# Children's Reactions to Verbal Arithmetical Problems with Missing, Surplus or Contradictory Data

EWA PUCHALSKA, ZBIGNIEW SEMADENI

## I. INTRODUCTION AND BACKGROUND

### 1.1. The aim of the paper

A typical verbal arithmetical problem satisfies the following conditions: there is exactly one number providing the answer to the problem and this number can be computed by using the information supplied in the problem; moreover, no numerical data are superfluous (there may be superfluous non-numerical information, e g. colours or names).

In this paper we consider problems that do not satisfy some of the above conditions, in particular, problems having either *missing data,* or *surplus data,* or *contradictory data.* In what follows, problems of these three types will be called *MSCD problems* (for Missing, Surplus or Contradictory Data Problems). [1]

A verbal problem may be considered either in a *logico-mathematical* setup or in an *applied-mathematics* setup In the former, solvers can use only the information given in the text of the problem; in the latter, they may (or are even supposed to) find some data themselves. For instance, suppose that a problem concerns Mary buying some items in a grocery. In the logico-mathematical setup, both Mary and the shop are typically unknown and there is no way of learning more about the situation, whereas in the applied-mathematics setup, Mary may be a girl from the solvers' class shopping in the nearby grocery and the solvers may go there and check prices, weights, etc Both setups should be the subject of regular teaching (because both kinds of reasoning are very important). However they should not be confused (by teachers or educators) and the children should understand which setup is meant when they are given a task In this paper we restrict our attention to logico-mathematical setups only.

The aim of the present paper is an exploratory investigation of how children react to MSCD problems. The topic has received little attention, particularly in the case of classroom instruction. Our intent is to challenge some conclusions that may be drawn from previously-known results

Problems with missing data, problems with surplus data, and problems with contradictory data, are logically different from each other. Still, they have some common features. First, in order to explain such a problem successfully, the pupil has to read the text meaningfully and critically and not to jump into computations before analysing the arithmetical relations involved. This feature is so significant that it justifies the joint investigation of children's reactions to problems of all three types

### 1.2 Survey of previous research

The most important work on how children deal with MSCD problems was done by Krutetskii [1968; see also 1969] as a part of his broad research on children's mathematical abilities. Krutetskii investigated problems (not necessarily arithmetical) with unstated questions, with incomplete information, or with surplus information One of the conclusions of his study was that able children can formulate a question to a given story and recognize missing or superfluous data, whereas — in a problem of a new type — an average child at first perceives only disconnected facts

Paige and Simon [1965] used problems with impossible data in order to examine high school and college students. These researchers were able to distinguish subjects who were primarily "physical" in their responses, that is, made extensive use of spatial and physical cues, from primarily "verbal" subjects, who relied chiefly on setting an equation for the problem as it was stated. Students attending to both verbal and physical cues were most likely to detect the contradiction.

Equipe Elémentaire de l'IREM de Grenoble [1979] reported results of a study on how children answered a test containing variations on the following absurd problem: "There are 26 sheep and 10 goats on a boat. How old is the captain? What do you think of the problem?" Almost 75% of children aged 7 to 9 and 20% of those aged 9 to 11 performed some arithmetical operations on the given numbers without expressing any doubt.

A 7 year old was given the following problem: "You have 10 red pencils in your left pocket and 10 blue pencils in your right pocket. How old are you?" The child answered: 20 years old", and after the remark "But you know very well that you are not 20 years old!" the child replied: "Yes, but it is your fault; you did not give me the right numbers!"

Decorte and Verschaffel [1985, page 11] gave first graders, among other tasks, the following problem with missing data: "Pete had some applies; he gave 4 apples to Ann; how many apples does Pete have now?" They found that more than half of the pupils did not perceive that the problem was unsolvable.

Radatz [1983, 1984], in his study of what kind of difficulties children have with verbal problems, included (among others) stories with unrelated data of the type: "4 persons in a Volkswagen left home at 9 o'clock and had to drive 200 kilometers", without stating any question. Many children attempt to make computations using the data. Radatz [1984] points out that, during problem solving, school beginners concentrate on stories rather than on numbers; during interviews it has been found that such children often augment the story with what follows from their own knowledge or experience. Older children, however, always try to reach some solution, perhaps by a trial-and-error strategy, and they often believe that nothing is unsolvable in mathematics. Children with little mathematical experience try to analyse the story more carefully, whereas older students have a specific attitude towards mathematics: It is viewed as an activity with artificial rules and without any specific relation to out-of-school reality.

Markovits, Hershkowitz and Bruckheimer [1984] investigated how children solve "pseudoproportionality" problems where there are no arithmetical relations between the data and therefore no conclusion follows, e.g. the height of a 10 year old boy was given and the question concerned his height when he would be 20. Many children gave absurd answers based on proportionality.

Bender [1985] gave children another kind of pseudoproportionality problem: "A postage stamp for a standard letter from Aachen to Munich costs 60 pfennig. The distance from Aachen to Munich is 600 kilometers. The distance from Aachen to Frankfurt is 300 kilometers. How much is the postage for a standard letter from Aachen to Frankfurt?" In contrast with the preceding example this question has a unique correct answer, provided that the solver knows that the cost of mailing an ordinary letter in West Germany is the same for all pairs of cities. However, many children answered: "30 pfennig".

Several persons: Bechtold [1965], Krutetskii [1968], Nesher [1976], Zweng [1979] and others [2] investigated how children solve problems with extraneous data and found that these problems were more difficult and took more time to solve than normal problems. (Some of these authors were concerned with more general questions and merely touched on the topics that interest us here).

The report from the Second National Assessment of the Educational Progress [NCTM, 1981] gives some results of testing how American students solve problems with surplus or missing data. Sample results: Only 47% of 9 year olds and 56% of 13 year olds correctly solved the problem: "One rabbit eats 2 pounds of food each week. There are 52 weeks in a year. How much food will 5 rabbits eat in a week?" In a problem with four prices on a menu, only 39% of 9 year olds responded correctly to a question about the total cost of a meal with three of those items; almost as many students added all four given prices. Sixty per cent of 9 year olds correctly answered a subtraction problem with extraneous information, but only 23% identified the extra information. It was concluded in the report that students were not accustomed to being asked such questions. In a problem with missing information, only 29% of 9 year olds identified what additional information was needed in order to find the total cost of a number of items.

Puchalska and Semadeni [1987] reported on individual interviews with children aged 7 to 12 years. One of the findings was that in the case of problems with missing data, a significant number of children said that the problems were unsolvable (and some of them gave acceptable explanations, too) and yet also believed that a better student or the teacher could solve the problem for sure ("Yes, she can, because she is a professor"). However, in the case of contradictory data, significantly fewer children said that the teacher could solve the problem.

A new dimension to the topic was given by E. Gruszczyk-Kolczynska [1986] who reported on her remedial program for 61 students in grades 1-3 in Katowice (Poland) from 1976 to 1983. In her efforts to overcome the children's emotional blocks she used intentionally ill-formulated problems (e.g. having contradictory data) with success. During conversation with the child she would formulate such a problem and pretend that this was *her* error; her intention was to convince the child that the teacher *can* also make a mistake and to reduce the child's fear of error. If her findings are confirmed on a broader scale, this may change the overwhelming opinion that MSCD problems can be given only to above average students; in fact she dealt with children whose failures were so serious that the school asked a remedial center for help.

**1.3 Survey of previous explanations of children's reactions**

In the preceding section we focused our attention on reports *describing* how children reacted when they had been given MSCD problems. We now compile various pertinent conceptions which either *explain* the children's reactions or may be useful for such an explanation.

We begin with a survey of possible reasons why the responses of children in tests with absurd problems (like the one concerning the age of the captain) were so poor. We note two tendencies: some people attribute the failures to the specific nature of the child's thinking whereas others attribute them to prior instruction (or to both).

As we mentioned in 1.2, Krutetskii [1968] has found that average children at first perceive only disconnected facts in a problem of a new type. This explains their poor performance on complex problems, but not so much the astonishing results to absurd problems as reported by the Grenoble IREM [1979].

Freudenthal [1982], commenting on the results from Grenoble, points out a possible lack of understanding of the context involved. Children usually know what to say when they are asked "How old are you?"; they count candles on a birthday cake, ask a cuckoo-bird how many years one will live, and so on, but they may still not have a clear idea what the age of a person is. This explanation, however, may not be valid in case of problems where the context is well known to the child.

Of particular significance is Freudenthal's conjecture of a possible magical context. Children (and some adults) may attribute a special secret meaning to numbers. Perhaps there is a secret way to find the age of the captain by counting his sheep and goats? (Children may also believe that the teacher knows the secret even if they cannot solve the problem.)

Decorte and Verschaffel [1985] have found that for a large number of school beginners, the crucial difficulty in solving word problems lies in the construction of an appropriate initial representation of the problem situation (rather than in the selection of the right arithmetic operation or in performing it). These authors present the concept of *word problem scheme* (WPS). It is a general mental scheme which involves the solver's knowledge of the role and the intent of verbal problems, of their typical structures, and the knowledge of a number of implicit rules, suppositions, and agreements inherent in the *word problem game,* needed in order to compensate for insufficiencies of the verbal text and to interpret it correctly. Lack of such knowledge results in errors or difficulties for the child.

Let us now gather together arguments referring to various possible drawbacks of instruction. In informal discussions the school is often blamed for students' failures. If a verbal problem — instead of being viewed as a description of a situation requiring a meaningful analysis — is viewed as a set of words, only one of which is the key to the selection of the proper operation, then absurd answers are quite likely. Textbooks are also blamed. IREM Grenoble [1979] gave samples of textbook problems which, indeed, could distort the child's conception of a verbal problem.

Dr. Edyta Gruszczyk-Kolczynska has informed us how she interprets children's responses to absurd problems of the kind administered, e.g., in the Grenoble experiment (the interpretation is based on her earlier experience with children). She believes that the child does notice that something is odd about the problem but does not trust his or her understanding of the statement. Rather than conceiving that something may be wrong with the problem itself, the child blames himself or herself and thinks that he or she perceives something wrong. In fact, for many children, there have been dozens of problems in the past which at first appeared strange, but the teacher's authority confirmed that they were unquestionable and that it was the child's fault not to grasp the right idea. Now the child prefers not to ask the teacher because it is generally known that better scores can be achieved by covering up any lack of understanding and not disclosing one's doubts. Thus whenever the child does not feel comfortable with a problem, he or she looks for any clues and tries to guess what to do.

Further potential factors related to the instruction will be considered in the following two sections.

### 1.4 The role of the social contract

Kilpatrick [1982, 1985] points out that research in the "social-anthropological" tradition (in particular, that of H. Mehan) has alerted educators to the social contract negotiated in classrooms (the terms "hidden curriculum" or "agenda" are also used in this context). "(...) the problem is given and received in a transaction. The mathematics classroom is a social situation jointly constructed by the participants, in which teacher and students interpret each other's actions and intentions in the light of their own agendas." [Kilpatrick, 1985, page 3]

> Students who are assigned a problem in class can bring to bear on the task a whole set of considerations that would not be operative if the students had formulated and posed the problems on their own: For example, they can usually assume that the problem will have a single, well-defined answer obtainable using procedures they have studied recently in class; they can assume that they will be evaluated by the teacher according to how hard they appear to try in solving the problem and how successful they appear to be; they may have some indication that the teacher knows the solution to the problem, and they may be able to get clues to the solution by asking the teacher certain questions; and they may be able to use gestural and postural cues from the teacher to tell whether or not they are on the right track in their solution (...) When someone sets a problem for us, we use all sorts of cues to assess whether that person thinks we can solve the problem, whether he or she is withholding vital information from us, whether a problem has a straightforward solution or involves some kind of trick, and whether it would be worthwhile to exert our maximum effort to come up with a solution. [Kilpatrick, 1982, pages 4-5]

> Researchers such as Carpenter and Lester (...) have noted that many children approach problems in an impulsive way, attending primarily to surface features of the problem statement in order to decide what action to take. The child's goal is to do something — anything. Much school instruction, rather than encouraging children to take a problem seriously and reflect on what the problem statement says, seems to reinforce their impulsivity. Because the children see the problem as a school task rather than as an intellectual challenge that is worth accepting, they grab at answers so as to escape from the task as fast as possible. [Kilpatrick, 1985, page 12]

The above descriptions fit very well the case of MSCD problems and may serve as an explanation of poor results on some tests.

### 1.5 Controversy about exposing the child to error

We now briefly recall the long-standing fear (widespread in the traditional teaching) of exposing the child to any error whatsoever. This fear is based on the empirical theory of reflection [see Aebli, 1951, chapter 1] and on psychologically formulated stimulus-response bonds [see Resnick and Ford, 1981, chapter 2]. On the other hand, it is clear that if the child is to learn mathematics by performing series of carefully devised similar tasks and imitating the correct way shown by the teacher, then any method appearing to involve possible confusion of concepts must be renounced. Consequently, in traditional instruction, problems with missing or contradictory data are out of the question.

This prevailing point of view has sometimes been challenged. Of particular interest are the following excerpts from Brownell [1942, pages 427-440]:

> The objection to so-called "absurd" problems is based upon *a priori* grounds Furthermore this objection fails to take cognizance of the necessity of teaching children to detect absurdities by exposing them to instances of absurdity as one part of the task of developing skills in problem solving. [Page 421]

> Maier accounted for errors in problem solving as the result of the inflexibility of the reasoner's set: The habitual response has the right of way and blocks out other possible reactions. [Page 427]

> Part of real expertness in problem solving is the ability to differentiate between the reasonable and the absurd, the logical and the illogical. Instead of being "protected" from error, the child should many times be exposed to error and be encouraged to detect and to demonstrate what was wrong, and why [Page 440; cf. also Suydam, 1980, page 46]

In some countries there have been attempts to include certain MSCD problems in regular classroom teaching.

In Poland, the official curriculum encourages teachers of grades 1-3 to give children occasional MSCD problems. This is in accordance with one of the goals of the curriculum. Mathematics lessons should contribute to the development of the child's critical thinking. A few such problems were scattered in the textbook by Puchalska and Ryger [1972] However, there has been no research concerning the effects of such problems on pupils' performance The idea goes back to L. Jelenska's textbook for elementary teachers published 60 years ago

> When we formulate problems, it is advisable that we occasionally pose them so as to make children realize that one has to find arithmetical relations between the givens If we present a problem of the form "Sophie is 5 years old, how old is her brother?", then the child has to note that the problem cannot be solved because no relation between the age of Sophie and the age of her brother is given. Then, posing problems with many irrelevant data (. . .) we habituate pupils to paying attention to the relations between the data. [Jelenska, 1926, page 201]

> It is also advisable (. ) to have redundant data. For instance, "A copy-book costs 60 zlotys [3] Hedwig paid 120 zlotys for two copy-books. How much should she pay for one copy-book?" Let the children amend the problem. [Jelenska, fourth edition of the previously quoted book, 1969, page 117]

Menchinskaya and Moro [1965, chapter IV, section 4] report that in a school the children were given *joke problems* of the following type: "Borya has 3 apples and Vera has 5 apples How many apples does their grandmother have?" After the children added the numbers as their response, their mistake was explained. They reacted very emotionally to the mistake and subsequently paid more attention to the formulation of the question.

Caldwell [1980, page 402] devised interesting examples of problems whose objective was:

a) Recognizing the absence of information necessary to the solution of problem and pointing out what is missing, or
b) Recognizing the presence of extraneous mathematical information in a word problem and eliminating it from the problem. (No work with children was reported.)

Kilpatrick and Radatz in their survey [1983] commented on findings by Radatz [1984] concerning problems with insufficient data:

> Teachers of mathematics might pose such insoluble problems to their pupils as a means of investigating how deeply they have internalized the view that school problems represent a special form of reality that has no relation to the real world problems. Otherwise many pupils get the feeling: In mathematics all tasks are soluble, even problems of the type "How old is the captain?"

Radatz [1984] concluded his study with a hint that one could possibly influence pupils' attitude by giving them special problems intended not for immediate computation but rather for verification of data, finding superfluous data, looking for all possible questions, and formulating problems by children themselves.

Working on another topic, Swan [1983] described an experiment with two styles of teaching decimal place value: The *conflict* teaching approach was intended to involve the pupils in discussion and reflexion of their own misconceptions and errors There was a "destructive" phase, in which old ideas were shown to be insufficient and inaccurate before new concepts and methods were introduced. The second teaching style, the *positive only* approach, made no attempt to examine errors and avoided them wherever possible. Swan concludes that a "conflict" approach may lead to a deeper conceptual understanding. [4] Markovits et al [1984] followed this line of thought in the context of problem solving: "We might be able to use the deliberate conflict situation, not only to make students more aware of the necessity of checking the reasonableness of the results, but also to remove methodological misconceptions."

## 2. EXAMINATION OF CHILDREN'S REACTIONS AND ATTITUDES

### 2.1 Organisation of the investigation

The aim of this exploratory investigation was to give children MSCD problems during regular lessons in grades 1 and 2 and to look for patterns in their behaviour as well as for particularly interesting individual responses We had in mind the following questions:

A. What kind of difficulties do children manifest when they are to deal with an MSCD problem?
B Do MSCD problems cause bewilderment, confusion of concepts, frustration or other negative effects on children?

C. Do children look for the meaning of the story? How do they react to absurd problems?

D Do children's responses improve after they have received some instruction on MSCD problems? What should be the role of pupils' attempts to amend the problem (under the teacher's guidance)?

E. Do children change their opinions about the solvability of a problem after a discussion or after an attempt to solve the problem?

We now report on experimental lessons organized in various schools in and around Warsaw in May and June 1985. At this time of the year typical Polish pupils in grade 1 are 7 or 8 years old. Nine teachers volunteered to give a lesson to their classes (6 teachers in grade 1 and 3 teachers in grade 2), selecting MSCD problems from a list prepared beforehand for this purpose. In Poland, normally, the teachers either do not give MSCD problems to children at all or do this only occasionally, several days apart from each other (in accordance with the curriculum recommendations). However, in the cases described in this paper, a whole 45-minute period was devoted to MSCD problems.

Problems were presented orally by the teacher (in grade 2 the pupils could also read them). Children raised their hands and the teacher called on them successively. In five classes (four at grade level 1 and one at level 2) the lessons were observed by one of the authors (sitting at the rear of the classroom, taking notes but not interfering with the lesson). A sampling of problems and responses (typical or particularly interesting) from those 5 classes is presented below. According to what other teachers reported, the reactions of their pupils were similar.

### 2.2 Description of children's reactions

The findings are arranged in the order corresponding to questions A and E. It should be noted, however, that the questions overlap and the findings — though labelled A, B, . . . — cannot be completely separated.

*QUESTION A*

In the case of most of the problems, there were pupils who did not grasp the structure of the problem and gave wrong responses. Some of them appeared to misunderstand what a verbal problem is and what is meant by its solution. Most errors resulted from performing operations on unrelated or irrelevant data. Samples:

*Problem 1*

> *Gapcio gave Dolly the following problem: "There were sparrows on a tree. I saw 5 sparrows and Dick saw 6 sparrows. How many sparrows were there on the tree?" What should Dolly say to Gapcio?*

In each of the classes observed, many children answered to the effect "There were 11 sparrows." Some pupils seemed troubled by the fact that Gapcio saw fewer sparrows than Dick and tried to explain it: "Perhaps one sparrow flew away", "Or was hidden", "Maybe Gapcio overlooked a sparrow". Few pupils noted that Gapcio and Dick could see the same sparrows and the question could not be answered.

*Problem 2*

> *Mary invited 5 girls and 3 boys to her birthday party. How old is she?*

In each class, there were pupils who gave the answer "eight years", sometimes with an explanation, e.g. "As there were 8 guests, she had just turned 8". Nevertheless, several pupils in each class appeared to recogize that the question was irrelevant and made various comments, e.g. "One has to know how many candles were lit on the cake".

*Problem 3*

> *Each day Olga puts money in her piggy-bank and keeps a record of how much she has there. On Monday she had 3 zlotys in her piggy-bank. On Tuesday she had 4 zlotys there. On Wednesday she had 8 zlotys in the piggy-bank. How much money did she collect?*

Many pupils said "fifteen" (conceivably they just added the three numbers), but there were also many answers of "eight". Some pupils tried to explain the meaning of the problem, e.g. "One does not have to make computations, it is just said!", "Why count? After all, Olga said how much she had."

It should be noted that none of these problems puzzled *all* the pupils in a class. There was always a number of children presenting reasonable explanations; some of them argued emotionally with their peers.

*Problem 4*

> *A farmer had 12 pigs. He went to the market and sold 4 hens. How many pigs did he have left?*

In each class several pupils answered "eight", but many others appeared to notice that the data were unrelated. A sample protest against the answer "eight": "Why, he sold hens, not pigs!"

Many times it occurred that a problem with surplus data was given and a pupil said: "That problem cannot be solved." Perhaps the child meant "I cannot solve it" or simply took his or her cue from colleagues who had discussed different problems.

*QUESTION B*

The first MSCD problem (with missing data) caused bewilderment and confused responses in all the classes observed. However, after the teacher had explained how to interpret this problem and the following ones, the perplexity gradually disappeared. In fact, pupils were more active and more lively than usual. They frequently laughed (particularly after absurd answers from their peers) and spoke spontaneously without being called on.

In the middle of the lesson some teachers asked whether pupils wanted more such problems read; the overwhelming reaction was "yes" ("We like this", "These are such funny problems", "We do not have to write, only to think, we prefer this"). On the following days, in some classes children urged the teacher to give them further problems of this kind.

*QUESTION C*

Some of the pupils whose answers were wrong appeared to employ superficial verbal cues in the text (e.g., in problems

2, 3, 4, above). However, many other pupils treated the problems meaningfully. Especially interesting were the frequent endeavours to rationalize absurd or impossible stories, either by reinterpreting the story or by accounting for a possible origin of the nonsense, e.g. "Maybe it's a slip", "Gapcio misused a word", "Dolly pretended to be clever, but she wasn't", "Gapcio forgot to say (. . .)" as well as "It's a stupid problem". Samples of reactions to specific problems:

*Problem 5*

> *Anna is 7 years old and Bob is 10. How much older is Anna?*

Responses: "Older??", "Three years older", "Zero, for she is younger", "Why, how can a smaller number be greater?", "Gapcio meant younger, not older" Pupils suggested how to amend the problem: "One should ask: "How much younger is Anna." Or: "Anna is 11 and Bob is 10. How much older is Anna?"

*Problem 6*

> *At the market, yesterday an egg cost 15 zlotys. Today an egg costs 14 zlotys. What will be the price of an egg tomorrow?*

Responses: "We can't know" "We can! Because yesterday it cost 15, today 14, so tomorrow it will cost 13. Because the price goes down". "We can't say how much the price goes down; it may go up as well"

*QUESTION D*

The help of the teachers was significant in two ways: First and foremost, they created a suitable atmosphere of free discussion; children were in the mood for expressing their thoughts. Second, they explained how to interpret a problem with missing data and problems of other types. Some teachers introduced (in grade 2) the terms "not enough givens" and "too many givens" Such explanations were indispensable to the children to grasp what it was all about.

The observations showed that pupils' attempts to amend problems with missing or contradictory data were of great importance for their learning. When the teacher continued to give further problems, pupils' responses gradually improved

*Problem 7*

> *Johnny and Mike are sitting in a classroom. There are girls standing at the blackboard. Johnny sees 3 girls and Mike sees 3 girls. How many girls are standing at the blackboard?*

Both the answer "three" and the answer "six" were frequent. To convince pupils that the answer "three" was reasonable, a teacher simulated the situation in the story. She asked 3 girls to come to the blackboard, pointed to a boy sitting down and said: "You are supposed to be Johnny", and to another boy "You pretend to be Mike" Then she asked the question: "How many girls does Johnny see?" After the answer: "three", she asked again: "And how many girls does Mike see?" "Three". "All right, so is it true that there are 6 girls at the blackboard?" "No, three girls. Mike sees them from another side."

When some children had troubles with problem 2 (quoted above), the teachers tried to help them. The traditional argument was used: "If we add girls to boys, shall we get years?" Another teacher was more successful when she asked a boy: "You Luke, you had a birthday some days ago. How many guests did you invite?" When the boy answered "three", she said: "So you are 3 years old?" The loud laughter of peers helped to identify the nonsense.

*QUESTION E*

The observation of the thinking of individual children was difficult; nevertheless, the behaviour of the pupils indicated that several of them became convinced by the arguments of the teacher and colleagues and changed their opinions.

A spontaneous change of childrens' responses was conspicuous in a class where the following sequence of MSCD problems was followed.

*Problem 8a*

> *Mike has a bicycle. Joan has a bicycle. Tom has a bicycle. How many bicycles do they have?*

*Problem 8b*

> *Mike wrote a letter to his uncle. Joan wrote a letter to her uncle. Tom wrote a letter to his uncle. How many uncles got letters?*

*Problem 8c*

> *Mike attends a school. Joan attends a school. Tom attends a school. How many schools do they attend?*

The aim of these problems was to surprise the children and provoke them to change their mind. Therefore the problems were presented in the following order: 8a, 8b, 8c and 8d again. As we had anticipated, problems 8a and 8b were both answered "three" quickly and without hesitation. When problem 8c had been read, the responses varied: first "one" and then "two" and "three", with some explanations. Children appeared to imagine concrete situations and base their answers on that. When the teacher came back to problem 8b, the children's responses were different: They pointed out that there could be three different uncles, or one uncle of all three children, or two uncles, that a letter might be lost in the mail, and so on

## 3. DISCUSSION

In 1.5 we quoted Brownell's opinion: "The objection to so-called "absurd" problems is based upon *a priori* grounds". Forty-five years later we find this statement still valid. There is no sound evidence against problems with missing or contradictory data as a means of instruction; there are many hints that such problems may be valuable. The results reported in the papers quoted above show that the majority of children give unsatisfactory, sometimes appalling, answers to MSCD problems. But in all cases known to us the investigation concerned children who had not been taught how to interpret such problems, so the results cannot serve as a basis of refutation. If the child is surprised by a drastic change of the way of posing a verbal problem and is perplexed by the new situation, how can we expect the answer to be a sound inference from the problem statement? Schoolchildren solve numerous problems meaningless to them, but are not told what to do when a problem given to them is ill-posed. No wonder they are bewildered and may react nonsensically.

The overall implication of our exploratory study is that MSCD problems, under suitable conditions, can be used as a means to develop the child's habit of reading the text of a problem meaningfully and critically; they may also contribute to better understanding of what a verbal problem is all about. "Suitable conditions" mean, first of all, that children have the opportunity to discuss the problems and express doubts.

The influence of peers during the lessons in Warsaw was significant. We believe that if the same problems were given to the same children individually, without the possibility of communication with other children, the results would probably decline. Moreover, in some cases the logical explanations of the teacher appeared less convincing that a burst of laughter or remonstrance from classmates.

However, our study shows that MSCD problems can be effective only if several of them are given to pupils consecutively. Isolated MSCD problems cause bewilderment and should not be used till the pupils are accustomed to such problems. Proper explanation of the initial MSCD problems is very important, but still more important is that the children should be encouraged to express their opinions and discuss the problems.

The pupils' conception of what is meant when the teacher tells them to solve a verbal problem should be considered within the framework of the social contract between them and the teacher. The nonsensical asnswers reported in IREM de Grenoble [1979], Radatz [1984] and others, appear to result from the childrens' belief that they were supposed to follow classroom routines.

Perhaps the most important single reason why students give illogical answers to problems with irrelevant questions or irrelevant data is that those students believe mathematics does not make any sense. Several authors have pointed out that students often see no connection between the symbol manipulation taught in school and the things that interest them. They do not want to analyse problems themselves and demand that they should be given "the rules of the day" or step-by-step instructions to follow [see e.g. Whitney, 1985].

A school beginner has troubles with verbal problems, does not understand their nature and the concepts involved. An objective of instruction should be to provide the child with certain mental schemes which are indispensable in problem-solving. However, these schemes may become too rigid and block the child's thinking in new situations. Thus, we should take care of both: to help the child develop certain schemes and then to make them flexible.

Using the terminology of Decorte and Verschaffel [1985] we may conjecture why so many school beginners do not give right answers to apparently very simple problems. These children interpret verbal problems in a specific, concrete way and often have troubles with sentences which are phrased differently from those they have experienced outside the school. Lack of WPS (Word Problem Schemes), a knowledge component intervening in the construction of a proper representation of the problem story, is an impediment to solving problems successfully. However, in our opinion, poor answers to problems of the type "How old is the captain?" [IREM de Grenoble, 1979] should be attributed to a distorted WPS rather than to the lack of such. It would be particularly helpful to learn more about how the child's thinking develops from the preschool lack of WPS to the distorted WPS of later on.

Let us comment on the response "But it was your fault; you did not give me the right numbers" of a 7 year old in Grenoble, quoted in section 1.2. We may laugh when we read this passage and think of the child's naiveté. But the case is not so simple. Suppose that the problem was posed differently: "You have 10 red pencils in your left pocket and 10 blue pencils in your right pocket. How many pencils do you have?" It would be wrong for a child to count the pencils that actually happened to be in the pockets. He or she is supposed to use the numbers given in the problem statement and not the numbers which happen to be true. And we do not expect the interviewer to say: "But you know very well that you do not have 20 pencils!", because in such a case the child's understanding of the social contract would be correct: when you solve a mathematical problem (in logico-mathematical setup), you use the numbers given in the story, you are not supposed to check what you have in your pockets. Consequently, when the question is "How old are you?", the child may understand that it is not a question about his or her true age, but a part of a "word problem game" where special rules have to be observed and one's true age has nothing to do with the mathematical reasoning. [5]

According to a well-known principle of psychological contrast, in order to help children understand a concept we should present them with contrasting examples. Thus to help children appreciate a well-formulated verbal problem we should confront them with problems having some obvious defects, in particular, with problems having insufficient data or contradictory data. This should not come only as a surprise during a test, but should be a carefully introduced new concept. The children should be aware of the change of the social contract: rather than performing some operations on numbers, the child has first to think whether the problem is solvable at all; the task is to recognize any deficiency and say how the problem could be amended.

A logical analysis of an MSCD problem (particularly of a complex one) can be expected from a formal-operational child (in Piaget's sense) only. We believe, however, that very easy cases can be meaningfully dealt with by children who began to learn problem solving not long ago. We expect that pupils will translate the problem into a real-life setup and will try to find an answer in the context of this setup. Thus the deficiency of the problems must be conspicuous in a real-life setup. If children are found to be at ease with simple MSCD problems, more sophisticated ones may then be considered.

## Notes

1] In a paper addressed to research mathematicians. Semadeni [1987] used the term *intentionally ill-posed verbal problems*. this term however, does not seem suitable for didactical purposes.

[2] Two papers by Cruickshank [1948] and Goodstein [1972] are quoted by Nesher [1976]; four more papers by Arter and Clinton [1974], Biegen [1972], Blankenship and Lovitt [1976] and Fafard [1977] are quoted by Suydam [1980].

[3] The złoty is the unit of currency in Poland Its actual value has varied since 1926 as a result of monetary reforms and inflation

[4] An exposition of various aspects of cognitive conflicts can be found in Berlyne [1965], chapter 9

[5] This is a hint that in a MSCD problem we had better avoid second person grammatical forms (and perhaps also first person) Thus, "Johnnie has . ." is preferable to "you have" or "I have". In other words, the problem should either be in logico-mathematical setup or the data should be compatible with the true ones

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