Teaching Mathematics via Problem Solving: a Course for Prospective Elementary Teachers [1]

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Jonathan is nearly two years old. The world, through his eyes, is a wonderful place, full of odd quirks and curiosities. His explorations often leave him "stuck," both cognitively and physically. His parents know this from his furrowed brow and barely audible, "Hmmm." When he first started getting stuck, it was not uncommon for him to cry to his parents for help. After many quiet responses of "You can figure it out, Jon," his favorite line has become, "Jon do it."

Children enter school with an almost insatiable curiosity and a determination to do things for themselves. It is as though they can hardly wait to be adults so that they no longer have to suffer the imposition of "Let me help you." Why, then, does it seem that so many of our university undergraduate students no longer share the wonderment of two-year-olds? Too many seem to have lost their excitement when confronted with new and curious situations. This seems especially true in the mathematics classroom where they are frequently unwilling to investigate any problem and its solution without the instructor's direct intervention. The favorite line is no longer "Jon do it," but rather "Teacher do it." It is possible, however, to create a classroom atmosphere that not only expects students to "do it," but actively encourages and supports students' explorations and autonomy.

In this paper we describe a mathematics course for prospective elemantary teachers that has teaching (and learning) mathematics via problem solving at its core. We believe that this view of mathematics instruction helps prospective teachers to begin to understand and appreciate the value of establishing a classroom climate that allows students to take charge of their own learning.

Teaching (and learning) mathematics via problem solving [2]

In the 1989 NCTM Yearbook, Schroeder and Lester wrote about developing mathematical understanding via problem solving. They made a case for the value of *teaching via problem solving*. According to them,

> In teaching *via* problem solving, problems are valued not only as a purpose for learning mathematics but also as a primary means of doing so. The teaching of a mathematical topic begins with a problem situation that embodies key aspects of the topic, and mathematical techniques are developed as reasonable responses to reasonable problems. [1989, p. 33]

> Central to our interest in teaching via problem solving is the belief that the primary reason for school mathematics instruction is to help students understand mathematical concepts, processes, and techniques. [p. 37]

> We believe that instead of making *problem solving* the focus of mathematics instruction, teachers, textbook authors, curriculum developers, and evaluators should make *understanding* their focus and their goal. . Children's learning of mathematics is richest when it is self-generated rather than when it is imposed by a teacher or a textbook.[p. 39]

With these ideas as our guide we have developed at Indiana University a new mathematics content course for preservice elementary teachers. *Mathematics for Elementary Teachers via Problem Solving* is a course in which students are actively involved in developing their own mathematics and where content is considered via problem solving in the spirit proposed by Schroeder and Lester. During each two-hour class period, students are presented with problems to work on in cooperative groups. The teacher's job is to circulate among the groups observing, asking questions, and helping students reflect on their thinking. Then, when most groups have found their solutions, the class shares solution strategies and answers. The teacher's role during this discussion is to help bring possible generalizations to light and draw attention to important points raised.

Thoughout the course, we discuss issues directly related to problem solving and, in particular, to the role of metacognition—monitoring one's progress, evaluating one's solution, assessing one's strengths and weaknesses, and a host of other activities associated with being reflective. The teacher becomes a guide, asking probing questions rather than leading questions. we expect students students to develop an internal monitor that, in the absence of the teacher or others, will act as a skeptic helping them clarify their thinking during problem solving [Hirabayashi & Shigematsu, 1987; Lester, Garofalo, & Kroll, 1989]. Our intent is for students to get to the point where the questions they begin to ask themselves and their fellow students are questions such as: *If I knew _____, how would it help? Is this approach getting us anywhere? Could I have solved it in a better way?*

At the beginning of the semester, students typically ask the teacher questions like, "Would you solve item four for us?" After two weeks or so, questions change to, "This is what I did for item four. Am I right?" All of us would agree that the second type is preferable to the first, but it is still not the kind of question that helps students propose their own conjectures and assess their own progress. Ideally students would ask questions, e.g. *If I knew ______, what could I predict about _______?*) and would begin to answer the questions for themselves. While many may be skeptical about this actually occurring in a mathematics

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classroom full of preservice elementary teachers, we have seen this happen in our mathematics course.

An example of the course in action

About ten weeks (2/3 of the way) into the semester, we typically begin a series of activities related to number theory. By this point in the semester, students have had plenty of exposure to a teacher who will not give answers and who will only ask probing types of questions. They have grown accustomed to questions from the teacher such as: *Why did you follow that plan?, What made you think of that?, Will your conjecture also hold true when....?* Some students are still asking the *Am I right?* question, but they are becoming used to the teacher saying: *Your solution may have merit; tell me more about why you think this. What do the other members of your group think?*

To kick off our consideration of number theory, we begin with the locker problem:

> Students at an elementary school decided to try an experiment. When recess is over, each student will walk into the school one at a time. The first student will open all of the first 100 locker doors. The second student will close all of the locker doors with even numbers. The third student will change all the locker doors with numbers that are multiples of three. (Change means closing lockers that are open and opening lockers that are closed.) The fourth student will change the position of the lockers that are multiples of five, and so on. After 100 students have entered the school, which locker doors will be open? [Adapted from Thompson, 1976, p. 79]

This problem acts as a springboard for discussing prime factorization. More specifically, it prompts consideration of numbers with an odd number of factors. As the class discusses the idea of prime factorization and begins to see patterns in the number of factors any given number would have, students start to form and articulate their own questions.

For example, in one of our classes the students were involved in finding the number of factors in very large numbers. They had begun work by using divisibility tests to list all the factors of certain numbers. When they found that one of the numbers had more than thirty factors and it appeared that the rest of the numbers might have an equally long list of factors, the students became unwilling to "crank and grind" through the problem. As a result, two student-generated questions came to the forefront:

1. If I were given a large number and found the prime factorization of it, could I predict whether it would have an even or an odd number of factors?
2. If I were given a large number and found the prime factorization of it, could I determine how many factors the number would have?

The students formalized their questions just before the usual class break time, and so they had the option of taking a ten-minute break and then returning to work to find an answer. Surprisingly, only a few students left the room, and those who did leave returned within a few minutes. Most students remained actively engaged in the process of finding patterns, looking for generalizations, and then testing their hypotheses. They were animated and noisy, and they were doing mathematics—their own mathematics.

Their attempts to answer the first question led to the following conjectures (Note: all p_ks are distinct prime numbers):

> $(p_1p_2p_3 \cdots p_n)^2$ has an odd number of factors because each p_i^2 has three factors. So, $(p_1p_2p_3 \cdots p_n)^2$ has 3^n factors, which is an odd number.
>
> $p_1^2p_2^2p_3$ has an even number of factors because p_3 will have exactly two factors.
>
> $(p_1p_2 \cdots p_n)^2p_{n+1} \cdots p_m$ has an even number of factors because p_{n+1} to p_m each has an even number of factors.

Whether or not the students' conjectures are correct is irrelevant. The point is that, although some of the students' reasoning was a bit shaky, their conjectures are the sort that mathematicians typically make when they do mathematics.

Exploring various special cases resulted in the following resolution to the second question:

p^2 has 3 factors	$(1, p, p^2)$
$p_1^2p_2p_3$ has 12 factors	$(3 \times 2 \times 2)$
$p_1^2p_2^2$ has 9 factors	(3×3)
$p_1^2p_2^2p_3^2$ has 27 factors	$(3 \times 3 \times 3)$

They said: "We see a pattern. We think we have theorem. We tried this with several numbers and it worked every time."

> <u>*Students' Theorem:*</u> *To find the number of factors a number has multiply the number of factors each individual prime factor has and you have the number of factors the (composite) number has.*

Although they were unable to find a "real" proof, it is clear that they were doing real mathematics. The two original student-generated questions and the conjectures that resulted from them are radically different from *Would you solve item _____?* and *Am I right?* Instead they are the type that drive inquiry. They are the type that indicate that students are developing the mathematical understanding advocated by Schroeder and Lester [1989].

At least three consequences immediately flowed from this experience. First, after class the teacher began to think about the students' theorem. She found the students' theorem intriguing enough to spend some time convincing herself that it is true. She also got two colleagues so interested in the problem that all three of them decided to try to prove it. Students often believe the teacher has (or must have) all the answers. By not having the answer readily available during class, the teacher demonstrated that it is acceptable to be at a loss momentarily. By continuing to formulate an answer, she demonstrated that teachers, too, solve problems and may need to take several days and have the help of others to find answers to their questions. In effect, she demonstrated that: (1) it is acceptable for the teacher not to know immediately, and (2) teachers sometimes engage in real problem solving themselves.

Second, this experience opened the door for a fair amount of discussion in our mathematics education department and considerable discussion in subsequent classes. In each of these settings, we discussed the fundamental principles of teaching via problem solving: a change from viewing teaching as an act of transmitting information to largely passive students to an act of helping students construct deep understanding of mathematical ideas and processes by becoming actively engaged in doing mathematics. We discussed the development of personal autonomy. When students begin to develop "reasonable responses to reasonable problems" [Schroeder & Lester, 1988, p. 33], they realize there are many solution paths, not just the path taken by the teacher, and they develop the personal strength to take the risk of offering their solution to the teacher and others. We also discussed the social norms necessary to encourage student exploration. [For a more complete discussion of the construction of social norms see Cobb, Wood, & Yackel, 1990 or Wood, Cobb, & Yackel, 1992]

Third, students reflected on the experience and wrote in their journals about it. Their comments included the following:

> *Anita*
> The more work that we do with factors and factorization, the more interesting it becomes. I never really was that attentive to many of the patterns and different ways to look at factorization.... I was younger and uninterested much of the time, and until this class I never looked at math in such a way as we do now. It only makes sense that we, as future educators, are trained this way. It's too bad it didn't start sooner.
>
> *Mary*
> I liked today's class. I can't believe we came up with a theorem all on our own. You didn't even point us in a direction. It makes me feel smart (I guess that's the word). We are doing math in its utmost form—originally. Not just computations with formulas that other people give you, real math. Funny, at the beginning of the semester I would have called that stuff (formulas and computations) real math.
>
> *Cameron*
> I must admit that I was not looking forward to coming to class today. It is so funny when I look back to the beginning of class that I was going to be content to just sit and not get involved in Mary's factorization excitement. . It would be so easy to just sit through class and let everyone else do the work, but when the people in my group are so interested and are working so hard to find an answer, it is contagious. Our group had such a good time—we laughed quite a bit at ourselves when we realized that we were way off track. . The only other comment I want to make is that the moment when that formula hit me (and worked with all our example #s) I was so ecstatic.
>
> *Teacher's reaction*
> I am ecstatic!!! Class was wonderful. These students are finally beginning to ask their own questions and are beginning to be able to answer their own questions. They are on the verge of not needing me.
>
> *Conversation with a colleague*
> It sounds like you've gotten them to the place where they can really begin to take learning into their own hands. It's too bad the semester is almost over.

Perhaps the most important result, however, was the lasting impression this experience had on the students. One student wrote on her final exam that she was able to answer the question on prime factorization because "I remember our theorem from class." Another class member, sitting in the teacher's office later in the semester, commented, "That class was awesome!" The students amazed themselves, and that amazement turned to confidence and an increased determination for the remainder of the semester.

No longer did the teacher have to remind them NOT to look in a book for an answer. Now they only rarely consulted the text for confirmation of their results. Nor did she need to act as an external monitor in the same way as earlier in the semester. The students were no longer requesting the same type of help. Instead, they busied themselves finding solutions and developing explanations to justify their thinking for other class members. They also began to view the idea of verification differently, as Cathy's journal entry suggests:

> To most people, verification is supporting an idea, concept, answer, belief, etc. However, to me it is more than that. Verification is support with understanding. I really like verification because it gives me a chance to support and show what I know, how I know it, and how much I understand it. If you verify something in order to do it correctly you have to understand it.

You can do it!

Developing the social norms in the classroom that are useful for promoting independent problem-solving behavior in students and making a firm commitment to adhere to them daily is no simple task. Students will pressure the teacher to "just tell us" and considerable determination is required to stand firm against the onslaught of students' complaints. However, this is not really so different from Jonathan's parents patiently reminding him that he can do it. Perhaps we need to remind ourselves, as educators, that we, too, can do it; we can stand firm and help our students develop the autonomy necessary to no longer rely on us.

As teachers, maintaining the energy level and the commitment to removing ourselves as the focus in the classroom can be difficult because many of us are accustomed to being the authority figure in the room. As Miller [1982] reminds us,

> Letting go—of someone or of some thing, some idea or belief or feeling—is a life task for many of us. We cling to that which is familiar or comfortable, or needed, oftentimes building convoluted rationales for our insistence on enveloping, holding on to that which we believe is love or friend or right situation. . We understand that the letting go allows us new freedom, a chance to soar, to move and change and grow. Yet, how difficult it is to uncurl the fingers, loosen the grip, feel the last hesitant touch of the familiar as it slides from our opened hands. [p. 181]

It is difficult, but it can be done.

Developing the materials with which to teach is yet another difficult task.[3] Most textbooks are filled with skill-and-drill exercises. The classroom we envision is based on learning mathematics through problem solving in a small-group, cooperative setting This requires developing and/or finding rich problems that require students to work together; problems that are not too hard but are complex enough to require cooperative work. It requires developing a curriculum and a teaching style that will encourage students to make mathematical connections and otherwise engage in doing, as Mary wrote, "real" mathematics. Initiating the development of curriculum such as this requires a firm commitment of massive amounts of individual time and energy.

Although our colleague commented that it was unfortunate that the semester is almost over, we suspect that it will never be over for many of these students. Perhaps Lucia, another student in the class, put it best:

> I would just like to say that I have enjoyed this class, even though sometimes I may grumble and whine It has helped me to see things (numbers, problems, reasons for math) in a different light . I feel like I have taken a good first step in possibly—can I say it—*liking* math again! I never have been able to stop thinking about a problem until I get it! Thank you—from me and all of my students to be!

Our hope is that we have planted in prospective teachers the seeds that will be watered by other teachers and, perhaps, by their future students. Our prospective teachers have grown; they've become their own question-askers and their own meaning-makers. They, like Jonathan, are able to say, "We'll do it!"

Notes

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[2] The course described in this section was developed by a team, which included—in addition to the authors—Carol J Fry, Philip Gloor. Diana Lambdin Kroll, Michele LeBLanc, Joanna O. Masingila, Francisco Egger Moellwald, Anne Raymond, and Vânia Santos The development was supported by National Science Foundation grant number TEI 8751478 to the Mathematics Education Development Center, John F LeBlanc, project director. The views expressed here are those of the authors and do not necessarily represent those of the National Science Foundation.

[3] Certainly, the materials developed by our team having a problem-solving spirit are not the only materials available for use with mathematics courses for prospective elementary teachers. However, our materials are not simply problem sets. Rather, they also include, among other things, extensive teacher notes, suggestions for further explorations of problems posed in class, and ideas for group problem-solving assessment. Persons interested in obtaining these materials should contact any *Kinko's* copy center and tell them that the books are available from the Bloomington, Indiana *Kinko's*, 113 N. Dunn St., Bloomington, IN 47401 (phone: 812 339-3773).

References

Cobb, P. Wood, T., & Yackel, E. [1990] Classrooms as learning environments for teachers and researchers In Robert B Davis, Carolyn A Maher, and Nel Noddings [Eds] *Constructivist views on the teaching and learning of mathematics* (pp 125-146) Reston, VA: National Council of Teachers of Mathematics

Hirabayashi, I., & Shigematsu, K [1987] Metacognition: The role of the "inner teacher' In J.C. Bergeron, N. Herscovics, & C. Kieran [Eds.] *Proceedings of the Eleventh Annual Meeting of the International Group for the Psychology of Mathematics Education* (Vol 2., pp 243-249). Montreal, Canada

Lester, F.K., Jr, Garofalo, J., & Kroll, D.L. [1989] *The role of metacognition in mathematical problem solving: a study of two grade seven classes* (Final Report of a project supported by the National Science Foundation Grant MDR 85-50346). Bloomington, IN: Indiana University Mathematics Education Development Center (ERIC Document Reproduction Service No. ED 314-255)

Miller, J.L. [1982] Feminism and curriculum theory part one: the breaking of attachments: feminism and curriculum theory. *Journal of Curriculum Theorizing*, 4, 181-186

Schroeder, T.L., & Lester, F.K., Jr. [1989] Developing understanding in mathematics via problem solving. In P.R. Trafton [Ed.] *New directions for elementary school mathematics*, 1989 Yearbook of the National Council of Teachers of Mathematics (pp 31-42) Reston, VA: NCTM

Thompson, M [1976] *Number theory*. Bloomington. IN: Indiana University Mathematics Education Development Center/Addison-Wesley

Wood, T., Cobb, P., & Yackel, E [1992] Change in teaching mathematics: a case study *American Educational Research Journal*. 28, 587-616

> Mathematical science shows what it is. It is the language of unseen relations between things. But to use and apply that language, we must be able fully to appreciate, to feel, to seize, the unseen, the unconscious Imagination, too shows what it is. The is that is beyond the senses. Hence she is, or should be, especially cultivated by the truly scientific — those who wish to enter the worlds around us.
>
> Ada Lovelace
