education, mathematics is a mandatory subject for every student, and up to university level it is often one of the subjects to which most time is allocated, usually second only to the local language. It is certainly a subject on which students spend much (if not most) of their study time. Given this dominant role of mathematics as an educational subject, the shape and means of mathematics curricula must be (and are) regularly examined, but so should be the fundamental justification of this dominance. However, whether one should teach mathematics at all is rarely questioned or discussed among professional mathematics educators

As one exception, one may cite the following from a delightful little book by Davis and Hersh (1988):

> There is no law of nature, God, or government that everybody must know the quadratic formula. Mathematics is interesting and important, but so are art, religion, literature, and many other things. (pp. 103-104)

This also touches an important aspect of the justification issue: it is not sufficient to argue the inherent value, importance and usefulness of mathematics, one must also consider the loss of taught wisdom in other areas that results from giving priority to mathematics. For the discussion to become fruitful, one needs an integrated view of mathematics among many ways to challenge, inform and enrich the human intellect.

The first purpose of this paper is to advocate that in addition to (and in fact prior to) the most commonly investigated questions in mathematics education, the 'how to' type of questions, we must ask ourselves 'why' and even 'if at all' type of questions, and that we must undertake a serious, open-minded discussion of them. The second aim is to propose a framework for this discussion which allows one to consider mathematics in its relations with the rest of human thought and knowledge that is (or could be) communicated by formal education.

Many mathematicians and mathematics educators will probably, at this point, put forward an objection: we do not need questioning of whether to teach the exact sciences, mathematics in particular, and certainly not coming from colleagues. For one thing, as mathematicians, we truly know the value of the subject and should not waste our time discussing it. Secondly, mathematics has already lost its incontestability, its invulnerability, for at least two reasons: the end of the cold war which used to justify the mass training of scientists, and - as a less tangible factor - a recent tendency to reject reason and science, in some sort of counter-reaction to the perceived rule of impenetrable technology over humans in modern society (e.g. Gross *et al.*, 1996). In other words, the justification issue need not be brought up by us, as we are already under attack from the outside. I wish strongly to contradict both of these responses. Mathematics both should and needs to take up the challenge of an academic environment in which its place is no longer likely to stay self-evident.

A final word about perspective: I have chosen to treat the subject without much reference to particular levels of mathematics teaching, since I believe that the justification problem should not be considered as isolated in such specific contexts, and that good arguments on this issue should refer and apply to mathematics teaching as a whole: its societal functions, and so on. It is nevertheless appropriate to inform the reader that I spent the last 10 years in universities in Denmark, U.S.A. and Japan as a student, teacher and researcher of mathematics. My point of view is evidently influenced by this particular background.

2. The problem

In the Western tradition of higher education, mathematics occupies a central, many-sided role. I do not intend to argue this point at length here; rather, I content myself with recalling that, from within the heritage of ancient Greece, Euclidean geometry was long thought of as the fundamental and perfect setting for a quantitative and logical description of the universe. Mathematics in this sense was not only a tool for the natural sciences (astronomy in particular), but above all it was an integrated part of a holistic view of the universe, revealed once and for all by God through people like Aristotle, Euclid and Thomas Aquinas. As we know, this universal model collapsed with the works of Galileo, Kepler, Newton and many others, and a crucial factor in this process was the rise of what we perceive today as 'modern mathematics'.

Mathematics lost some of its eternal, absolute character with these developments, particularly after the definite defeat of the formalist attempt to establish a logical framework for all conceivable mathematics to be discovered. Mathematics gained in vitality, applicability and influence, and is certainly an indispensable tool in many ways in our present-day society.

Due to the rising complexity of these tools, however, it becomes more and more hopeless even for experts of mathematics to keep a global perspective, and more and more evident that what is taught in general public education is only a very small corner of the mathematical knowledge needed to understand and develop even those applications of mathematics that influence the plainest aspects of human daily life It is, in fact, a depressing paradox, known to most of those who are engaged in both research and teaching of mathematics, that while new developments in the subject take place at ever increasing heights of abstraction and presupposed knowledge, the level at which it is taught and learned seems to be mostly decreasing.

With the loss of the innocent times when Euclidean geometry was still the unaffected basis of quantitative reasoning, we have not only lost a firm answer to the natural question about what is relevant to teach in mathematics It also becomes less obvious whether mathematics teaching is at all relevant, at least in its traditional forms. The recent appearance of computers has made the question even more urgent, as it can be argued (cf. Kemeny, 1988) that many of the standard skills that are still being taught world-wide are now absolutely useless - pocket computers can do even advanced tasks better. Much of what is left of mathematical work to be done by humans is at a level of abstraction and technicality far beyond standard school curricula

In addition, the scope of elementary school education, which in mathematics is usually up to some familiarity with symbol manipulation, has been extended to the entire population in most industrialised countries The conclusion that to this broad audience, one should teach only elementary arithmetic, rather than make everyone struggle with some useless rudiments of more abstract topics, is not far from imposing itself when reflecting along these lines.

To summarise, the following points necessitate a complete revision of the theoretical grounds on which one can base a justification of mathematics as a central part of general education:

- the historical development of mathematics itself;
- the change of mathematics' role in society and among other fields of human thought;
- the availability of computer technology;
- the broader scope of education in general

For such theorising to be useful, it should address the problems in universal terms that allow the same questions to be considered for other subjects along with mathematics, and (to the extent this is possible) allow for non-specialists of mathematics to validate the arguments put forward to justify the role of mathematics in education.

3. Some current approaches

As mentioned in the introduction, the teaching of mathematics is often tacitly assumed to be justified Nevertheless, from the on-going public debate on the state and means of this teaching, one can in fact extract at least three typical arguments in its favour; these are listed below as 1 - 3. In addition to these, I list two more important aspects for the discussion, which come from recent treatments of the issue in question.

1. *The training purpose*
   Mathematics provides training in logical and deductive analysis (which is then assumed to be desirable)
2. *The prosperity purpose*
   Mathematics is a pre-requisite for technological developments which in turn provide financial prosperity.
3. *The aesthetics purpose*
   Mathematics is a beautiful intellectual structure that learners should be offered to appreciate.
4. *The filter purpose* (cf., Davis and Hersh, 1988)
   The degree of success in learning mathematics can be used as a just way to evaluate how gifted students are and hence which professional and educational opportunities they should be given.
5. *The democracy purpose* (cf., e.g., Niss, 1994)
   A certain level of mathematical capacity should be obtained by all citizens in a democracy, because such capacity is crucial to qualified democratic participation at many levels (technical, economic, etc) in modern society.

The first three are more or less classical, while the last two are different in nature both from the first three and from each other. Point 4 is an observation of an actual function of the mathematics discipline in many contexts, though not a function desired by its authors; point 5 expresses an ideologically-shaped ideal of the author rather than a common educational policy.

Before going into a more detailed discussion of these viewpoints, let me point out a common feature: they all emphasise some claimed property of mathematics that could be (and is) claimed by many other subjects, and hence, as they stand, they are not very useful in a dialogue or comparison with other disciplines.

The training purpose is probably the oldest one, as old as the subject itself. It emphasises the intellectual benefits obtained from the study of mathematics, rather than the specific knowledge obtained through this study. In my opinion, this is an apology for teaching mathematics, rather than a justification This is not meant to deny that a serious undertaking of mathematical studies entails, as a side-effect, a certain sharpening of the intellect, but it would be unjustified academic chauvinism to claim this to be a characteristic property of mathematics; of course, any study forms and trains the mind in some way. We ought to be more specific about what mathematics does for the learner and his or her society.

The prosperity purpose certainly meets this demand. A related purpose, which is still around to some extent, but which seems less dominant now than in the days of the cold war, is that of the external power a nation can build on military technology developed by scientists with a strong mathematical background. In both cases, the objection that just a rather small number of such experts suffices to meet the demand still does not eliminate the need for a massive training in mathematics at more elementary levels

The problem here, at least from a humanistic viewpoint, is that the interests of the individual learner are only considered in so far as they agree with those of the nation, or as they can be satisfied by being part of a prosperous or powerful group. Again, this does not invalidate the argument completely, but if pursued to its extremes, one does end up with a conception of education and learners which is very far from (Western?) values like democracy, integrity of the individual and so on Mathematics should not be presented as a mere tool to design technology just because this is an easy way to convince decision-makers.

The *aesthetics purpose* probably offers some attraction to most of those who have gained a more advanced knowledge of the subject. The idea is certainly popular among professional mathematicians, but is largely incomprehensible to others. Effectively, mathematics does have difficulties competing on this ground, because other areas with aesthetic appeal (e g., music and the arts) are readily accessible, and also they seem to reach 'deeper' in this sense The party-trick or 'brainy puzzle' face of mathematics does have some general appeal, but it is rather different from the aesthetic experience of the professional mathematician (somewhat like the Disney version of a piece of literature relative to the original) Thus, a main problem with this argument, in contrast to the previous two, is that it is difficult to promote, at least in its true form, and even the true form is not too convincing

The *filter purpose* refers to the function of mathematics as a filter to sort students, especially within the educational system It is rejected by Davis and Hersh (1988):

> We do not really want to be gate keepers and agents of exclusion (p 104)

Before leaving it there, one should still consider that the argument has a socio-cultural side, dating back to the rationalism of the eighteenth century, to the effect that mathematics is thought to be a field of competition in which favoured social background is beaten by pure intelligence and just reasoning

Although this has not been sufficiently investigated, it does seem plausible that the general socio-cultural bias in academic recruitment (for this, evidence exists in abundance) is less pronounced in mathematics and other exact sciences than in, say, the humanities This, however, does not support the original thesis: that certain formal requirements in mathematics serve to suppress the general phenomenon of dependence between socio-cultural background and professional achievement

The democracy purpose is present, under various forms, in several current directions of research in mathematics education Indeed, as formulated above, it does seem attractive to me, except that it needs to be based on grounds which are specific to mathematics Evidently, the knowledge of many quite different areas can be relevant to a qualified participation in the forming of today's society. We need to exhibit an accessible yet truthful model of what the learning of mathematics can do for the learner and his or her capacities as a citizen in our modern democratic society. An attempt is made below

4. Mathematics distinguished by a linguistic register

The first task in an attempt to solve the justification problem is to locate mathematics among other subjects that are, or might be, taught in a given context. It should be clear by now – at least, it was argued in the previous sections – that this is not settled by the conventional nomination of mathematics as the most certain and 'pure' form of human thought. The works of Wittgenstein, Lakatos, Hersh and many others have provided ample demonstration that mathematics is a human construction under constant change, just as are philosophy, music, military technology, and so on Now, how can human activities which are so distant in nature, methods and uses, ever be evaluated against each other? Evidently, choices have to be made, if not by exclusion, then at least in the sense of giving priority In the words of mathematician-philosopher Blaise Pascal:

> *Il faut parier.*

This forms a central theme of no 233 of his *Pensées*

One may invoke here that the choice is to some extent left to learners, the extent being an increasing function of age This does not contradict that 'official choices' have to be, and are indeed being, made Even if choices are ideally to be made on an individual basis, we are still left with the problem of how the educational system can put learners in a position to make this choice.

To meet the needs pointed out at the end of Section 2, I shall take a linguistic approach to this task. I shall use the term 'language' in a very broad sense, simply to mean forms of human expression; language as found in dictionaries, novels and ordinary speech will be called 'verbal language' here, and will thus be just one group of 'linguistic registers' to be found within the entire body of languages

Mathematics can be singled out, among other forms of human imagination and ingenuity, by the very specific linguistic register in which its ideas are formulated. The conceptual description that I need here, and which is outlined in the following, is to a large extent based on the more detailed accounts of Skemp (1982) and Rotman (1988).

One should, however, admit right away that the structural analysis of this 'mathematical register' is still far from complete, the main problem probably being that it is hard to separate such a description from the more speculative, and general, analysis of the nature of mathematics (often called the philosophy of mathematics). I shall thus rely on a pragmatic and descriptive view on this matter: in fact, the situation here is not so different with other linguistic registers.

On a superficial level – which usually allows us to recognise a given text or talk as one on mathematics, prior to any detailed consideration – it consists of a certain mixture of symbols along with some parts formulated in verbal language; often, drawings of a geometrical nature are also present, and for many purposes they can be regarded as parts of the symbolic inventory That the symbolic side of the register is indispensable can be seen by a short observation of mathematical discussion, which will reveal how spoken communication eventually has to be complemented by writing symbols (typically on blackboards or paper) In transmitting non-trivial mathematical ideas, writing is unavoidable only because the symbolic parts of the message are best perceivable through vision, hence because these parts are crucial Also, verbal language has several functions in the mathematical register, from heuristic or explanatory remarks to statements which could be formulated using symbols as well; for instance, the statement:

> A function is continuous at points where it is differentiable.

can certainly be re-written without any use of wording, but the message would for most purposes be obscured by doing so.

At the level of content, we have logical deductions and speculations on more or less abstract quantitative notions, which however are deeply related to the aforementioned symbolic structures to be observed on the textual surface; the distinction signifier–signified is at least problematic in this case The need for writing alluded to above often comes from the *constitutive* function of symbols in mathematical communication; for instance, just writing 'X' may, in some contexts, be all it takes to introduce a topological space.

As in the early stages of learning a verbal language, one begins in mathematics by introducing the student to some basic vocabulary and some rules on how they interact; mathematics has syntax, semantics, and so on, and by working with some elementary objects of the register one begins to see the need for each of these sets of rules. As the study progresses, one also gets to know more of the inventory of the register and some of its more subtle rules, and one gets to see how learned rules are sometimes broken - such as the identity $ab = ba$ in the context of mappings - more complex, nuanced rules are then constructed in the learner's mind All this is a very complicated procedure in which human interaction plays a most central role, as does the learner's own reflection, experience and gifts.

A linguistic register is used to communicate certain types of information Verbal language is, of course, employed in other intellectual disciplines like literature, philosophy and so on The linguistic register of mathematics is well known to be applicable in (and was historically often designed for) other fields as well, physics in particular. Certainly the purist view of mathematics as being almost characterised by its remoteness from tangible reality is a negative and fruitless one Separating a linguistic register from its applications does not advance its analysis, but makes it literally meaningless. Insight is obtained through experience with applications of the learned linguistic structures.

Another facet of the mathematical register is that it is a much more recent human construction than verbal language Maybe it is also more apparent that it is a construction, as it does certainly not 'happen' without the engaged will of its constructors Virtually any human being is exposed to verbal language from the first moments of life. The mathematical register can be learned only through directed study; Gauss-type myths about small children discovering deep mathematical structure out of fresh air are probably often just myths. Also, the admiration gathered by such claims to exception only enforces the general rule that gaining proficiency in the mathematical language demands considerable time, energy, and ability, both from the student and from the instructor.

Any linguistic register has its limitations. In spite of the widespread use of 'loan-words' from the mathematical register in other disciplines, it has its weak sides; it is particularly useless in describing human feelings, judgements and interactions. This text, for instance, does not and could not apply the mathematical register The fact that a characteristic of the mathematical register is the necessity for self-reference (cf., Rotman, 1988) had already been noticed by Aristotle:

> mathematics is [...] a science which deals with permanent things, but not with things which can exist separately. (in *Metaphysics*, 1064)

In particular, the mathematical register does not apply to express qualitative entities like emotion or opinion:

> Mathematics takes no account of good and bad. (*ibid.*, 996)

The feeling seems to be widespread that too much exercise within this register has some kind of emotional mutilation as a side effect For instance, the French novelist Flaubert (1881) claimed, in his novel *Bouvard et Pécuchet*, that:

> *les mathématiques déssèchent le coeur.*

This criticism addresses some perceived social patterns in the 'mathematical culture' which of course are relevant to the mathematical register (this being constructed by the culture) and the way it is taught and practised; but, for the justification problem, it is like arguing for a ban on hammers by their irrelevance to musical performance.

In conclusion, let me summarise the essential features of the mathematical register:

- it consists of a certain combination of symbolic and verbal inventory with certain rules governing its use;
- it is a relatively recent, ever-changing human construction;
- it takes time, will and energy to access;
- it is applicable and necessary in many contexts in which quantity and form need formal articulation;
- it does not allow for reference to external, 'undefined' objects; in particular, it cannot serve to express emotion.

5. A linguistic approach

Mathematical statements pertain to objects which are non-measurable, mental constructions. This does not mean that they do not represent knowledge which can be universally agreed upon, such as the solution of a given equation; such consensus depends only on the fluency obtained within the relevant part of the mathematical register One may object to the grounds on which the statement is argued, such as the axiom of choice for many parts of advanced analysis, but this kind of dispute is different from the questioning of whether the argument itself is correct.

Since I have taken a pragmatic point of view - linguistics is indeed concerned with describing and analysing language as practised rather than with normative criticism or engineering of such practice - I note here that practising mathematicians, even at advanced levels, are rarely engaged in long-lasting disputes about the validity of a mathematical argument. As I have already pointed out, history presents continuous changes in what is considered acceptable mathematical proof, and such changes will continue to occur.

Thus, the universal agreement that we are talking about is not universal in 'time direction', that is, it is not eternal. It refers to the actual 'state-of-the-art', it is only meaningful within a given version of the mathematical register, which is also the basis for what is taught in mathematics Thus, when arguing the relevance of such teaching, one should not focus on a certain set of mathematical statements (or theories), but on the fluency within the register that is needed to treat them.

Consider a concrete example, inspired by Kemeny (1988) which occurs at different levels of complexity in most teaching of mathematics: solving equations On a superficial level, the taught matter here consists of algorithms or formulae in which one fills in the data to the given equation in order to produce a solution. The use of such an algorithm requires in principle no intellectual activity, and can indeed be left to computers; it is thus not what should be stressed from a mathematical point of view In the linguistic approach, the focus is on:

(a) gaining familiarity with the conceptual structures - concretely, a part of the mathematical register - that go into formulating and solving the problem, including the relevant terms to express well-posedness, existence of solutions and proofs;
(b) possible interpretations of the problem and its solution: in other words, possible uses of the acquired language.

This may sound less concrete than solving 100 equations of the same type, to which some algorithm applies; indeed, such solving exercise takes place on a lower level of the register (typically, elementary arithmetic) than the level required to master the subject matter

The worth of mathematics teaching should generally be found in the versatility within the mathematical register that it provides learners with. The learners' ability to organise non-mathematical structures by use of the register depends on this, as does his or her overall ability to engage in creative practice within the register. The analogy with learning a foreign language is particularly enlightening when we compare 'factual knowledge' of standardised problem-solving tools with proficiency within the register at the level in question: it is like comparing a (possibly large) stock of memorised standard phrases with fluency in the language at the same level

The idea of teaching foreign languages as a set of standard phrases, each applicable to certain situations, is evidently absurd to anyone who has seriously attempted to learn (or teach) one It cannot be justified because it is useless On the other hand, the counterpart of this idea is widely practised in the teaching of mathematics. One cannot deny that standard methods are of some use in certain applications of mathematics, but they become more and more insufficient as educational goals; the availability of mathematical and numerical software aside, cookbook-mathematics in this sense is also irrelevant to most of the purposes listed in Section 3, particularly to the democracy aspect.

Moreover, in contrast with *communicative* abilities in mathematics, such automatic skills are rigid and accommodate changes in the demands of applications very poorly; thus, cookbook-mathematics teaching is useful only under the assumption that mathematics does not change, which, as we have already argued, is far from true. Taking a linguistic viewpoint on the justification issue implies a change of focus towards communicative aspects of mathematics, while factual knowledge in the sense of solving tools is de-emphasised. Thus, this choice of viewpoint has strong implications as to what kind of mathematics teaching we seek to justify, and should consequently practice.

These consequences are certainly non-trivial as they exclude a considerable fraction of present-day mathematics teaching from justification (at least for my agenda), so that my reflections on the justification issue, far from merely providing endorsement for a continuation of current practice, point to a radical revision of the aims and methods of mathematics teaching at all levels. Another example of this is indicated later in this section.

How, then, are we to argue for the value of providing students with fluency in the mathematical register? I wish to discuss four groups of arguments here which I believe are particularly important:

- the *global communication function* in its relation to verbal language;
- the *noise-free human interaction aspect*;
- the *human-being–nature dialogue* in a contemporary setting;
- the *readiness-of-learners aspect*, as a complement to the former three.

I start again from the analogy between learning mathematics and learning a (very) foreign language. Why should one take the trouble to do so? In the case of a verbal language, one would immediately point to the enhanced potential for communication the learner obtains - a certain fraction of the human population and a certain body of textual material that become approachable Once a thorough acquaintance with the new language has been reached, the learner will find him- or herself not only able to 'translate' - to express and understand equivalent statements in the foreign language as well as in the mother tongue - he or she will also experience how cultural environment shapes a language, how subtle differences make some ideas more easily, or even exclusively, expressible in one of the two.

However, the contemporary need for global communication enforces the translation aspect as a main motivation for learning a foreign verbal language The 'global village' is rapidly approaching a situation where a few languages are - in addition to being mother tongues for a large number of people - universal means of communication in a variety of forums where multi-national interaction takes place. In this situation, the teaching of these few languages is certainly essential for everyone.

Mathematics has its role in this realm of interaction, but the relation between 'translation' and 'expression' is quite different here. The impact of local culture on mathematical *practice* is not negligible, and many studies have been devoted to it; but in essence, what is taught and devised in mathematics around the world today, takes place in the *same* linguistic register, thus enabling universal expression of quantitative and structural ideas which are deep parts of common human experience. In the history of mathematics, there are many examples of how isolated groups of researchers fought with the same problems at different places and times, gaining parts of insight without knowing of relevant discoveries done long (in some cases, centuries) before.

Today, the mathematical community consists of tightly spun networks of researchers who exchange information as easily and as quickly as they could if they were all living in the same village. Therefore, today, the mathematical register presents practically no local variations, and as mentioned this also affects the taught forms It is, more than any verbal language, common ground on which human action and interaction takes place To the extent that this fact become visible through our teaching of mathematics, it is already a significant argument for the importance of such teaching, and it then becomes important to others than the small community of experts engaged in pushing forward the limits of human knowledge in the subject. Indeed, the historical and contemporary practice of mathematics, in particular the momentum it has gained with the internationalisation of the past few centuries, is an invaluable lesson to humanity about its potential to develop common meaning based on cross-cultural communication.

It is in fact the 'weak side' of the register, discussed in Section 4, which turns out to be its real strength in this relation. Namely, the closedness of the register - the necessity of self-reference, the exclusion of emotional complication - means that it is a perfect channel for *noise-free communication* in the following sense: while any verbal communication contains several layers of intended, implied and perceived meaning, involving directly as well as indirectly stated notes of emotion, the mathematical register does not allow for such equivocation Here, the use of the term 'noise' for indirect meaning is not meant in a pejorative sense; 'noise' is both necessary and inevitable in other linguistic registers: in fact, the ability to perceive indirect meaning is probably at the heart of human intelligence.

The value of the mathematical register in this context is to reveal that *there is something more*: it represents a channel for intelligent communication which is essentially different from both verbal and artistic registers, and which can still not be reduced to the inanimate binary streams that can be handled by technological devices. The experience of this side of human intelligence is another main point of learning the mathematical register. Of course, emotional energy could and should be involved in constructing mathematical knowledge, but it is foreign (or external) to the register itself.

To see what I mean by this, imagine a mathematician who has just solved a long-standing problem, the Riemann conjecture, say. His paper will be both written and read with strong emotions involved, but the mathematical statement and proof is there, clean and limpid for anyone to appreciate, whether or not they have any prior feeling about it. The proof may be argued about, but the argument that takes place within the register will not involve emotion.

The fact that emotions do not interfere with messages formulated within the mathematical register is also the *raison d'être* of its capacity to enable cross-cultural communication which was discussed before Its relevance for students is accentuated by the general noise-level in contemporary mass-communication, as the mathematical register - unfortunately, not always the presentation of it in teaching - is not itself touched by the general tendency towards commercialisation, over-simplification and superficiality. Mathematics can and should exploit the means of modern technology to facilitate the learning of its language, and also to enable this learning to take place at higher levels than otherwise possible - but one should take care not to confuse means and goal by presenting mathematics as merely one (perhaps not so entertaining) way to apply technology

It is also an important experience for democratic purposes to gain knowledge of a linguistic register in which statements can be formulated in the limpid form that was mentioned before. Of course, constructing mathematical or statistical models always implies a certain view of a situation, which could certainly contain political points of view; but without proper knowledge of the mathematical register, one could easily be led to believe that not only the mathematical treatment, but also the model itself, has some validity which is exempt from ordinary debate. In contrast, knowing that the mathematical register cannot itself provide endorsement of subjective meaning, one is instead led to look for it in the assumptions of the model. The criticism of models is an important aspect of being a qualified participant in the democratic process, but it is impossible without a thorough knowledge of how the mathematical register works.

For three years, I have been teaching a freshman course on mathematics for biology students. The justification issue becomes very concrete in this setting as mathematics is on the one hand mandatory for these students, and on the other hand not what they *a priori* feel they have come to study: real living nature, not dead formulae. Indeed for much of their curriculum, mathematics is really irrelevant - but this is only the descriptive part of the field, nature described as humans have discovered it and explored it with their senses. Using a wide range of simple examples that can be phrased with the modest inventory of mathematical language that we get to study, I am trying to convince them that mathematics can be thought of as the language of nature: when we go beyond the description of our own, static impressions, to consider process and change in nature, our understanding of matters is deepened considerably by realising how these processes follow mathematical patterns, and consequently how nature in a sense communicates its regularities in mathematical terms. We can read the following verses of Lamartine as an invitation to be attentive to this kind of message

> Adore ici l'écho qu'adorait Pythagore,
> Prête avec lui l'oreille aux célestes concerts.
> (v. 55f of Le Vallon, in *Méditations Poétiques*, 1820)

Learning mathematics provides, in this way, a new sense by which nature can be perceived

In fact, mathematical models are present in modern natural science whenever quantitative (as opposed to descriptive) analysis is needed. The 'unreasonable effectiveness of mathematics in the natural sciences' (Wigner, 1960) really means that the mathematical register has proven to be a surprisingly efficient tool in the *dialogue between humanity and nature* Many mechanisms in nature can be expressed mathematically, and humans can impose control over these mechanisms by intelligent use of such expression, often involving quite complicated mathematical analysis.

A folklore phenomenon, which I take as an example because of its simplicity, is the repeated occurrence of the exponential function in this dialogue The exponential function is in itself an elementary mathematical object that does not carry any emotional or political meaning. Being interpreted as say, a forecast of demographic or ecological developments, its implications in this direction can be far-reaching.

This example points to one of the most intriguing discoveries of our time: not only are humankind and nature engaged in a fragile interaction which can be observed and described in mathematical terms, but it is also possible for mankind to direct the process by making use of such knowledge. Gaining mastery of the mathematical register is a necessary step to becoming engaged in this dialogue. The dangers of leaving it to an exclusive élite have become sadly apparent in more recent history, and represent another main argument for the popularisation of mathematical fluency

I have argued for the importance of versatility in the mathematical register at a non-trivial level as well as how demanding the acquisition of such versatility is. This leads me to point out a complementary reason for promoting mathematics teaching for children and adolescents, the *readiness-of-learners* aspect. It is a widely-held opinion, based on experience and, to some extent, on research, that addressing the basics of mathematical knowledge is most likely to be successful when done early in life. For instance, Robitaille and Travers (1992) note as one conclusion of the two first IEA international surveys on mathematics education in various countries:

> 13-year-olds in all of the participating countries had a more positive view of mathematics as a process than did the senior students [..] Similar trends were noted in the second IEA mathematics study, [..] these findings may be an accurate reflection of a decline in students' interest in and attitude toward the continued study of mathematics (pp 691-2)

The age factor may also be particularly important for the purposes of mathematics teaching which were emphasised above; Robitaille and Travers conclude from the same study that:

> Senior students tended to rate the importance of the role of mathematics in contemporary society less highly than did the 13-year-olds, [..] it may be that senior students were indicating that they did not see very much in the way of applications of the mathematics they were studying in school to everyday life That would seem to be a realistic view considering the nature of many of the topics included in the curriculum at the time. (p. 692)

Just as it is becoming more and more common to start the teaching of the first foreign language at an age where children have still not achieved literacy (or even oral fluency) in the mother tongue, communicating mathematically would probably become more natural if trained earlier. Here, we are not talking about elementary arithmetic alone, but also of some rudiments of algebra, geometry, probability, and so on. Much of what is often a source of painful struggles for teenagers could probably be perceived as intriguing games at an earlier age. Because early acquaintance with mathematics at this level of ambition is, to my knowledge, not really practised anywhere, I must leave it as a thought which I believe deserves be tried out.

One could imagine that the familiarity with two languages found among bilingual children could be to some extent reproduced in the context of the mathematical register. It certainly seems a blatant lack of timing to me that university students even in developed countries are often still unfamiliar with elementary notions of geometry, manipulation of fractions, and the like. At that point, it may be too late to approach such items with the open mind of a child, as they do not really meet the demands for intellectual challenge which are natural for the age, nor do they enable the learner to deal with real-life problems at his or her level of maturity.

Another important argument to accelerate the acquisition of mathematical fluency is the need for it in other science subjects, particularly in physics. Everyone who has tried to teach (or learn) classical mechanics without the availability of calculus will know what I mean by saying that, in such subjects, the difficulties and complexity can grow immensely if the relevant parts of the mathematical register are not mastered. Like the learning of a language must be co-ordinated with, and to some extent precede, the reading of literature written in that language, so must curricula be devised so that difficulties with mathematical language do not obscure those subjects which use it as a means of expression.

Conclusion

The view of mathematics as characterised by a specific linguistic register, including an analysis of the nature and functions of this *mathematical register*, enabled me to provide several arguments for mathematics teaching in an ambitious sense which emphasises communicative abilities developed thereby, and to formulate the arguments in a way which indicate the position of mathematics among other forms of communication in which human knowledge (and school subjects) appear. In particular, I have stressed the importance of general fluency in the mathematical register for global, noise-free human interaction, for participation in human society and for popular engagement in the dialogue between humanity and nature. I have pointed out the contours of possible need for a complete revision of mathematics teaching, both in the choice of emphasis and in its temporal placement within the educational system

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