Images of Mathematics Outside the Community of Mathematicians: Evidence and Explanations*

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Some preliminary observations about the image of mathematics among mathematicians

Mathematicians from time to time discuss the nature of mathematics and the mind-set that characterizes research in their field. I believe we can say that this has happened particularly in the modern age, after mathematics had established itself as an autonomous discipline distinct from physics, philosophy, etc., and that the crisis in foundations at the end of the nineteenth century as well as the renewed interest in the philosophy of mathematics stimulated this kind of discussion.

An echo of this "existential" research among mathematicians may frequently be found in the pages of *The Mathematical Intelligencer*. On this theme see, for instance, Borel [1983] and Dehn [1983] that, even if somewhat dated in parts, offer good ideas for reflection. Borel foreshadows, among other things, one of the central themes of the discussion, i.e the duality between mathematics as a self-contained discipline with its own inner valency and beauty—the "aesthetics of mathematical thought" of Dreyfus and Eisenberg [1986]—and mathematics as an instrument with applications to other disciplines.

A historical analysis of the development of mathematical theories seems to prove that in the past this duality was serenely accepted as a part of the nature of mathematics and its relationship to the other sciences—mathematics as the "queen and servant of science" as Bell [1951] puts it—while now we tend to think that it forces us to act in a somewhat schizophrenic manner. It may seem that accepting one or other of these two aspects—the applicable or the theoretical—involves a peremptory rejection of the other, even a contempt for it. This duality is encountered again in the discussion on educational problems relevant to the teaching of mathematics; as we shall see, it is on this duality, exaggerated and trivialized (arid reckoning versus rigorous reasoning that, moreover, is an end in itself), that the image of mathematics outside the mathematical community is built

Doubtless such philosophical positions as neo-idealism (on which generations of teachers of mathematics were moulded), which attributed to science only a practical and non-cognitive value, encouraged the schizophrenia induced by this duality. We should add to this that the development of theories of the philosophy of mathematics, such as neoempiricism and fallibilism, further deepened the crisis of the image of mathematics by increasing the loss of certainty. Researchers in mathematics education are becoming sensitive to the links between these problems in the philosophy of mathematics and the problems of teaching/learning mathematics, and they are trying to study the relationships between different conceptions of mathematics and educational problems. For a more in-depth view of these points see Hanna [1983] and Lerman [1989].

The image of mathematics among professional mathematics is tortuous and controversial; it should not surprise us, therefore, that for mathematics teachers, deciding what image to transmit to their pupils is a source of doubt.

Linked to the discussion on the nature of mathematics is the discussion about the figure of the mathematician and about what creativity—and therefore research—means in mathematics. Certain classical works, such as Hadamard [1945] and Newman [1956], illustrate this theme. Moreover such writings as Halmos [1968], [1981], [1985] and Hardy [1969] are good material for study.

In fact I would say that, generally speaking, the most suggestive and exhaustive sources on this theme are the biographies of great mathematicians. As I have already had occasion to say in Furlinghetti [1988], they offer exceptional suggestions, not only about the nature of the various branches of mathematics and the discoveries that belong to them, but also about what it means to be a mathematician or, more broadly, a research scientist. My experience of research in mathematics education has shown me that, as far as the students are concerned, biography (a good biography) better unveils the mysterious (to the students) nature of the discipline than targeted explanations of the subject. In order to lead the discussion towards my goal, as we shall see later on, I would like to remark here that, of course, in a multi-media society such as ours, the transmission of biography and, more generally, any knowledge, may occur through media other than books. For instance, in 1990 when I was in Leicester (UK) to attend the conference HIMED 90 organized by the British Society for the History of Mathematics (see Fauvel [1991]), I saw the Swedish film *The hill on the dark side of the moon* (1983), on the life of the Russian mathematician Sofy Kovalevskaya, effectively presented by Ivor Grattan-Guinness, the well known historian of mathematics. A good screenplay, acceptably faithful to historical reality, and a good director were excellent starting-points towards understanding something about mathematical creativity and the figure of the professional mathematician. The idea of the cinema as a medium for relating the biographies of mathe-

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maticians is not new. In the past, the Italian national television network has broadcast biographies of Descartes and Pascal filmed by the famous Italian director Roberto Rossellini.

Mathematics and mathematicians for the non-mathematician: images and stereotypes

Mathematics is a discipline that enjoys a peculiar property: it may be loved or hated, understood or misunderstood, but everybody has some mental image of it. It is one of the fundamental bases of the school formation of an individual and constitutes the main unifying educational element across different cultures: it has its own intrinsic quality due to the fact that, unlike other school subjects such as literature, history, etc., it presents some features that are more or less independent of the cultural context and of territorial factors. Ethnomathematical studies along the lines of Fasheh [1982] and D'Ambrosio [1985], even when they point to anthropological and historical differences in the developments of the various mathematical cultures, confirm a substantial pervasiveness of mathematics throughout all civilizations. Since the means used for transmitting mathematics are the natural language together with icons, symbols, figures, the differences between the various languages affect this transmission less than they do in the case of other subjects. This feature emphasizes the universal character of mathematics

This intrinsic character on the one hand confirms my statements about the fact that mathematics is one of the fundamentals in the formation of the individual, while on the other hand leads us to think how much the (virtual or actual) learning of mathematics is a school experience that marks a person from a cultural point of view.

I believe that the image of mathematics (and the consequent image of mathematicians that is conditioned by it) that is present in anyone who has had some schooling is subject to stereotyping (a term that I use here without giving it a negative connotation) in a form that I shall briefly analyze below. For this purpose I think it is profitable to quote certain literary passages that may lead us to some of these stereotypes.

Let us begin with the English poet Samuel Taylor Coleridge (1772-1834), who at the age of seventeen writes from Christ's Hospital to his brother (the quotation is from page 5 of Cundy, H.M. and Rollett, A.P.: 1951, *Mathematical models*, Clarendon University Press, Oxford):

> I have often been surprised that Mathematics, the quintessence of truth, should have found admirers so few and so languid. Frequent consideration and minute scrutiny have at length unravelled the cause; viz that though Reason is feasted, Imagination is starved; whilst Reason is luxuriating in its proper Paradise, Imagination is wearily travelling on a dreary desert. To assist Reason by the stimulus of Imagination is the design of the following production (The work to which he alludes is a problem of Euclid's expressed in verse.)

In this passage Coleridge gives a poetical view (as perhaps we should expect from a poet) of a stereotypical image of mathematics outside the mathematical community: mathematics is, by definition, truth but, unfortunately, it is also an arid subject, based on reasoning devoid of fantasy. Those who have worked in mathematical research know how badly this image matches the real nature of the activity. On the contrary, I think that if comparisons could be made between the fantasy involved in all creative activities, not only the sciences but also the arts and technologies, we would have to conclude that music and mathematics are in the lead since the results obtained in these creative activities are an intrinsic product of the spirit, i.e. they are not the discovery of something already present in nature (e.g. physics), or something born, with all the intermediate transfiguration due to the artist, out of nature (e.g. the figurative arts), or having an aim driven by functionality (e.g. as with technology and architecture). *We are the poets*, James Sylvester reminded Leopold Kronecker—a view shared by Karl Weierstrass.

A more positive opinion was expressed by the Italian writer Umberto Eco [1932], who in *Il nome della rosa* (The name of the rose) writes:

> Only in the mathematical sciences, as Averroës said, are the things known to us and those that are known in an absolute manner identified [...] Mathematical knowledge consists of propositions constructed by our intellects so as to operate always as true, either because they are innate, or because mathematics was invented before the other sciences. And the mathematical library [in which the events take place] was designed by a human mind that thought in a mathematical way, because without mathematics you cannot build a labyrinth

The stereotype to which we may make recourse is of mathematics as an activity of perfect reasoning, and therefore as a synonym for truth and certainty. This stereotype may be found embodied in many idioms and clichés in the common language. The above passage highlights the importance of modelling the problem to be solved as a means of arriving at the solution

The idea of modelling, or rather of mathematics as one of the keys to understanding physical reality, was very well stated in a passage by the German writer Heinrich Böll (1917-1985), who, in *Gruppenbild mit Dame* (Group portrait with lady) says:

> [...] on the other hand Leni was very unlucky with two closely related subjects: religion and arithmetic (latter, mathematics). If just one of her teachers had had the idea of making little Leni, when she was six, understand that the starry sky, that she loved so much, offered possibilities of physical and mathematical approaches, she would not have resisted the abacus or the multiplication table, for which she felt the same horror others feel for spiders

This passage, in a more semantically correct way than the young Coleridge used, introduces a poetical flavour to mathematical discourse; I like it because in a concise but pregnant way it illustrates certain educational possibilities and a correct teaching approach to the learning of mathematics.

Böll is not the only author to show an unconscious sensitivity towards the problems of learning mathematics. The Italian poet Giacomo Leopardi (1798-1837) in his *Zibal-*

done (Notebook) writes a few pages about mathematics, in particular a dissertation about ordinal and cardinal numbers that is worth reading (as are all his writings). In it the poet hypothesizes about the possibility of introducing the cardinals before the ordinals, on the basis of hypotheses about their use by primitive men. We should remember that the passage was written many decades before the logico-mathematical discussions on numbers and the foundations of arithmetic.

Of course, for a good teaching approach to mathematics offered by men of letters we must mention the works of Lewis Carroll, which are not only pervaded by mathematics but are also rich in teaching cues (especially in regard to the introduction of logic), as I observed in Furinghetti [1988]. On the other hand, the Reverend Carroll (Charles Lutwidge Dodgson, 1832-1898) was an educator and a teacher of mathematics so this side of his literary production is certainly not surprising.

In literature the idea that school experience conditions the image of mathematics is expressed very effectively by other authors. The Austrian Robert Musil (1880-1942), in his famous novel *Der Mann ohne Eigenschaften* (The man without qualities) devotes some pages to a discussion of the nature of mathematics and its image in society. He was trained as an engineer and has a knowledge of the field (another engineer who was a great writer is the Italian Carlo Emilio Gadda (1893-1973)). The following passage by Musil is from volume 1:

> There is really no need to say too much about this subject as everybody now realizes very well that mathematics has entered like a demon into all the applications of life. Perhaps not everybody believes the story that you can sell your soul to the devil, but those who must be experts on the soul, because as priests, historians, and artists they make a good living from the business, bear witness to the fact that the soul was ruined by mathematics and that mathematics is the origin of the perfidious reason that makes man master of the world and the slave of machines. The inner sterility, the monstrous mixture of rigour in minutiae and indifference to the whole, the desolate loneliness of man in a tangle of details, his anxiety, his wickedness, the fearful aridity of his heart, the thirst for money, the coldness and violence, that mark our times, are, according to these views, only and simply consequences of the damage that logical and rigorous reasoning causes to the soul! And so even then, when Ulrich became a mathematician, there were people who predicted the collapse of European culture because man harboured in his heart neither faith nor love, neither innocence nor kindness; and it is significant to remark that all these men, when they were boys and schoolchildren, were bad at mathematics. From this they later held it as proven that mathematics, mother of the exact sciences, grandmother of technology, was also the matrix of that spirit that later produced poison gas and bomber planes.

The image that emerges from other passages in the same work is very complex and tortured: it may refer to the stereotype of mathematics as an arid science, but with the addition of a background of enormous ill-will towards the subject, an ill-will that, I believe, is not widespread in such an exaggerated form (except perhaps among engineers who suffered so much from the propaedeutic teaching of mathematics to which they were subjected in the first two years of college).

Another quotation, this time more serene, from Musil's *Die Verwirrungen des Zöglings Törless* (The confusions of young Törless, from *Young Törless*, translated by Eithne Wilkins & Ernest Kaiser, Pantheon Books, New York, 1955, 111):

> Mathematics is a world in itself and one has to have lived in it for quite a while in order to feel all that essentially pertains to it.

The following passage from Stendhal (Marie-Henri Beyle, 1783-1842) establishes in a very pregnant manner an attention to mathematical education and to school experiences as fundamental for the formation of the individual. No wonder that passages of this book are extensively quoted in [Hefendehl-Hebeker, 1991] since it offers good material for a "clinical analysis" of the feeling of a pupil faced with the difficulties in learning algebra in an authoritarian environment. My quotation is from the English edition of *La vie d'Henri Brulard* (The Life of Henry Brulard, translated by Jean Stewart and B.C.J.G. Knight, Harmondsworth, 1973, 299):

> I loved mathematics all the more because of my increased contempt for my teachers, MM. Dupuy and Chabert. In spite of the grandiloquence and urbanity, the suave and dignified air that M. Dupuy assumed when he spoke to anyone, I had enough shrewdness to guess that he was infinitely more of an ignoramus than M. Chabert. M. Chabert, who in the social hierarchy of the bourgeoisie of Grenoble stood so far below M. Dupuy, sometimes on a Sunday or Thursday morning would take a volume of Euler or... and resolutely tackle difficulties [...]
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> My enthusiasm for mathematics may have had as its principal basis my loathing for hypocrisy, which for me meant my aunt Séraphie, Mme Vignon and their priests.
>
> In my view, hypocrisy was impossible in mathematics and, in my youthful simplicity, I thought it must be so in all the sciences to which, as I had been told, they were applied. What a shock for me to discover that nobody could explain to me how it happened that: minus multiplied by minus equals plus (- × - = +)! (This is one of the fundamental bases for the science known as *algebra*).
>
> Not only did people not explain this difficulty to me (and it is surely explainable, since it leads to truth), but, what was much worse, they explained it on grounds which are evidently far from clear to themselves.
>
> M. Chabert, when I pressed him, grew confused, repeating his *lesson*, that very lesson against which I had raised objections, and eventually seemed to tell me: "But it's the custom; everybody accepts this explanation. Why, Euler and Lagrange, who presumably were as good as you are, accepted it!"

In my opinion all the stereotypes about teachers of mathematics and their (presumed) coldness, on mathematics as intellectual nonsense—and let's not forget the play *La leçon* [1951] (The lesson) by Eugen Ionescu—are here hinted at and explained. Perhaps this passage, as with the other pages on the school experience of the novel's protagonist, should be suggested reading for prospective teachers so as to make them think about their pupils' cultural expectations and how the disappointment of such expectations brands the individual intellectually, especially when he is sensitive.

Let me finish my review on the images of mathematics emerging from literature with a quotation that shows that the serenity and wisdom of a life (here the life of Mahatma Mohandas Karamchand Gandhi, 1869-1948) may be transferred into a perception of the nature of mathematics; in it the image of geometry as "pure deduction" is emphasized. The passage is from the book Gandhi, M.K., *An autobiography or The story of my experiments with truth*, Navajivan publishing house, Ahmedabad, 1991 (first edition 1927) translated from the Gujarati by Mahadev Desai, 14:

> Geometry was a new subject in which I was not particularly strong, and the English medium made it still more difficult for me. [...] When, however, with much effort I reached the thirteenth proposition of Euclid, the utter simplicity of the subject was suddenly revealed to me. A subject which only required a pure and simple use of one's reasoning power could not be difficult. Ever since that time geometry has been both easy and interesting for me.
>
> Samskrit, however, proved a hard task. In geometry there was nothing to memorize [...]

I like to compare the different images that emerge from reading with images coming from another art that, as I mentioned before, I consider very effective and topical, and that I love very much: the cinema. I think that the cinema should be considered, more often than it is, not only as entertainment but as an expression of culture and a vehicle for information about society. For example, in the specific instance of surveying the images of mathematics held by lay people, the cinema offers many unexpected glimpses.

It may be surprising, considering the results about the image of mathematics indicated before, that something about this subject is present—not as an object of the narration, as in the case described by Emmer [1983], or in the many movies produced with didactic purposes, but as a narrative expedient—in a form of communication that is generally considered more suitable for entertainment, Actually, the claim is justified by concrete objective facts. First, as was observed previously, a certain image of mathematics is present in each of us, and the fact that it more or less unconsciously surfaces in certain circumstances is understandable. Moreover, the characteristic of mathematics of being a unifying element across various cultures and of permitting quick and clear narrative hints through the inconography provided by its symbols, plays a fundamental role in a communication form, such as the cinema, that has to make use of transmission devices different from novel-writing or oral narration.

We should moreover add to what has been said already that the cinema shows much interest in the world of school. A director that I will mention again, Peter Weir, recently directed a good film on the world of the school (*Dead Poets' Society*, 1989). The director Ramon Menendez made the film *Stand and Deliver* (1988), from a successful novel, about the real-life experiences of the mathematics teacher, Jaime Escalante. Through his teaching of mathematics, Escalante brings back into society a few young people from one of the poorest districts of Los Angeles, enabling them to be admitted to prestigious American universities. This film aroused much interest in the world of school (see, for instance, the notices in *Mathematics Teaching*, 1989, no. 129, p. 44).

I could quote from many different films. I shall start again with poetry. In *The wizard of Oz* (Victor Flemming, 1939) there is a scene in which the Scarecrow, one of Dorothy's three friends, having received a brain from the wizard, in order to test its effectiveness enunciates a version of the Pythagorean theorem (perhaps without excessive concern for correctness). This is an example of mentioning mathematics as a synonym for correct reasoning.

In two quite different films, united only by a similar reference to mathematics, we find the stereotype of mathematics as extraneousness and abstraction from life, and, metaphorically, as purity. In the scene in *Doctor Zhivago* (David Lean, 1965) in which the protagonist, an adolescent and innocent student, is going to be seduced, the director, in order to give an idea of Lara's unawareness of what is going to happen to her, closes up to the book she is studying: it is a book of Euclidean geometry. The same narrative device is used in *Picnic at Hanging Rock* (Peter Weir, 1975), when the director wants to project an idea of how far the teachers and students are from what is about to happen. In both these instances geometry intervenes as a paradigm of perfection and abstraction, a stereotype present in many situations: in *Il convivio* the great Italian poet Dante says that geometry is unblemished by error and quite sure of itself.

The positive role of mathematics, synonym for truth, integrity, and justice, repeatedly emerges in the film *Torn Curtain* (1966), by Alfred Hitchcock, a director who also studied engineering: for example, "π" is chosen as the symbol of the group of "good" characters that help the protagonist. The film attempts a (rough) explanation of the meaning of this symbol.

Associated with the image of mathematics that we have illustrated previously is the figure of the mathematician as a meek man (until he rebels against vexation and injustice), absent-minded, naive and, generally speaking, biddable, but also endowed with clear (and sometimes strict) ideas about ethics and morals. These characteristics appear in the character of the university professor protagonist of *Straw Dogs* (Sam Peckinpah, 1971) and in the high school mathematics teacher protagonist of *Bianca* (Nanni Moretti, 1985).

Another commonplace view about mathematics (the incompatibility between femininity and mathematics) may be found in the film *A Streetcar named Desire* (Elia Kazan, 1951), from a play by Tennessee Williams (1914-1989). The protagonist, Blanche, in order to make herself respectable, lies about her past, and tells her suitor that she is a teacher, but when asked whether she teaches mathematics, she denies it with disdain as if it were a slur on her femininity.

Among the many negative references that lead us to think about bad school relationships with mathematics, I remember an unusually trivial Woody Allen who in the film, *Radio Days* (1987), in order to let us know how disagreeable a schoolmate is, makes him recite mathematical formulae during a chance meeting in the amusement park.

How and why a certain image of mathematics is formed at school

On the basis of the evidence reported in the previous section we may schematize the images of mathematics held by adults in two streams: those who were autonomously able to elaborate mathematical ideas regret the bad approach they had in school, the others (the majority) harbour feelings of refusal and repulsion towards the discipline. In the examples of the first kind it is possible to observe a substantial harmony between mathematics and the other cultural aspects of the personality; in the other examples the sentiment of refusal may develop into the "terror" felt toward mathematics that is so impressively expressed in the verses of Victor Hugo (1802-1885), translated into English in Adda [1987].

The basic point emerging from the considerations in the previous sections is that the adult's image of mathematics is conditioned (unfortunately, usually in a negative direction) by the school experience of the individual in a more radical way than happens with other subjects. In part, this fact is linked to the almost total lack of obvious external referents that makes the school mathematical experience transferable only with difficulty to other environments. For this reason, as discussed in Howson, Kahane and Pollak [1988], it is difficult to develop any form of popularization of mathematics; this discipline is the Cinderella of the science museums all over the world.

Nevertheless I point out that the development of the ability to find mathematical referents in the real world should be one of the main aims of mathematics education: in this schools often fail because of the way mathematics instruction is organized. In Taylor [1981] the usual way mathematics is learned in school is compared to learning Latin poetry by copying out the works of the minor poets without ever arriving at being able to read (and understand) the poetic masterpieces of Virgil. Why should one enjoy Latin poetry after such an approach? A similar criticism can be made of mathematics teaching. Traditionally it contemplates:

- the introduction of ideas and techniques;
- the application of ideas and techniques to the solution of problems;
- the consolidation of ideas and results through formalization, generalization, and abstraction.

In school practice it happens that problems, ideas, and applications are held back and prominence given to the acquisition of techniques and manipulative skills. While this activity is the less rewarding, in every respect, for teachers and students alike, the attention and the effort of the students concentrate on it because it is usually at this stage that learning is evaluated. Unfortunately, I believe it is the case that this orientation towards formalism and mechanicism as ends in themselves is more and more emphasized as the school level increases. As with Latin poetry, why should one enjoy mathematics after such an approach?

On these foundations, the negative image of mathematics gels and mathematics becomes a discipline extraneous to the student's expectations and understandings. The following description by Jonathan Swift (1667-1745) of what happens in the fantastic lands visited in *Gulliver's Travels* gives an idea how students live the learning of mathematics (the quotation is from the English-Italian edition *I viaggi di Guilliver*, A. Mondadori, Milano, 1982, 398):

> I was at the Mathematical school, where the Master taught his Pupils after a Method scarce imaginable to us in *Europe*. The Proposition and Demonstration were fairly written on a thin Wafer, with Ink composed of a Cephalick Tincture. This the student was to swallow upon a fasting Stomach, and for three Days following eat nothing by Bread and Water. As the wafer digested, the Tincture mounted to his Brain, bearing the Proposition along with it. But the success hath not hitherto been answerable, partly by some Error in the *Quantum* or Composition, and partly by the Perverseness of Lads, to whom this Bolus is so nauseous that they generally steal aside, and discharge it upwards before it can operate, neither have they been yet persuaded to use so long an Abstinence as the Prescription requires.

The idea of the school mathematical experience as alien to the students' way of being originates not only in the choice and the organization of the mathematical topics to be taught, but also in the style of teaching these subjects. Among the factors affecting this style I attach considerable importance to the role of the concept images pre-existing in students' minds when a certain subject is introduced (here I refer to the terminology and the definition in Tall and Vinner [1981]. In this connection let us consider a paradigmatic example which mathematics educators struggle with, the teaching of limit.

This concept encompasses epistemological obstacles as well as psychological difficulties. An explanation of the latter lies in the fact that the student has a personal experience of the use of the term "limit" in contexts outside mathematics having very different semantics: this experience leads to concept images to which, consciously or unconsciously, the student refers, that are different in each student because the perceptions are absorbed and generalized in different ways. In the research discussed in Furinghetti and Paola [1988a] and Furinghetti and Paola [1988b] we have found that the majority of students perceive the *limit* as a "boundary" (that cannot be crossed), because usually in the common language the word *limit* has this meaning; other aspects of this kind of difficulty are discussed in Monaghan [1991]. The importance attached to exterior interferences (in this case, language) is linked to the fact that concept images built on experiences out of school have a longer life that those built in school, and are absorbed more deeply, because they are a "strict production" of the intellectual activity of the students themselves and because "reinforcement" is more frequent outside school than in school. The result of this situation is that the teacher often has to deal at the same time with given theoretical constructs (here, the formal definition of limit) and with the concept images of students which, in contrast to the theoretical constructs, do not allow for control of the process of formation of concepts. In this situation students perceive the concept definition given by the teacher as an authoritarian and extraneous element

What should be left of mathematics when one has forgotten mathematics? Some advice to teachers.

After having drawn attention to the fact that it is necessary for teachers to consider the presence of pre-existing concept images in the minds of the students, we will try to single out what should be left of mathematics when one has forgotten the mathematics. I shall only provide some general ideas, to be discussed and developed according to the age of the students, the cultural milieu, and all the other environmental factors inherent in teaching, in the light of a balance between manipulative skills, abstraction, and application.

I think it a good objective, for a continuation of study at university, training for work, and, of course, for life, that the following elements should remain from a mathematical education:

- The basic techniques with operations. These techniques have to be associated to such concepts as approximation and orders of magnitude that permit forecasting and the evaluation of results
- The concept of an axiomatic system, at least at the semi-intuitive level of a game rule
- The univocal and non-ambiguous use of language (at least the natural language)
- The habit of deductive reasoning, at least in the more elementary meaning of "good reasoning" or, according to the terminology used by Peano [1901], "natural logic"
- The concept of modelling a problem.

The term "concept" is used here to mean the general idea underlying a notion apart from the various techniques associated to it.

All this having been stated beforehand, the teacher, in his turn, should forsake some illusions and incline towards a considerable cultural tolerance. In Davis and Hersh [1988, 103] it is said that "There is no law of nature, God or government that everybody must know the quadratic formula. Mathematics is interesting and important, but so are art, religion, literature, and many other things." The teacher has to reflect on this statement and approach mathematics as a set of human activities that students have to experience in school, as advocated in Plunkett [1981], and not as a body of rigidly defined knowledge. Eventually, the teacher has to accept that it is possible to live happily or unhappily independently of one's relationship to mathematics.

On this basis I believe it is possible to offer students the opportunity of developing what I shall call an "ECOLOGICAL" IMAGE OF MATHEMATICS, meaning by that an image respectful of the peculiarities of this protean discipline

Acknowledgment

I dedicate this paper to my mother, Teresa Arvigo, who much encouraged me when I was a student of mathematics, and to my father Augusto (1913-1982) who, when I had studied long hours, used to take me to the cinema

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School came to bore me. It took up far too much time which I would rather have spent drawing battles and playing with fire. Divinity classes were unspeakably dull, and I felt a downright fear of the mathematics class The teacher pretended that algebra was a perfectly natural affair, to be taken for granted, whereas I didn't even know what numbers really were They were not flowers, not animals, not fossils; they were nothing that could be imagined, mere quantities that resulted from counting To my confusion these quantities were now represented by letters, which signified sounds, so that it became possible to hear them, so to speak. Oddly enough, my classmates could handle these things and found them self-evident. No one could tell me what numbers were, and I was unable even to formulate the question. To my horror I found that no one understood my difficulty. The teacher, I must admit, went to great lengths to explain to me the purpose of this curious operation of translating understandable quantities into sounds. I finally grasped that what was aimed at was a kind of system of abbreviation, with the help of which many quantities could be put in a short formula But this did not interest me in the least. I thought the whole nusiness was entirely arbitrary. Why should numbers be expressed by sounds? One might just as well express *a* by apple tree, *b* by box, and *x* by a question mark *a*. *b*, *c*, *x*, *y*. *z* were not concrete and did not explain to me anything about the essence of numbers, any more than an apple tree did But the thing that exasperated me most of all was the proposition: If $a = b$ and $b = c$, then $a = c$, even though by definition *a* meant something other than *b*, and being different, could therefore not be equated with *b*, let alone with *c*. Whenever it was a question of an equivalence, then it was said that $a = a$, $b = b$, and so on. This I could accept, whereas $a = b$ seemed to me a downright lie or a fraud. I was equally enraged when the teacher stated in the teeth of his own definition of parallel lines that they met at infinity. This seemed to me no better than a stupid trick to catch peasants with, and I could not and would not have anything to do with it. My intellectual morality fought against these whimsical inconsistencies, which have forever barred me from understanding mathematics. Right into old age I have had the incorrigible feeling that if, like my schoolmates, I could have accepted without a struggle the proposition that $a = b$, or that sun = moon, dog = cat, then mathematics might have fooled me endlessly — just *how* much I only began to realize at the age of eighty-four. All my life it remained a puzzle to me why it was that I never managed to get my bearings in mathematics when there was no doubt whatever that I could calculate properly. Least of all did I understand my own *moral* doubts concerning mathematics

Equations I could comprehend only by inserting specific numerical values in place of the letters and verifying the meaning of the operation by actual calculation. As we went on in mathematics I was able to get along, more or less, by copying out algebraic formulas whose meaning I did not understand, and by memorizing where a particular combination of numbers, for from time to time the teacher would say, 'Here we put the expression so-and-so', and then he would scribble a few letters on the blackboard. I had no idea where he got them and why he did it — the only reason I could see was that it enabled him to bring the procedure to what he felt was a satisfactory conclusion I was so intimidated by my incomprehension that I did not dare to ask any questions

C G. Jung