Challenges to the Importance of Proof

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An informed view of the role of proof in mathematics leads one to the conclusion that proof should be part of any mathematics curriculum that purports to reflect mathematics itself, and furthermore that the main function of proof in the classroom reflects one of its key functions in mathematics itself: The promotion of understanding. Yet developments in both mathematics and mathematics education have now caused the very place of proof in the teaching of mathematics to be called into question. Examining these developments, this paper concludes that proof is alive and well in mathematical practice, and that it continues to deserve a prominent place in the mathematics curriculum This paper also argues that the most important challenge to mathematics educators in the context of proof is to enhance its role in the classroom by finding more effective ways of using it as a vehicle to promote mathematical understanding

Proof and understanding

For some time I have been asking myself what role proof ought to play in mathematics education My initial inquiries led me to publish a critique of the view of proof adopted by the "new math" movement of the 1950s and 1960s, in which I examined the belief implicit in the "new math" that the secondary-school mathematics curriculum better reflects mathematics when it stresses formal logic and rigorous proof [Hanna, 1983].

I found that this belief rested upon two key assumptions: (1) that in modern mathematical theory there are generally accepted criteria for the validity of a mathematical proof, and (2) that rigorous proof is the hallmark of modern mathematical practice. Both of these I came to reject. First of all, after examining the major accounts of the foundations of mathematics (logicism, formalism, intuitionism and quasi-empiricism), I concluded that these significant schools of mathematical thought hold widely differing views on the role of proof in mathematics and on the criteria for the validity of a mathematical proof

Secondly, through an examination of mathematical practice I came to the conclusion that in the eyes of practising mathematicians rigour is clearly secondary in importance to understanding and significance, and that a proof actually becomes legitimate and convincing to a mathematician only when it leads to real mathematical understanding Specifically, I concluded that mathematicians accept a new theorem only when some combination of the following holds: The theorem and its implications are understandable and not obviously in doubt, the theorem is significant enough to warrant study, the theorem is consistent with the body of accepted results, the author has an unimpeachable reputation in the field, and there is a convincing mathematical argument for the theorem, rigorous or otherwise.

In light of the theory and the practice of mathematics, then, I argued that we can impart to students a greater understanding of proof and of a mathematical topic by concentrating our attention on the communication of meaning rather than on formal derivation I stated that:

> the primary implication of this conclusion for curriculum planning is that a secondary-school mathematics which aims to reflect the real role of rigorous proof in the theory and practice of mathematics must present rigorous proof as an indispensable tool of mathematics rather than as the very core of that science [Hanna, 1983, p. 89]

Several mathematicians have expressed similar points of view [Davis and Hersh, 1981, 1986; Kline, 1980; Manin, 1977] In particular, I have been both gratified and reassured to find corroboration of my views in a recent paper by William Thurston [1994] Along with 15 other mathematicians, Thurston was responding to an article by Jaffe and Quinn [1993], who had cautioned against weakening the standards of proof. Jaffe and Quinn had proposed that heuristic work be labelled "speculation" or "theoretical mathematics", to distinguish it from what they regard as proper mathematics, in which theorems are to be proven rigorously

Thurston maintains that in attempting to answer the question "What is it that mathematicians can accomplish?" one should not begin with the question "How do mathematicians prove theorems?". He points out that the latter question carries with it two hidden assumptions, (1) that there is a uniform, objective, and firmly established theory and practice of mathematical proof and (2) that progress made by mathematicians consists of proving theorems [p. 161], and he goes on to state that these assumptions will not stand up to careful scrutiny One will note that these hidden assumptions are in effect the same as the assumptions of the "new math" which I discussed earlier.

For Thurston the right question to ask is "How do mathematicians advance human understanding of mathematics?". And he adds: "We [mathematicians] are not trying to meet some abstract production quota of definitions, theorems and proofs. The measure of our success is whether what we do enables *people* to understand and think more clearly and effectively about mathematics" [p 163].

My own research has led me to stress the importance of understanding, but I have never seen this as a criticism of formal proof as such Thus I find myself in agreement with Thurston:

> I am *not* advocating the weakening of our community standard of proof; I am trying to describe how the process works Careful proofs that will stand up to scrutiny are very important ... Second, I am *not* criticizing the mathematical study of formal proofs, nor am I criticizing people who put energy into making arguments more explicit and more formal These are both useful activities that shed new insights on mathematics [Hanna, 1983, p. 169].

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Challenges to proof from mathematics

Not all agree with Thurston on this point, however. A number of recent developments in the practice of mathematics, all of them reflecting in some way the growing use of computers, have caused some mathematicians and others to call into question the continuing importance of proof.

The computer has acted as a leavening agent in mathematics, reviving an interest in algorithmic and discrete methods, leading to increased reliance on constructive proofs, and making possible new ways of justification, such as those that make use of computer graphics [Davis, 1993]. The striking novelty of its uses, on the other hand, has lent a tone of urgency to the discussions among mathematicians about its implications for the nature of proof [Jaffe & Quinn, 1993; Thurston, 1994; Tymoczko, 1986]

Indeed, the use of the computer has led some to announce the imminent death of proof itself [Horgan, 1993]. On the basis of interviews with several mathematicians, John Horgan makes this prediction in a thought-provoking article entitled "The death of proof" that appeared in the October 1993 issue of *Scientific American*. He claims that mathematicians can now establish the validity of propositions by running experiments on computers, and maintains that it is increasingly acceptable for them to do mathematics without concerning themselves with proof at all.

One of the developments that prompted Horgan's announcement is the use of computers to create or validate enormously long proofs, such as the recently published proofs of the four-colour theorem (Appel and Haken) or of the solution to the party problem (Radziszowski and McKay) These proofs could not conceivably be constructed in any other way, because it is impossible for any human being to perform the long computations which they require. Nor could a human being hope to verify these computations after they have been carried out by a computer. Because computers and computer programs are fallible, then, mathematicians will have to accept that assertions proved in this way can never be more than provisionally true.

A second and particularly fascinating development is the recently introduced concept of zero-knowledge proof [Blum, 1986], originally defined by Goldwasser, Micali and Rackoff [1985]. This is an interactive protocol involving two parties, a prover and a verifier. It enables the prover to provide to the verifier convincing evidence that a proof exists, without disclosing any information about the proof itself. As a result of such an interaction the verifier is convinced that the theorem in question is true and that the prover knows a proof, but the verifier has zero knowledge of the proof itself and thus is not in a position to convince others.

In principle a zero-knowledge proof may be carried out with or without a computer. In terms of our topic, however, the most significant feature of the zero-knowledge method is that it is entirely at odds with the traditional view of proof as a demonstration open to inspection. This clearly thwarts the exchange of opinion among mathematicians by which a proof has traditionally come to be accepted

Another interesting innovation is that of holographic proof [Babai, 1994; Cipra, 1993]. Like zero-knowledge proof, this concept was introduced by computer scientists in collaboration with mathematicians. It consists of transforming a proof into a so-called transparent form that is verified by spot checks, rather than by checking every line. The authors of this concept have shown that it is possible to rewrite a proof (in great detail, using a formal language) in such a way that if there is an error at any point in the original proof it will be spread more or less evenly throughout the rewritten proof (the transparent form). Thus to determine whether the proof is free of error one need only check randomly selected lines in the transparent form

By using a computer to increase the number of spot checks, the probability that an erroneous proof will be accepted as correct can be made as small as desired (though of course not infinitely small). Thus a holographic proof can yield near-certainty, and in fact the degree of near-certainty can be precisely quantified. Nevertheless, a holographic proof, like a zero-knowledge proof, is entirely at odds with the traditional view of mathematical proof, because it does not meet the requirement that every single line of the proof be open to verification.

Zero-knowledge proofs, holographic proofs, and the creation and verification of extremely long proofs such as that of the four-colour theorem are feasible only because of computers. Yet even these innovative types of proof are traditional, in the sense that they remain analytic proofs. More and more mathematicians appear to be going beyond the bounds of deductive proof, however, using the computer to confirm mathematical properties experimentally.

Horgan quotes several mathematicians who concede that experimental methods, though perhaps not new, have acquired a new respectability. They have certainly received increased attention and funding following the development of graphics-oriented fields such as chaos theory and non-linear dynamics. As a result, more mathematicians have come to appreciate the power of computational experiments and of computer graphics in the communication of mathematical concepts

They are going well beyond communication, however. In a clear departure from previous practice, it now seems to be quite legitimate for mathematicians to engage in experimental mathematics as a form of mathematical justification that does not include pencil-and-paper proof. Horgan claims, in fact, that some mathematicians think "the validity of certain propositions may be better established by comparing them with experiments run on computers or real-world phenomena" [1993, p. 94].

A case in point is the University of Minnesota's Geometry Center, where mathematicians examine the properties of four-dimensional hypercubes and other figures, or study transformations such as the twisting and smashing of spheres, by representing them graphically with the aid of computers. Horgan also cites the so-called computer-generated video proof called "Not Knot" prepared by the mathematician William Thurston at Berkeley.

Exploration itself is not inconsistent with the traditional view of mathematics as an analytic science. But in drawing general conclusions from such explorations these mathematicians would appear to be turning to the methods of the empirical sciences. Indeed, the Geometry Center has helped found a new journal called *Experimental Mathematics*. Such a radical shift in mathematical practice is entirely justified, according to Philip Davis, a mathematician who strongly advocates greater use of computer graphics. He argues that the concept of visual proof is an ancient one that was unfortunately overshadowed by the rise of formal logic and deserves to regain its important place in mathematics [Davis, 1993].

One must admit that visual proofs and other experimental methods are easier to grasp than some new methods such as holographic proof. Babai himself points out that holographic proofs "may not provide either insight or illustration." On the other hand, as he adds, they do enjoy the advantage of yielding near-certainty, and in this respect they "differ fundamentally from experimental mathematics" [1994, p 454]

The developments I have tried to describe here certainly pose intriguing questions for practitioners and philosophers of mathematics [Horgan, 1993; Krantz, 1994]. In discussing zero-knowledge and holographic proofs, for example, Babai [1994] asks the following questions: "Are such proofs going to be the way of the future?", "Do such proofs have a place in mathematics? Are we even allowed to call them *proofs*?"

Others have posed similar questions. Should mathematicians accept mathematical propositions which are only highly probably true as the equivalent of propositions which are true in the usual sense? If not, what is their status? Should mathematicians accept proofs that cannot be verified by others, or proofs that can be verified only statistically? Can mathematical truths be established by computer graphics and other forms of experimentation? Where should mathematicians draw the line between experimentation and deductive methods?

Mathematicians continue to debate these and other questions in meetings, on the Internet and in the Forum section of the *Notices of the American Mathematics Society* Yet the very existence of this debate is a confirmation of the central role that proof is seen to play. It is difficult to know just how mathematicians will eventually answer such questions, and in any case one should not expect unanimity If a loose consensus does evolve, it will undoubtedly redefine the concept of proof to some degree. Perhaps this consensus will recognise a multiplicity of types of justification, perhaps even one that is hierarchically ordered

But the point we must not lose sight of, I believe, is that the existence of such a new consensus, even one with large remaining areas of disagreement, would not create a situation which would differ in principle from that which has prevailed up to now. As discussed above, there has never been a single set of universally accepted criteria for the validity of a mathematical proof. Yet mathematicians have been united in their insistence on the importance of proof This is an apparent contradiction, but mathematics has lived with this contradiction and flourished. Why would one expect or want this to change?

This is why I think Horgan's article is mistitled. It actually provides evidence that proof is thriving, albeit under a number of exciting disguises To be sure, some of the things being done today, in the name of proof or as an alternative to proof, may in the end lose the battle for general acceptance, but proof itself appears to be alive and well In using the power of technology, mathematicians are certainly creating new ways of proving and even new ways of thinking about mathematics. But by no means are they abandoning the idea of proof.

Challenges to proof from mathematics education

The influence of educational theories

Since the demise of the "new math," with its exaggerated emphasis on formal proof, we have witnessed in the North American mathematics curriculum a gradual decline in the use of any kind of proof at all This can be attributed in large part, I believe, to the curriculum reforms and the theories of mathematics education which have come to dominate the scene since the 1960s. In their views of proof, mathematics educators have probably been influenced more by these developments in their own field than by innovations in mathematics itself.

The first important movement, back-to-basics, was followed by a number of others, such as instruction by discovery, cooperative learning, focus on problem-solving, and classroom interaction. None was ever universally accepted, but all exercised significant influence on the curriculum. While none of them attacked the teaching of proof specifically, they did shift the emphasis away from it, relegating it to heuristics [Polya, 1973; Silver, 1985].

The most influential theory of education at present, judging by the attention it receives from mathematics educators, is undoubtedly constructivism in its various forms, all of which subscribe to the central tenet that knowledge cannot be transmitted, but must be constructed by the learner [Cobb, 1988; Kieren & Steffe, 1994; von Glasersfeld, 1983]. It too may come to be seen as having had a deleterious effect on the teaching of proof, if only because it has been interpreted in a way that undermines the importance of the teacher in the classroom.

Paradoxically this has happened at the very time that a number of experimental studies have confirmed just how important the teacher really is. In exploring new ways to teach proof, these studies have shown the value of such approaches as debating, restructuring, and preformal presentation, all of which posit a crucial role for the teacher in helping students to identify the structure of a proof, to present arguments, and to distinguish between correct and incorrect arguments [Alibert, 1988; Balacheff, 1988; Blum and Kirsch, 1991; Leron, 1983; Movshovitz-Hadar, 1988]

These interesting new methods encourage students to interact with each other, but they also require the active intervention of the teacher. They are thus inconsistent with the passive role seemingly inspired by constructivist theories. Where classroom practice is informed by these theories, there is evidence that teachers tend not to present mathematical arguments or take a substantive part in their discussion. They tend to provide only limited support to students, leaving them in large measure to make sense of arguments by themselves. The idea clearly seems to be to let students "form their own intuition about the structures of mathematics" without the intervention of the teacher [Koehler & Grouws, 1992; Pirie, 1988].

Lampert, Rittenhouse and Crumbaugh [1994], for example, report with approval a classroom in which fifth graders engaged in group discussion and where the context of instruction was such that it was possible, as they put it, "for the teacher to step out of the role of validator of ideas and into the role of moderator of mathematical arguments."

It should be said that there does not appear to be anything in the constructivist theory of learning itself that would deny to the teacher an active role. In fact Cobb [1994], one of its proponents, decried just such an interpretation of constructivist theory, and added:

> Pedagogies derived from constructivist theory frequently involve a collection of questionable claims that sanctify the student at the expense of mathematical and scientific ways of knowing. In such accounts, the teacher's role is typically characterized as that of facilitating students' investigations and explorations. Thus, although the teacher might have a variety of responsibilities, these do not necessarily include that of *proactively* supporting students' mathematical development. Romantic views of this type arise at least in part because a maxim about learning, namely that students necessarily construct their mathematical and scientific ways of knowing, is interpreted as a direct instructional recommendation. As John Dewey observed, it is then a short step to the conclusion that teachers are guilty of teaching by transmission if they do more than stimulate students' reflection and problem solving [p. 4].

Indeed, Yackel and Cobb have been quite specific about the active role of the teacher:

> ... when students give explanations and arguments in the mathematics classroom their purpose is to describe and clarify their thinking for others, to convince others of the appropriateness of their solution methods, but not to establish the veracity of a new mathematical "truth." ..... The meaning of what counts as an acceptable mathematical explanation is interactively constituted *by the teacher and the children*." [Yackel & Cobb, 1994, p. 3].

Yet the constructivist theory of learning *has* been translated into classroom strategies which are inimical to the teaching of proof. As mentioned, recent studies confirm that it is crucial for the teacher to take an active part in helping students understand why a proof is needed and when it is valid. A passive role for the teacher also means that students are denied access to available methods of proving: It would seem unrealistic to expect students to rediscover sophisticated mathematical methods or even the accepted modes of argumentation.

Current theories of mathematics education have undoubtedly made us more aware of the construction of knowledge by the learner, of the social and cultural environment, and of the need to foster among students a critical perspective. We do need to ensure, however, that students develop the ability to assess each step in a proof and make an informed judgment on the validity of an argument as a whole. It would seem unwise to avoid methods that promise to help do this effectively, simply because they require active intervention by the teacher.

The influence of Lakatos

Some current initiatives influenced by the work of Lakatos [1976] and set out by NCTM in the *Professional Standards for Teaching Mathematics* advocate classroom discourse among students. I have already pointed out that this idea can unfortunately be interpreted in such a way as to downplay the role of the teacher. But there is also an inherent problem with these initiatives, in that they recommend having students develop among themselves "agreed upon rules" for appropriate mathematical behaviour. Underpinning this approach is the belief that it is possible and desirable to emulate in the classroom the heuristic proof analysis described by Lakatos in *Proofs and refutations*.

The publication of *Proofs and refutations* provoked much discussion among philosophers, mathematicians, and mathematics educators. Fascinated by this new and engaging way of looking at mathematical discovery, however, mathematics educators may have assumed that Lakatos' approach is more widely applicable than in fact it is. The case for heuristic proof analysis as a general method rests only upon its successful use in the study of polyhedra, an area in which it is relatively easy to suggest the counterexamples which this method requires [Anapolitanos, 1989; Hacking, 1979; Steiner, 1983]. Should one really generalize from a sample of one?

It is not difficult, in fact, to find examples where the way in which a proof is found or a mathematical discovery is made would be radically different from the process of heuristic refutation described in *Proofs and refutations*. Even the proof of Euler's theorem cited by Lakatos, for example, is a case in which refutation is redundant; as soon as adequate definitions are formulated the theorem can be proved for all possible cases without further discussion. In fact, whenever mathematicians work with adequate definitions (or with an adequate "conceptual setting," to use Bourbaki's term), the process of proof is not one of heuristic refutation. In "A renaissance of empiricism in the recent philosophy of mathematics" (1978, p. 36), Lakatos himself says:

> Not all formal mathematical theories are in equal danger of heuristic refutations in a given period. For instance, *elementary group theory* is scarcely in any danger; in this case the original informal theories have been so radically replaced by the axiomatic theory that heuristic refutations seem to be inconceivable.

The application of Lakatos' ideas to the classroom raises additional issues. One must question, for example, the extent to which a group of students can emulate in the course of a lesson or series of lessons the drawn-out process of examination and discussion through which new mathematical results are subjected to potential refutation [Hanna & Jahnke, 1993].

More relevant here, however, is that the application of Lakatos' ideas may convey in the classroom a misleading picture of mathematical practice. His concepts of "informal falsifiers" and of the "fallibility" of mathematics seem to have led many mathematics educators to believe that we should eliminate any reference to "formal" mathematics in the curriculum and in particular that we should downplay formal proof [Dossey, 1992; Ernest, 1991].

In my opinion this attitude is misguided. In the first place, formal proof arose as a response to a persistent concern for justification, a concern reaching back to Aristotle and Euclid, through Frege and Leibniz. There has always been a need to justify new results (and often previous results as well), not only in the limited sense of establishing their truth, but also in the broader sense of providing grounds for their plausibility. Formal mathematical proof has been and remains one quite useful answer to this concern for justification.

Secondly, it is a mistake to think that the curriculum would be more reflective of mathematical practice if it were to limit itself to the use of informal counterexamples. The history of mathematics clearly shows that it is not the case, as Lakatos seems to have implied, that only heuristics and other "informal" mathematics are capable of providing counterexamples. Indeed, formal proofs themselves have often provided counterexamples to previously accepted theories or definitions. For instance, as Mark Steiner [1983] points out, Peano provided a counterexample to the definition of a curve as "the path of a continuously moving point" by showing *formally* that a moving point could fill a two-dimensional area.

Gödel's famous incompleteness proofs are another example, with an interesting and ironic twist. In this case *formal* proofs were employed to demonstrate that the axiomatic method itself has inherent limitations. Gödel could not have produced these proofs without using a comprehensive system of notation for the statements of pure arithmetic and a systematic codification of formal logic, both developed in the *Principia* for the purpose of arguing the Frege-Russell thesis that mathematics can be reduced to logic. His proofs could certainly not have been produced in informal mathematics or reduced to direct inspection.

Nor does it seem reasonable to assume that Gödel's conclusions could have been arrived at through a discovery of counterexamples ("monster-barring") followed by a denial ("monster-adjusting"), or by finding unexplained exceptions ("exception-barring") or unstated assumptions ("hidden-lemmas"). Curiously enough, however, when some mathematics educators make a case that formal proof and rigour should be downplayed in the curriculum they rest their case on the most formal of Gödel's proofs.

Informal methods clearly have an important place in the mathematics curriculum. Those who would insist upon the total exclusion of formal methods, however, run the risk of creating a curriculum unreflective of the richness of current mathematical practice. In so doing they would also deny to teachers and students accepted methods of justification which in certain situations may also be the most appropriate and effective teaching tool.

The influence of social values

Proof has also been challenged by the claim that it is a key element in an authoritarian view of mathematics [Confrey, 1994; Ernest, 1991; Nickson, 1994]. This claim owes much, including its terminology, to Lakatos [1976], who was attempting to offer what he saw as a "long overdue challenge" to the "Euclidean programme," as he termed it, a programme which in his opinion aimed to create an "authoritative, infallible, irrefutable mathematics."

Supporters of this claim would argue that this so-called "Euclidian" view of mathematics is in conflict with the present values of society, which dictate that we not bow down to authority and not regard knowledge as infallible or irrefutable. They appear to see proof in general, and rigorous proof in particular, as a mechanism of control wielded by an authoritarian establishment to help impose upon students a body of knowledge that it regards as predetermined and infallible.

Now, it may be true that mathematics has sometimes been presented as infallible and taught in an authoritarian way, but I do not think there has been a recent consensus among educators that it should be. Whatever the case, I find it rather strange that proof should have become the main target of what in the end may be no more than a misguided desire to impose a sort of political correctness on mathematics education.

I am not sure, in the first place, what it means to say that mathematics or a mathematical proof is "authoritative," to use a term taken from Lakatos. Certainly a proof offered by a very reputable mathematician would initially be given the benefit of the doubt, and in that sense the fact that this mathematician is considered an "authority" by other mathematicians would play some role in the eventual acceptance of the proof. But the claim seems to be that the very use of proof is authoritarian, and I must say I am at a loss to understand this.

In fact the opposite is true. A proof is a transparent argument, in which all the information used and all the rules of reasoning are clearly displayed and open to criticism. It is in the very nature of proof that the validity of the conclusion flows from the proof itself, not from any external authority. Proof conveys to students the message that they can reason for themselves, that they do *not* need to bow down to authority. Thus the use of proof in the classroom is actually *anti*-authoritarian.

Of course one could claim that the use of proof requires that the students accept certain "authoritative" rules of deduction, and so move the argument to a new, meta-mathematical plane. But one would hope that those who challenge the role of proof are not also challenging the very idea of rules of reasoning. It would be disturbing indeed to see mathematics teachers ranging themselves on the side of a revolt against rationality.

It has also been claimed that the use of proof strengthens the idea that mathematics is infallible. Looking at this first from the point of view of theory, however, it is clear that any mathematical truth arrived at through a proof or series of proofs is contingent truth, rather than absolute or infallible truth, in the sense that its validity hinges upon other assumed mathematical truths (and upon assumed rules of reasoning). If we look at this claim from the point of view of mathematical practice, we know that mathematicians, much as they would like to avoid errors, are as prone to making them as anyone else, in proof and elsewhere. The history of mathematics can supply many examples of erroneous results which were later corrected. Thus it is hard to see how proof strengthens "infallibility" in any way.

The use of proof in the classroom has also been called into question on the grounds that it would encourage the idea that mathematics is an *a priori* science. The supporters of this claim appear to see a conflict between this idea and their view that mathematics is "socially constructed" [Ernest, 1991]. Their use of the term *a priori* is not clear to me, but I suspect that what they reject is not the idea that mathematics is *a priori* in the sense of being analytic, non-empirical. One presumes that what they have in mind is *a priori* in the sense of given, pre-existing, waiting to be discovered, a view of mathematics which of course they might well see as standing in opposition to "socially constructed."

On this point I would agree with Kitcher [1984], however, when he says that the pursuit of proof and rigour does not carry with it a commitment to looking at mathematics as a body of *a priori* knowledge. Nor do I believe that the value of proof in mathematics education hinges upon a resolution of this ongoing philosophical debate. As Kitcher put it: "To demand rigor in mathematics is to ask for a set of reasonings which stands in a particular relation to the set of reasonings which are currently accepted" [page 213]. Whether the set of reasonings currently accepted is regarded as given *a priori* or as socially constructed has no bearing on the value of proof in the classroom.

Those who challenge the use of proof in general would challenge even more strongly the use of rigorous proof in particular. Yet rigour is a question of degree, and in mathematical practice the level of rigour is often a rather pragmatic choice. Kitcher explains that it is quite rational to accept unrigorous reasoning when it has proven its worth in solving problems (as has been the case in physics). He adds that mathematicians begin to worry about defects in rigour only when they "come to appreciate that their current understanding ... is so inadequate that it prevents them from tackling the urgent research problems that they face" [p. 217].

Mathematics educators could profitably ask themselves the question Kitcher asked about mathematicians: When is it rational to replace less rigorous with more rigorous reasoning? Kitcher's answer is: "when the benefits it [rigorization] brings in terms of enhancing understanding outweigh the costs involved in sacrificing problem-solving ability." A more rigorous mathematical argument may sometimes be more enlightening. It is the teacher who must judge when more careful proving might be expected to promote the elusive but most important classroom goal of understanding.

In sum, I find myself out of sympathy with those who have challenged the use of proof in the classroom as being an expression of authoritarianism and infallibility. There are no grounds for the belief that proof is in conflict with present-day social values or with the reality of mathematics as a human enterprise open to error. Nor does a significant role for proof in the classroom require mathematics educators to embrace a specific, *a priori* view of mathematics.

The true function of proof in the classroom

In trying to define the true function of proof in mathematics education it is helpful to look at the function of proof in mathematics itself. Its main role is clearly that of justification and verification. Mathematicians are not inclined to accept new results without seeing a proof, even though other types of demonstration may play a supplementary role in convincing them.

But, as we have seen, mathematicians expect more of proof than justification. As Manin has said [1977], they would also like a proof to make them wiser. This means that the best proof is one which also helps mathematicians understand the meaning of the theorem being proved: to see not only *that* it is true, but also *why* it is true. Of course such a proof is also more convincing and more likely to lead to further discoveries. There may also be other valuable benefits: A proof may demonstrate the need for better definitions or yield a useful algorithm; it may even make a contribution to the systematization of results, to the communication of results, or to the formalization of mathematical knowledge.

But not all these functions of proof are relevant to learning mathematics, so they should not be given the same weight in instruction [de Villiers, 1990; Hersh, 1993]. While in mathematical practice the main function of proof is justification and verification, its main function in mathematics education is surely that of explanation. (For this very reason proof should not be undertaken in the classroom as a ritual, aimed vaguely at reflecting mathematical practice, but rather as a meaningful instructional activity.)

To say that a proof should be explanatory is not to say it cannot take different forms. It might be a calculation, a visual demonstration, a guided discussion observing proper rules of argumentation, a preformal proof, an informal proof, or even a proof that conforms to strict norms of rigour, all depending on the grade level and the context of instruction. What is common to all levels and contexts, however, is that the students are learning mathematics that is new to them but consists of known results [Hanna & Jahnke, 1993]. They know *that* the results are true. Clearly the challenge is to have them understand *why* they are true.

Prerequisites for the successful use of proof

To meet this challenge one must make sure the students understand the concepts used, one must structure and present the proof in such a way that is clear and convincing, and one must equip the students with the tools that will allow them to understand a proof.

In *Proof as a source of truth*, Michael D. Resnik [1992] asks how it is that a proof induces us to believe its conclusion, and says that this happens because in the course of our mathematical education we have been prepared "to understand its component statements and to follow its reasoning. We have been prepared to evaluate its inferences as well as data upon which it is based" [p. 30].

Resnik states that a proof addresses psychological, logical, and cultural challenges It succeeds in the psychological task because it is clearly stated and uses a notation that is understood by the reader. It is logical because it presents the ideas in an order which follows certain accepted rules, and so makes clear how the argument leads to the conclusion Finally, it fits the cultural context because it is aimed at an audience that has the required level of experience, understands the language and has been taught to follow a mathematical argument.

A proof would not succeed with students who had never learned to follow an argument, however Resnik says we believe a proof in part because we have been prepared through our mathematics education to follow its reasoning. Here the role of the teacher is crucial. In addition to concepts specific to the mathematical topic at hand, the teacher has to make the students familiar with patterns of argumentation and with terms such as assumption, conjecture, example, counterexample, refutation and generalisation. Only as the students learn modes of logical thinking will they acquire the ability and the confidence needed to evaluate and to construct a proof.

Thurston [1994] points out that "people familiar with ways of doing things ... recognize various patterns of statements or formulas as idioms or circumlocution for certain concepts or mental images . . " [p 167]. The problem, as he says, is that " .. to people not already familiar with what's going on the same patterns are not very illuminating; they are often misleading. The language is not alive except to those who use it " Thus mathematics educators risk confusion if they assume that their students are conversant even with the tools of argumentation in the most general sense, much less with specifically mathematical ones

Proofs that prove and proofs that explain

A proof that we propose to use in the classroom must be well structured, and almost any proof could presumably be restructured to make it more teachable. Yet proofs do differ greatly in their inherent explanatory power In previous papers [Hanna, 1989; 1990] I pointed out that it is useful to distinguish between proofs that prove and proofs that explain. A proof that proves shows that a theorem is true A proof that explains does that as well, but the evidence which it presents derives from the phenomenon itself [Steiner, 1978] This distinction has been expressed in different ways by many others, and goes back at least to the 18th-century mathematician Clairaut [Barbin, 1988]

As an illustration, let us look at different ways of proving that the sum of the first n positive integers, $S(n)$, is equal to $n(n + 1)/2$ As we know, this theorem can easily be proved by mathematical induction. Such a proof has little explanatory value, however. It demonstrates that the theorem is true, but gives the student no inkling of *why* it is true.

A proof that explains, on the other hand, could show why the theorem is true by basing itself upon the symmetry of two representations of that sum, as follows:

$$\begin{aligned} S &= 1 + 2 + \dots + n \\ S &= n + (n-1) + \dots + 1 \\ 2S &= (n+1) + (n+1) + \dots + (n+1) \\ &= n(n+1) \\ S &= n(n+1)/2 \end{aligned}$$

Other explanatory proofs of this theorem could be based upon the geometric representation of the first n integers by an isosceles right triangle or by a staircase-shaped area.

We have seen that even the most experienced mathematicians prefer a proof that explains. For teachers it is all the more important to take the time to search out those proofs which best promote understanding. Such a proof is much more likely to yield not only "knowledge *that*," but also "knowledge why." Unfortunately there is no guarantee that every theorem we might like to use will have a proof that explains But let us reserve proofs by mathematical induction, or other proofs which are non-explanatory, for those limited situations in which we simply cannot find a proof that "makes us wiser."

A good proof, however, must not only be correct and explanatory, it must also take into account, especially in its level of detail, the classroom context and the experience of the students. To achieve the key goal of understanding it may be necessary to emphasize some points at the expense of others, and it may even be appropriate to leave some out entirely where that can be done without loss of integrity Related ideas have been put forward convincingly by Blum and Kirsch [1991] in their plea for "doing mathematics on a preformal level," as well as by Wittmann and Müller [1988] in supporting the concept of the *inhaltlich-anschaulicher Beweis*, in which the demonstration makes use of the meaning of the terms employed rather than relying on abstract methods.

A summing-up

I have identified some challenges to the status of proof in mathematics and presented some assessments that have even predicted the death of proof. But I hope to have shown that proof continues to thrive, sometimes in unfamiliar but fascinating forms I have also looked at some developments in mathematics education that seem to challenge the position of proof as a necessary and respected classroom activity. I hope to have shown that these challenges are unfounded, and that giving proof the attention it deserves in the curriculum is not in conflict with any of our shared educational or social values.

In my initial research on proof I found myself criticizing the excessive emphasis on proof in the curriculum and arguing that it was misguided. Little did I know then that the time would come when I would need to deplore the neglect of proof, but this is precisely what I am doing now. This paper is in large part a plea for recognizing the rightful place of proof in the mathematics curriculum as a key tool for the promotion of understanding

In reply to the question "Do we need proof in school mathematics?" Schoenfeld [1994] gives an unequivocal reply: "Absolutely. Need I say more? Absolutely." I could not agree more.

Note

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References

Alibert, D. [1988]. Towards new customs in the classroom. *For the Learning of Mathematics*. 8(2), 31-35, 43

Anapolitanos, D. A. [1989]. Proofs and refutations: A reassessment. In K. Gavroglu, Y. Goudaroulis, P. Nicolacopoulos (Eds.) *Imre Lakatos and theories of scientific change*, 337-345. Dordrecht: Kluwer

Babai, L. [1994]. Probably true theorems, cry wolf? *Notices of the American Mathematical Society*, 41(5), 453-454

Balacheff, N. [1988]. *Une étude des processus de preuve en mathématiques chez les élèves de collège*. Thèse d'État. Grenoble: IMAG, LSD2, Université Joseph Fourier, Grenoble

Barbin, E. [1988]. La démonstration mathématique: significations épistemologiques et questions didactiques. *Bulletin de l'A P M E P* no 366, 591-619

Blum, M. [1986]. How to prove a theorem so no one else can claim it. *Proceedings of the International Congress of Mathematicians*, 1444-1451

Blum, W., & Kirsch, A. [1991]. Preformal proving: Examples and reflections. *Educational Studies in Mathematics*. 22(2), 183-204

Cipra, B. [1993]. New computer insights from "transparent" proofs. *What's Happening in the Mathematical Sciences*, 1, 7-12

Clairaut ([1986)] *Elements de géometrie* (Originally published in 1765) Republished Laval: Siloë

Cobb, P. [1988]. The tension between theories of learning and instruction in mathematics education. *Educational Psychologist*, 23, 87-104

Cobb, P. [1994]. Constructivism in mathematics and science education. *Educational Researcher*, 23(7), 4

Confrey, J. [1994]. A theory of intellectual development. *For the Learning of Mathematics*, 14(3), 2-8

Davis, P. J. [1993]. Visual theorems. *Educational Studies in Mathematics*. 24(4), 333-344

Davis, P. J. & Hersh, J. [1981]. *The mathematical experience*. Boston: Houghton Mifflin

Davis, P. J. & Hersh, J. [1986]. *Descartes dream*. New York: HBJ Publishers

De Villiers, M. [1990]. The role and function of proof in mathematics. *Pythagoras*, 24, 17-24

Dossey, J. [1992]. The nature of mathematics: Its role and its influence. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning*. 39-48. New York: Macmillan

Ernest, P. [1991]. *The philosophy of mathematics education*. London: Falmer

Goldwasser, S., Micali, S. & Rackoff, C. [1985]. The knowledge complexity of interactive proof-systems. *Proceedings of the 17th ACM Symposium on Theory of Computing*, 291-304

Hacking, I. [1979]. Imre Lakatos' philosophy of science. *British Journal for the Philosophy of Science*, 30, 390-39

Hanna, G. [1983]. *Rigorous proof in mathematics education*. Toronto: OISE Press

Hanna, G. [1989]. Proofs that prove and proofs that explain. In G. Vergnaud, J. Rogalski, & M. Artigue (Eds.), *Proceedings of the International Group for the Psychology of Mathematics Education*. Vol II, 45-51, Paris

Hanna, G. [1990]. Some pedagogical aspects of proof. *Interchange*, 21(1), 6-13

Hanna, G. & Jahnke, H. N. [1993]. Proofs and applications. *Educational Studies in Mathematics*. 24(4), 421-437

Hersh, R. [1993]. Proving is convincing and explaining. *Educational Studies in Mathematics*, 24(4), 389-399

Horgan, J. [1993]. The death of proof. *Scientific American* 269(4), 93-103

Jaffe, A. & Quinn, F. [1993]. "Theoretical mathematics": Towards a cultural synthesis of mathematics and theoretical physics. *Bulletin of the American Mathematical Society*, 29(1), 1-13.

Kieren, T. & Steffe, L. [1994]. Radical constructivism and mathematics education. *Journal for Research in Mathematics Education*, 25(6), 711-733

Kitcher, P. [1984]. *The nature of mathematical knowledge*. New York: Oxford University Press

Kline, M. [1980]. *The loss of certainty*. Oxford: Oxford University Press

Koehler, M. & Grouws, D. [1992]. Mathematics teaching practices and their effects. In D. Grows, (Ed.), *Handbook of research on mathematics teaching and learning*, 115-126. New York: Macmillan

Krantz, S. G. [1994]. The immortality of proof. *Notices of the American Mathematical Society*, 41(1), 10-13.

Lakatos, I. [1976]. *Proofs and refutations*. Cambridge: Cambridge University Press

Lakatos, I. [1978]. *Mathematics, science and epistemology (Philosophical papers vol 2)*. J. Worrall & G. Currie, (Eds.) Cambridge: Cambridge University Press

Lampert, M., Rittenhouse, P. & Crumbaugh, C. [1994]. From personal disagreement to mathematical discourse in the fifth grade. Paper presented at the annual meeting of the American Educational Research Association, New Orleans

Leron, U. [1983]. Structuring mathematical proofs. *American Mathematical Monthly*, 90(3), 174-185

Manin, Yu. [1977]. *A course in mathematical logic*. New York: Springer-Verlag

Movshovitz-Hadar, N. [1988]. Stimulating presentations of theorems followed by responsive proofs. *For the Learning of Mathematics*, 8(2), 12-19

National Council of Teachers of Mathematics, Commission on Standards for School Mathematics [1989]. *Curriculum and evaluation standards for school mathematics*. Reston, VA: Author

Nickson, M. [1994]. The culture of the mathematics classroom: An unknown quantity. In S. Lerman (Ed.), *Cultural perspectives on the mathematics classroom*. 7-36. Dordrecht: Kluwer

Pirie, S. [1988]. Understanding: Instrumental, relational, intuition, constructed, formalised ...? How can we know? *For the Learning of Mathematics*, 8(3), 2-6

Polya, G. [1973]. *How to solve it. A new aspect of mathematical method*. Princeton, NJ: Princeton University Press

Resnik, M. D. [1992]. Proof as a source of truth. In M. Detlefsen (Ed.), *Proof and knowledge in mathematics*, 6-32. London: Routledge

Schoenfeld, A. [1994]. What do we know about curricula? *Journal of Mathematical behavior*. 13(1), 55-80

Silver, E. A. (Ed.) [1985]. *Teaching and learning mathematical problem solving: Multiple research perspectives*. Hillsdale, NJ: Lawrence Erlbaum Associates

Steiner, M. [1978]. Mathematical explanation. *Philosophical Studies*. 34, 135-151

Steiner, M. [1983)]. The philosophy of Imre Lakatos. *The Journal of Philosophy*, LXXX, 9, 502-521

Thurston, W. P. [1994]. On proof and progress in mathematics. *Bulletin of the American Mathematical Society*, 30(2), 161-177

Tymoczko, T. [1986]. The four-color problem and its philosophical significance. In T. Tymoczko (Ed.), *New directions in the philosophy of mathematics*, 243-266. Boston: Birkhauser

Von Glasersfeld, E. [1983]. Learning as a constructive activity. In J. C. Bergeron & N. Herscovics (Eds.) *Proceeedings of the North American chapter of the International Group for the Psychology of Mathematics Education*, 1, 41-69. Montreal: PME-NA

Wittmann, E. C. & Müller, G. [1990]. When is a proof a proof? *Bull Soc Math Belg*. 1, 15-40

Yackel, E. & Cobb, P. [1994]. The development of young children's understanding of mathematical argumentation. Paper presented at the annual meeting of the American Educational Research Association, New Orleans

Editorial note

The article by William Thurston, referenced and cited by Gila Hanna, was reprinted in FLM 15(1): 29-37

To take what there *is*, and use it, without waiting forever in vain for the preconceived—to dig deep into the actual and get something out of *that*—this doubtless is the right way to live.

Henry James