On Topological Properties of Functions

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It is known in the math education research community that understanding the concept of function is not trivial for many students, whether they are in high school or are college undergraduates [Briedenbach *et al.*, 1992; Cuoco, 1993; Goldenberg, 1988; Leinhardt *et al.*, 1990; Piaget *et al.*, 1977; Sfard, 1992]. This paper focuses on the understanding of the concept of function as well, presenting a discussion on topological properties of functions It contains an analysis of the mathematical concepts involved and of students' responses to questions dealing with topological properties More specifically, in this paper I discuss students' mathematical thinking when they are asked to determine whether a property of a function is a topological property or not

Section 1 presents a discussion about topological properties and introduces the term a *super-property*, which refers to a property of properties of some entity Section 2 describes the data sources for the cognitive discussion described in Section 3 The cognitive discussion is based on an analysis of the data according to three levels of abstraction Briefly these three abstraction levels are related to three mathematical objects that students deal with (a set, a function, and a property-of-a-function) when they are asked to decide whether a property of a function is topological or not. The conclusion contains suggestions for further research

1. Topological properties and super-properties

Both sets and functions can be characterized by topological properties A property of a *set* X is called a *topological property* "if it is equivalent to a property whose definition uses only the notion of open set[1] of X and the standard concepts of set theory (element, subset, complement, union, intersection, finite, infinite, etc.) " [Chinn and Steenrod, 1966, p 53] For a property of a *function* to be topological, it should be equivalent to a property "whose definition uses only the notions: open sets of X and Y [the domain and the range], images and inverse images, and the standard concepts of set theory " [Chinn and Steenrod, 1966, p 57] Thus, determining whether a certain property-of-a-set or a certain property-of-a-function are topological properties or not, is a mathematical activity based on the analysis of the property

To illustrate, let's compare the term *a topological property of a function* with the term *a characteristic property of a person* In this illustration, the "person" is the analogue of the "function" Just as we describe some properties of a person as *characteristic* properties, we can describe some properties of a function as *topological* properties Let's call "a property of a property" a *super-property* Using this terminology, the adjective *characteristic* is a super-property of a person (because it describes properties of a person, like "sensitive"), and the adjective *topological* is a super-property of a function (because it describes properties of a function) Another super-property of a person can be, for example, *external* (describing a person's properties, like "tall", "has blue eyes", etc), while an additional super-property of a function can be, for example, *describing the domain* And thus, the properties "the domain of the function is two open intervals" and "the function has two points of discontinuity" are describing-the-domain properties of a function, while "the range of the function is the set $\{-4, 3\ 8\}$" is not a describing-the-domain property

2. Research background

The data in this paper were collected during a topology course taken by 18 undergraduate students who were studying for their mathematics teaching certificate The course was based on activities, discussions, and intuitive explanations, before formal proofs were introduced Because of time limitations, only subsets of R^n were under discussion during the course In relation to subsets of R^n, the following concepts were discussed: open and closed sets, set properties (e g , boundary, compactness, connectedness), and homeomorphism By the end of the course, topological properties of sets and topological properties of functions were under discussion[2] At this stage, the following two homework assignments were given to the students [Chinn and Steenrod, 1966, p 60]:

1 Determine which of the following properties [of sets] are topological and which are not For each property given that is not topological, find two topologically equivalent sets[3], one with the property and one without

a X is unbounded

b X is a finite set

c X is a curve of length 2

d X is convex polygon

2 Which of the following properties of a function $f: X \to Y$ are topological?

a The image of each open set of X is an open set of Y

b f is a similarity

c f is a translation

d The inverse image of each point is a finite set

e The inverse image of each point is a compact set

f The inverse image of Y is bounded

g The inverse image of each point is a connected set

Solving these assignments required the students to deal with properties of unknown entities — sets and functions It turned out that the students did not have any difficulties while solving Question 1, which deals with properties of *sets* However, they faced many difficulties when trying to solve Question 2, which deals with properties of *functions*. In other words, they succeeded at the task when it involved an arbitrary, not specific *set*, but they did not succeed at the task when they had to base their analysis on an arbitrary *function*

A class discussion took place after the students had worked on these assignments. During the discussion the students reflected on the difficulties they had had. At the end of the discussion, as a result of the issues and questions raised by the students, they were asked to answer the following questionnaire:

1. How is it possible to determine that a given property of a function is a topological property?
2. How is it possible to determine that a given property of a function is not a topological property?
3. Is the property "being a homeomorphism" a topological property of a function?

In the next section, students' answers to the homework assignments and to the questionnaire are analyzed. In addition, responses from a conversation with three of the students are presented.

3. Analysis according to three levels of abstraction

As mentioned above, students dealt successfully with properties of sets in general and with topological super-properties of sets in particular. Yet, they encountered difficulties in working with properties of functions and in discussing topological super-properties in relation to functions. From the students' responses (both verbal and written), it turns out that these difficulties can be analyzed by exploring three types of arguments students used in response to the above questions. Indeed, the various arguments can be grouped according to three levels of abstraction relative to three mathematical concepts: a set, a function, and a property-of-a-function. At the first level of abstraction, the students based their discussion on the sets involved in the function, i.e., its range and domain. At the second level of abstraction, their discussion was based on the function itself, and at the third level it was based on the function's property in question.

In what follows I'll illustrate these kinds of arguments. Then, connections between abstraction levels and the three mathematical concepts — a set, a function and a property-of-a-function — will be discussed.

Level 1. Discussion focuses on sets: The domain and the range

At this level of abstraction, students discussed the sets involved in a function. They referred to the range and the domain of a function, to the relationships between these sets, and how the function maps one set to the other. The following example demonstrates this way of thinking. It is Sharon's answer to Question 2: "How is it possible to determine that a given property of a function is not a topological property?"

> Sharon:
> Let's select a one-to-one continuous function f: $X \to Y$ and a function f^{-1}: $y \to X$, also continuous (a homeomorphism), which possesses only the not-topological property. We'll calculate $f(X)=Y$
> If X and Y are equal and topologically equivalent, then we will say that the property is topological.
> If X and Y are topologically equivalent but not equal, then we will say that the property is not topological, because something "moved", and such a thing is not topological.

At the beginning, Sharon selects a homeomorphism (which actually ensures that the ests are topologically equivalent) in such a way that it possesses the *not-topological property*. So, she has already decided that *the property is not topological*. This contradicts the continuation of her answer, in which she distinguishes between two situations: she concludes that the property is or is not a topological property according to whether the range and the domain are two (equal or two different) sets that are topologically equivalent.

Another question raised by Sharon's answer is: How can she be sure that the function has *only* the non-topological property? And if she can, as was mentioned above, what is the purpose of her analysis if she *already* knows that it is not a topological property?

This way of thinking (i.e., the analysis of the sets involved in the function instead of the analysis of the function property under discussion) was also dominant during conversations with three of the students who were asked to reflect on the class discussion and the difficulties which had been expressed by the students. In the following quotation, Tammy says explicitly that she looks at the sets when analyzing topological properties of functions. She explains it by saying that, when she discusses functions she sees the domain and the range.

> Tammy:
> The problem is when you come to discuss it deeply. When you talk about a property of a function, you first imagine the domain and the range of the function [...], and when you start to discuss the domain and the range the problem appears, because it is impossible to compare a property of a function with a property of sets. And when we talk only about the property of the function, whether it is topological or not, we use only the definition of this property and not how the function maps. I mean, how it maps and which sets it maps, and here is the source of the problem. I [...], for example, [when I saw] a similarity function, I immediately said: wait a minute, a similarity function maps one set to a second set in this way.

Level 2. Discussion focuses on a function

At the second level of abstraction, when students are asked about *properties of functions*, the subject of their answers is the notion of *function*. Here are two examples:

> Nancy:
> *Answering the question: Given a function f: X → Y, is the property "f is a similarity" a topological property?*
> "f is a similarity": that means it is possible to pass from X to Y by some stretching (which preserves angles but changes lengths), and hence it is a topological function (preserves topological properties).

Nancy does not speak about the property of "being a similarity function", but rather about a similarity function. She also assigns the adjective "topological" to a function. A similar phenomenon connected with the function's property being "a homeomorphism" is demonstrated by Gill.

Gill:

Answering the question: Is the property "being a homeomorphism" a topological property?

No, because it is possible that a function is a homeomorphism but has properties that are not topological

Like Nancy, Gill does not speak about the property of "being a homeomorphism", but about a *function* which is a homeomorphism

Level 3. Discussion focuses on properties of functions

At this level of abstraction, students deal with properties of functions as they are asked to do. They focus on the *topological* super-properties of functions and discuss whether a property of a function is a topological property or not.

Here are Dana's answers to the following two questions:

1. How is it possible to determine that a given property of a function is a topological property?
2. How is it possible to determine that a given property of a function is not a topological property?

In addition to her answers to what she is asked, she adds a reflective comment which compares the mathematical activities presented in the above questions

Dana:

1. If it is possible to define the function's property only in terms of open sets in the domain and range, images and inverse images, and concepts from set theory, then it is possible to say that the property of the function is topological.

2. If it is possible to show that a definition of the property of the function, which uses only the terms mentioned in 1, does not exist, then it is possible to say that the property of the function is not topological

It is more difficult to prove that a given property of a function is not topological because usually it is impossible to be sure if a better definition, which uses only the "permitted" terms, exists

Connections between the three mathematical concepts in question and levels of abstraction

In this subsection I will discuss two issues:

a) How the three concepts of a set, a function, and a property-of-a-function can be viewed as representatives of three different levels of abstraction; and
b) How the phenomenon described in this paper can be viewed as an illustration of a more general mathematical way of thinking.

At the first abstraction level, the concept of set is discussed. The concept of set can be considered a less abstract mathematical entity than the concept of function, according to the mental constructions that one must create in the process of understanding these concepts. Actually, the concept of set is one component of the concept of function. Hence, when one manipulates the concept of *function* beyond simple numerical computations, he or she has to treat the concept of *set* as an object — "[...] being capable of referring to it as a real thing — a static structure, existing somewhere in space and time." [Sfard, 1991, p. 4] Next, property-of-a-function is a more abstract concept than that of function. This can be observed by examining the term *property-of-a-function*. When one talks about a property-of-a-function, or declares that a function has a particular property, s/he has now to conceive of the concept of function as an object. Otherwise, what "thing" would own the property?

It is not surprising that several abstraction levels are involved in the discussion. It is known that the history of mathematics is reflected in learners' thinking. In other words, since earlier mathematicians were faced with barriers during the development of the mathematics, it is reasonable to expect learners to experience similar difficulties. Consider for example, what Bourbaki [1966] says about the previous mathematical knowledge that was required in order to deal with topological spaces:

> Before a general theory of topological spaces [...] could be developed, it was necessary that the theory of real numbers, of sets of numbers, of sets of points on a line, in a plane and in space should be more systematically investigated than they had been in Riemann's [who is considered as the creator of the topology] time [p. 164]

Thus, it is natural that students tend to deal with the concept of set and the concept of function before dealing with a more advanced mathematical concept, like property-of-a-function

The student's thinking at different levels of abstraction can be explained from the constructivist point of view [Kilpatrick, 1987; Sinclair, 1987; Davis *et al.*, 1990; Confrey, 1990]. In the process of problem solving, one focuses and hangs on to concepts for which one has already constructed a mental structure and hence can make some sense of. Thus, it is reasonable to assume that students who focus on the sets involved in the function (at the first level), have made sense of the concept of set and hence use this concept when answering the questions. At this time, they may not have a mental structure for the concept of function with which they can respond to the questions. A similar explanation may clarify the students' tendency (at level 2) to focus on a function rather than on the property under discussion (at level 3). While they do have a mental structure for the concept of function, they do not yet have a mental structure for the concept of property-of-a-function that they can think with.

As was found, students replaced a discussion focused on properties of a function with a discussion focused on a function or on the sets involved. This can be seen as an unconscious action of reducing abstraction [Hazzan and Leron, in preparation], that is,

> [S]tudents' show a tendency to work on a lower abstraction level than the one on which the mathematical concepts were presented to them and which is expected by the instructor. [...] When students lack the mental apparatus for making sense at the level expected by the instructor, they marshal all their resources to make sense in any way then can. This involves the wide-ranging phenomena that we have termed "reducing abstraction". [...] Obviously, this point of view cannot be taken by the student herself — students mostly are not aware of the existence of a higher abstraction level that the one on which they are working [...] In many cases reducing abstraction [...] helps the students to make sense

of the situation [L]ooking from the students' perspective, this is the most sensible thing to do: If you can't deal with the course concepts on the expected level, you deal with them on whatever level you feel capable of. Unfortunately, this often leads to answers that from a purely mathematical point of view are classified as errors

The source of the need to reduce abstraction can be explained by the following students' responses Some students from the topology class, in addition to answering questions, tried to express their understanding of the mathematics involved Here are two responses to Question 3 (Is the property "being a homeomorphism" a topological property of a function?) which demonstrate this idea:

Tammy:
I think that it is *a topological property)*, but I cannot present a proof or an explanation

Iris:
"Being a homeomorphism" — A topological property which is defined (the function itself) by concepts from set theory. This is very unclear as a definition, and it is difficult to see it intuitively

4. Conclusion

The questions presented to the students in this study revealed their understanding of the concepts: property-of-a-function and super-property Giving students the opportunity to deal with properties and super-properties of mathematical concepts may help them think about the global common ideas behind various mathematical topics, and treat mathematical concepts as real things In order to better understand the contribution that dealing with properties and super-properties (of math concepts) can make to our mathematical thinking, I would like to suggest three directions in continuation of the discussion:

(a) *Understanding other super-properties of functions* "Topological" is one super-property of a function In future research it will be possible to discuss other super-properties of functions

(b) *The fourth level of abstraction* This level follows the three abstraction levels described in Section 4 At this level, understanding the properties of super-properties is discussed

(c) *Understanding the properties and super-properties of other mathematical concepts* In this direction, we could focus on, for example, understanding the properties and super-properties of the concept of a group

Notes

1 A is an *open set* in R^n if, for each point x in A, there is a number $r > 0$ such that A contains all the points in R^n whose distance from x is smaller than r The students during the course discussed only subsets of R^n

2 I distinguish between two kinds of properties of (mathematical) concepts The first are *essential* properties which appear in the definition of the (mathematical) concept For a certain object to be recognized as a particular concept, it must satisfy these essential properties. The second kind of properties are properties that not all the instances of a particular mathematical concept must satisfy. These properties provide more information and describe more accurately certain incidentals of the concept In this paper, when I use the term property-of-a-set and property-of-a-function I refer to the second kind — properties which not all the sets/functions necessarily hold

3 The following concepts relate to the determination of whether two sets are topologically equivalent:

Homeomorphism: Let X be a subset of R^n and Y be a subset of R^m A function $f: X \to Y$ is called a *homeomorphism* if it is one-to-one, continuous, and $f^{-1}: Y \to X$ is also continuous

X and Y are called *topologically equivalent* (or *homeomorphic*) if there is a homeomorphism between them

A different way than the one presented in Section 1) for determining whether a property P is a topological property of a set is the following "A property P of a topological space X is a *topological property* if whenever Y is a topological space such that $Y \cong X$ [Y and X are homeomorphic] then Y also has property P." [Conover, 1975, p 52]

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