Communications

Research Problems in Mathematics Education — I

Early last year (January 1983) I sent a letter to about 60 mathematics educators asking them to suggest research problems whose solution would make a substantial contribution to mathematics education. Some twenty replied The text of the letter and a short selection of responses follows More responses will be published subsequently

I have deliberately not edited the responses, nor commented on them here I believe it may be interesting for readers to see some of the communications I received in the form in which I received them. Later on I may make some observations and offer some of my own suggestions. In the meantime perhaps others will be moved to join in the conversation, by supplying their own set of problems or by commenting on these.

It should be stressed that the authors did not write for publication, though they have subsequently given their permission for me to use their contributions in this way. I am grateful to them for this. "First thoughts" may not always be "best", but they are instructive in their way.

The Editor

The letter

I am writing to you, and about 60 other mathematics educators in several countries, to invite you to participate in a small "exercise". The exercise consists in formulating a number of *specific problems* whose solution would be likely to advance substantially our knowledge about mathematics education.

The famous example of Hilbert's 23 problems, announced in an address to the IMU in 1900, is at the back of my mind. There are clearly important ways in which we cannot expect to emulate his achievement. Our field does not have the advantage of a long and successful history, and we have as yet no universally accepted criteria for determining which problems in our field are solvable or not. Nevertheless, I believe the attempt to follow Hilbert's example could be instructive.

I remind you that Hilbert tried "to lift the veil behind which the future lies hidden". He proposed we "let the unsettled questions pass before our minds and look over the problems which the science of today sets and whose solutions we expect from the future". He demanded of his problems that they could be expressed clearly in a way that was easy to understand, and that they should be difficult but not completely inaccessible.

I do not expect anyone to produce *23* problems! The exercise will not work very well, though, if you confine yourself to only those problems you intend to work at. You should include problems whose solution seems important even if you do not feel particularly moved by them or particulary competent to tackle them. One of the outcomes of the exercise may be to show how we perceive the extent of our field I leave you to decide the boundaries of "mathematics education": you may include problems which could be labelled "psychological", "sociological", etc., provided they meet the criterion of promising, by their solution, an advance in mathematics education.

Please send me your "several" problems, mailing them to me not later than March 15, 1983. You may use, as Hilbert did, one or two paragraphs to set the scene for each one. You may invite colleagues to collaborate with you if you wish. The exercise is quite serious, but I cannot ask you to spend days in preparing your answer, so please do not neglect to write even though you may feel your response is only provisional and not as deeply considered as you would like.

I will prepare some analysis and summary of all the replies and circulate it to you. If the replies are as interesting and significant as I expect them to be, I would like to be free to write an article based on them for "For the Learning of Mathematics". Please indicate when you reply if you put any restrictions on my use of your words.

Thank you for reading this far. I very much hope to hear from you.

Sincerely,

DAVID WHEELER

GEOFFREY HOWSON

I was intrigued by your letter on Hilbert's problems.

First I asked myself to what degree Hilbert's problems were significant in the development of mathematics in the twentieth century. This sent me back to his talk. Certainly it was a *tour de force,* but I wonder just how influential it was in shaping 20th century maths. The problems on the continuum and on the consistency of arithmetic gave rise to great pieces of mathematics, yet they have scarcely influenced the development of mathematics (despite Kline's *Loss of certainty*). Had Gödel proved that one could establish the consistency of the arithmetical axioms, then our lecture courses would not have been substanially changed! Hilbert scored a good number of hits, Lie groups, p.d.e.s, etc., but looking down the list one notes how many problems were in the theory of numbers—now not very fashionable—and how few in algebra. Of course, Hilbert could only talk about what he knew—that Einstein was round the corner and that computers would hit the world were not in his ken. He had no means of telling how scientific and technological innovations would affect mathematics—but in view of recent developments, Schubert calculus seems very small beer.

Since writing this opening paragraph, I have had the opportunity quickly to scan the AMS Volumes (1976) on *Mathematical developments arising from Hilbert's problems.* I note that Browder says, with much more authority than I can muster, but nevertheless in the same vein:

"We should not fail to note that there is no hint [in Hilbert] of such decisive developments *in the following decade* [my italics] as the development of topology, both combinatorial and set-theoretic, or of functional analysis [even though Hilbert was to contribute so greatly to this]"

Let us praise famous men—but let us not create idols.

Now to mathematics education.

My second reaction was to write down: "What is a problem in mathematics education?". The next question will surely be "What is a solution?". Paradoxically, the second is almost easier to answer, because we can say that a "solution" will not be like a "solution" in mathematics having universal and eternal (?) standing, but it will have a range of validity limited both in space and time.

A colleague who recently returned from the Arab states found that the biggest "problem" he faced was that he and his students did not share the same view on the purpose and nature of education. For them, education was sitting there whilst the "truth" was revealed to them. There is no universally held concept of education. It follows that there will be difficulties in defining the rôle of mathematics within, say, the *general* education of an adolescent.

Thus a recent meeting in Saudi Arabia decreed (I am quoting from memory but could supply details [1]) that the purpose of primary school mathematics is to move children's thoughts from the concrete world around them to the abstract, and so facilitate their movement from a preoccupation with matters temporal to thoughts of things spiritual.

These would not appear to have been the views of those who wrote the chapter in Cockcroft [2] on primary mathematics. The difference in philosophy is so significant that it is unlikely that the two systems will share common "solutions" to the problems they encounter.

Moreover, even if we restrict the problem to consideration of affairs in England then we see that a committee sitting in 1980 is bound—because of social and technological changes—to arrive at a different conclusion from one sitting in 1900, 1950 or even 1970.

Cockcroft is somewhat disappointing in that it talks of the knowledge we wish pupils to have, but it does not speak of metaknowledge—of the view which students will (or should be helped to) form of knowledge. This metaknowledge will itself be changing continuously and, of course, should govern the selection of knowledge at the "content" level. I am teaching my grandmother to suck eggs—but an awfully large number of grandmothers don't seem to have mastered that art!

Changing motivation, sex rôles, employment patterns, etc., are all going to ensure that solutions soon go out of date.

Even at a very mundane level, *How best to introduce the notion of a variable?,* say, is affected both by the child's preceding education, by the time (age) at which it is attempted, and by technological development. Nunn's "solution" [3] becomes invalid in a "calculator" age. *How best to prepare textbooks?* has a solution which is dependent upon teacher competence (at a "general", national level); something again which is a function of place and time. *How best to organize a classroom?* is dependent on the expectations of pupils (and research has demonstrated differences according to social origins, etc.), on material provisions, etc.

So far I have not been encouraging! What can one salvage? I do not believe "problems" in a mathematical sense have a true maths education analogue. Rather I should say:

It would help me to know more about ...

> the motivation of pupils and the way it is being affected by economic, political & social changes,
>
> the way in which teachers perceive their rôle—for this would help me understand their aims and fears and the problems to be tackled if one wishes to accomplish change,
>
> "mathematical ability",
>
> .....
>
> (Here one could supply further "areas" on demand!)

It would help me to have

> a better theoretical framework within which to consider/study/investigate mathematics education. (Here I am for the moment not asking for a *theory,* just the framework which might help to develop one. Ed Begle made a start with his *critical variables* [4] but I find this linked to a limited, idiosyncratic view of mathematics which gave insufficient weight to sociological considerations.)

Finally a question which might permit a global solution! Is mathematical ability governed by innate neurological/physiological/genetical factors, and if so, to what extent? (Is our lack of success with class 4D really due to their lack of mathematical ability? Is Atiyah better than me not because he went to Manchester Grammar School and Cambridge, received better teaching, was more motivated, came from a richer social environment, but simply because his brain has a bigger RAM and rather better circuitry?)

Here then is a very rough and ready response to your letter. Should you wish to come back at me, fine.

Notes

[1] "Recommendations of Second Conference", *Times Educational Supplement*. 5 September, 1980

"The objective is to make students ... understand abstractions and be steeped in the area of symbols. It is good training for the mind so that they may move from the concrete to the abstract, from sense experience to ideation, and from matter-of-factness to symbolization. It makes them prepare for a much better understanding of how the Universe which appears to be concrete and matter of fact is actually *ayatollah* signs of God—a symbol of reality".

[2] Committee of Inquiry into the Teaching of Mathematics in Schools *Mathematics Counts*. HMSO, London. 1982

[3] Nunn T P *The teaching of algebra* 2nd Edn London: Longmans. 1919

[4] Begle, E.G. *Critical variables in mathematics education* MAA/NCTM 1979

Centre for Mathematics Education
University of Southampton
Southampton S09 5NH. U.K.

THOMAS KIEREN

My contribution follows two of your suggestions but violates a third. I have been thinking for a considerable time about personal mathematical knowledge and how such knowledge gets built up. Thus, what follows relates in one way or another to this aspect of mathematics education and may not reflect on the full scope of the field. The problems or hypotheses proposed are certainly provisional in nature and not as systematically structured as they could be. But I shall take the opportunity to preface such problems/hypotheses with some explanatory remarks.

I have proposed that personal mathematical knowledge building involves five facets. Because of the nature of mathematical activity one such area is *psychological* with *factors* such as information processing, perception, memory and developmental mechanisms (e.g. correspondence, transitive reasoning). I have titled a second facet *mathematical variation*. This is not the Dienes use of the word but looks rather to the outcomes of considering the consistency of a mathematical system. Usually there are several models for such a system. While it might be logically economical to consider one model or interpretation, I think personal knowledge building demands experience with several variations. These variations (as well as levels at which they are addressed) each carries its own informal mathematical symbol system. Thus, a third facet of knowledge building focuses on levels of *symbolic control*. There also seem to be distinct thinking tools (e.g. counting, forming ratios) which can be used to build up mathematics. I have termed these *constructive mechanisms*. The final facet of knowledge building is *imagery* which I will simply describe here as mental pictures.

These ideas on mathematical knowledge building are certainly neomentalist in nature. In general I hypothesize that useful personal mathematical knowledge is built through the coordination of these five facets. A curriculum for mathematical knowledge building would have to deliberately take these into account. The theorems/problems posed below come from this kind of thinking.

These hypotheses/problems are also influenced by what I consider to be the nature of mathematics. I believe that Lakatos has termed mathematics the science of the invented. If this is so, it is curious that mathematical ideas have such a universal (used in many senses) character. I believe that mathematics is about the brain's response to pattern (defined broadly). Bohr suggested that physics is *what we can say* nature is. Following this structure it may be better to say that mathematics is *what we can say* about the impact of patterns in the brain. Thus, mathematical knowledge building is a very personal thing (inventedness), but its external expressions reflecting brain functioning have this universal character.

Problems/Hypotheses

1. The axioms of a mathematical system point to principal knowledge building problems. (This hypothesis is a knowledge building conjecture related to the question: "Why are axioms axioms?")

   Related hypotheses:

   (a) The Peano axioms point to the roles of unit, successor, counting on, counting back and "and so it goes" (Freudenthal) in developing whole number knowledge. A practical consequence of this in the research field is that the works of Steffe *et al* and Gelman are complementary rather than contradictory and that they in turn complement the work of persons like Brainerd, Nelson and Lovell.

   (b) Quotient field axioms point to the role of dividing up equally (partitioning) in rational number knowledge development.

   (c) These axioms also suggest that rational number knowledge building is distinct from and not a mere extension of whole number knowledge building. (Off the record, I suspect that the brain functioning needed is substantially different, if we could observe it.)

   One can think up a large number of subquestions which would contribute to the study of the main question here.

2. What are the constructive mechanisms of mathematics and how do they function in knowledge building? This question has been asked and tackled with respect to whole numbers (counting). To me these mechanisms embrace and have as a consequence of their use the

protomathematical "nothings" of which Gattegno has spoken

Again there are many related subquestions or problems. One which I have been pursuing is:

(a) In what ways is partitioning a constructive mechanism in rational number knowledge building? As a consequence of the thinking under hypothesis 1(b) above partitioning should be used in fractional number settings in both additive and multiplicative ways.

Here is a practical related conjecture:

(b) Curriculums which deliberately develop and use constructive mechanisms will "produce" "better" mathematical knowledge than those that do not. An important aspect of *mathematics education* is evaluation. I have used "better" in the previous conjecture. I suspect that we should be evaluating according to the *shape* of student "achievement" instead of its "length" (see the work of Tom Maguire at CERA in June).
A question, Eval (1): Given various mathematical content areas, what is meant by the *shape* of a person's knowledge, how can we determine (measure) this and how does this relate to mathematical knowledge building at a higher level in the same area, to knowledge building in other areas?

One of the major potential influences on mathematical knowledge building is computer use. Still in this category is the following question.

(c) To what extent are computer procedures (particulary in Lisp-like languages) constructive mechanisms for persons developing or executing them in a mathematical context?
(d) What is the nature of mathematical intuition? What are the curricular/instructional means to foster this intuition? What are the consequences of such a programme?
This question has certainly been addressed in may ways (Dedekind, Poincaré, Hadamard, Higginson). Some recent starts towards an answer come from Fischbein's talk at Berkeley (and now in FLM): "intuition is the homologue of perception at the symbolic level"; and from Friedman: "intuition is innate but improveable". This discussion fits here because of the following conjecture formulated as a partial response to the questions raised above:

Constructive mechanisms are major tools of mathematical intuition.
I think such mechanisms could be seen (if we looked properly) in the work of Sanja (in the Soviet Studies) or in the work of modern cosmologists or quantum physicists (e.g. Heisenberg's use of matrices as a thought tool).

3. The breadth of extent (related to van Engen's idea of understanding) of a person's mathematical knowledge is directly related to experience with mathematical variations.

A person using a variation-rich environment will have a more extensive mathematical knowledge (within a given general area) than one who has not had such experience

4. Northrop Frye talks about a development of the use of language and symbolism across the history of western civilization as involving hieroglyphic (identification of word/symbol and object), hieratic (the word metaphorically put for an object) and demotic (the word used analytically with no necessary object in mind) phases.

Conjecture:

(a) Informal mathematical language involves symbol-object identification and metaphoric use of symbols. (Thus adding fractions is put for laying two lengths end to end) It should be noted that "object" need not refer to physical objects but to number sets or patterns, for example.
(b) Given a mathematical topic what are appropriate informal symbolic structures?
(c) Given a mathematical topic what sequence of levels of symbolism is appropriate for a novice learner?
(d) Many people have discussed roles of oral language in mathematics instruction and knowledge building (e.g. the Hendersons, J. Payne, Behr). In what ways does oral language used by the "teacher" and the "knowledge builder" function in mathematical knowledge building? Why? (It may be that oral language constructions are by nature more metaphoric in character for the user.)

5. The connectedness of a person's mathematical knowledge (related to Morgenau's construct space ideas, Skemp's reflective thinking and understanding) is dependent on instructional patience. (Allowance for, not confrontation of, van Hiele levels, knowledge and tolerance of levels of language use, deliberate development of constructive or procedural mechanisms in a particular mathematical topic domain.)

(a) The van Hieles have described levels of instructional and knowledge building development in geometry. What are the uses of van Hiele levels in geometry curriculum development?
(b) What is the meaning of such levels in other mathematical topics (e.g. rational numbers, calculus)?
(c) Hypothesis: In using computer languages in mathematics instruction one needs to be aware of van Hiele-like levels

6. Hypothesis: mathematical knowledge building requires the deliberate coordinate development of imagery, symbolism and constructive mechanisms.

(a) Mathematical imagery and constructive mechanisms are dual in nature. (e.g. A "linear" image of rational numbers is suggestive of counting and vice versa.)
(b) Because of the inter-relationship between image, procedure (constructive mechanism) and symbols, LOGO does provide an egosyntonic mathematical knowledge building environment (Papert).

7 This question is related to the other six but represents a different thrust of the "theory" above. What is the interaction between memory structures and the person's structural (self-built) mathematical knowledge? (I have observed a young student with a weak short term memory. In subtraction for which she had almost no knowledge structure her knowledge of facts seemed random. Yet she seemed to understand addition with carrying and appeared faultless in her knowledge of related addition facts.)

I think that the 20-odd conjectures given above can be fitted into a general knowledge building structure. Further, the categories (1 — 7) are by no means independent and hence there is a certain redundant character in this list. The categories could also contain more major questions and certainly all imply a large number of specific questions some of which are currently undergoing investigation.

I hope this communication is useful.

Department of Secondary Education
University of Alberta
Edmonton, Alberta
Canada T6G 2G5

NICOLAS BALACHEFF

Pour ma part j'éprouve quelques difficultés à jouer le jeu de Hilbert. A y bien réfléchir, il me semble que cela tient à ce que nous n'avons pas derrière nous 2000 ans de recherches scientifiques sur l'ensiegnement des Mathématiques, mais tout au plus disons ... quelques décennies.

Ainsi, et c'est à mon sens là l'essentiel, nous manquons d'un cadre théorique précis, assez commun, et spécifique de ce que nous appelons en France les recherches en Didactique des Mathématiques. Dans une telle situation il est bien difficile de formuler des problèmes à-la-Hilbert avec quelques garanties sur la réelle communication de leur signification. En d'autres termes, peut être est-ce là le premier problème à évoquer:

> *Quel cadre théorique, quel cadre expérimental pour les recherches sur l'enseignement des Mathématiques?*

A ne pas considérer cette question en tant que telle, on court le risque de continuer à collectionner des travaux, des résultats, dont l'interêt restera trés ponctuel et qui, faute de pouvoir prendre racine dans une problématique spécifique de l'enseignement des Mathématiques, chercheront ailleurs leur validation (psychologie, psycho-pédagogie, linguistique, mathématiques ...).

Les problèmes qui nous préoccupent, s'articulent sur trois termes:

> *les Mathématiques, l'élève, la situation scolaire.*

L'élève n'est pas seulement un enfant, mais un individu engagé dans un situation sociale particulière avec un objectif déterminé: l'acquisition de connaissances mathématiques. La complexité de notre tâche tient à ce qu'il n'est pas possible de séparer ces trois termes. Ils doivent être compris tous les trois dans un même cadre théorique et expérimental rendant compte de la complexité de leurs relations mutuelles (en particulier l'interaction des fonctionnements épistémologiques, psychologiques, sociologiques). La réflexion dans cette direction s'est largement développée en France ces dix dernières années. Nous en avions fait une présentation lors du congrés ICME IV*; on peut en particulier se reporter aux textes d'Yves CHEVALLARD (The didactics of mathematics: its problematic and related research) et de Guy BROUSSEAU qui présentaient des questions centrales du point de vue théorique.

Pour terminer je me risquerais à évoquer une ou deux questions relevant des recherches sur les processus de résolution de problèmes.

On a maintenant une connaissance assez solide des principaux invariants heuristiques, par contre nous savons assez peu de choses de leur fonctionnement. En d'autres termes

> *Quels sont les processus à l'origine d'une décision heuristique déterminée?*

On peut s'orienter vers la recherche d'heuristiques du second ordre, mais à l'évidence le problème ne sera résolu que provisoirement. Il me semble plus raisonnable de chercher à préciser les processus de contrôle à l'oeuvre dans la résolution d'un problème; j'en vois deux types principaux:

— les contrôles par la signification, ou contrôles sémantiques; ceci soulignera le caractère central des aspects conceptuels (et non exclusivement des aspects procéduraux. Il n'y a pas de savoir-faire sans savoir).

— les contrôles par des procédures de preuve dont la forme la plus achevée en Mathématiques et l'élaboration d'une démonstration. Ces procédures peuvent être trés variées et dépendent de façon décisive de ce qui contitue le système de rationalité de l'élève. Elles sont à la source du questionnement des processus de résolution de problèmes et des notions sur lesquels ils portent, par là elles peuvent susciter des décisions heuristiques.

Il y a pour la démonstration un problème particulier. Du fait même des stratégies le plus souvent adoptées dans l'enseignement, la démonstration apparaît moins comme un outil de preuve que comme une forme rhétorique propre à la classe de Mathématiques. La vérité de l'énoncé à démontrer n'est pas en question; par la démonstration l'élève montre ce qu'il a compris, appris; il vise à satisfaire une demande de l'enseignant qui porte d'abord sur l'évaluation de son apprentissage. Ainsi le problème est-il de savoir comment d'une part redonner sa place à la démonstration

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* Address by members of the G.R.D.M. (France) at the ICME IV. *Recherches en Didactiques des Mathématiques*. 1981, Vol. 2, No. 1, pp. 129-158

dans le processus de résolution de problèmes, c'est à dire la replacer dans une "dialectique de la validation" (au sens de BROUSSEAU); et d'autre part comment la replacer dans une pratique sociale; car pour être restaurée dans sa signification la démonstration doit apparaître comme un moyen dans un débat où l'enjeu est la valeur de vérité d'une assertion.

Voilà les quelques réflexions dont je souhaitais vous faire part pour contribuer à votre enquête Je serai trés intéressé par les suites que connaitra votre entreprise.

IMAG
B.P. 68
38402 Saint Martin d'Hères
France

JEREMY KILPATRICK

You undoubtedly know that when the program committee for ICME 4 invited Freudenthal to be a plenary speaker at Berkeley in 1980, their hope and expectation was that he might attempt a Hilbertian list for our field. He didn't, and I've about decided that such a list doesn't make sense for mathematics education since our problems are never clearly defined, let alone solved. Each generation of mathematics educators ends up wrestling with many of the same problems the preceding generations thought they had "solved," and I think that's likely to be a permanent condition of our field, not simply a product of our limited history and our lack of agreed-on criteria for what problems are "solvable." We don't *solve* problems of mathematics education, we *inter* them Like Dracula, they come back to haunt us because we never quite manage to put a stake through their heart.

Nevertheless, I have attempted to define three problems that, even if not well expressed or solvable, seem to me to be central to our field. All three problems are at the interface between curriculum and instruction.

The first problem concerns skills and "automaticity of response." One reason mathematics teachers provide "drill and practice" for pupils is that they want the pupils to be able to respond automatically to certain questions (e.g., what is the product of 5 and 9?). The argument is that, when such responses are automatic, the pupil's attention is free for consideration of more complex questions (e.g., do I next add the remaining number or multiply by it?) In the January issue of the *Journal for Research in Mathematics Education,* Bob Gagné argues that "automaticity of skills" has been undervalued by mathematics educators. One can ask, however, what price automaticity? Les Steffe and Rick Blake, in the May *JRME,* contend that too great a stress on automatic responses is likely to leave pupils confused as to the meaning of what they are doing. It's an old debate—how are "meaning" and "automaticity" to be orchestrated? Should one teach for automaticity and let meaning follow—running the risk of finding what Kath Hart found in the CSMS project with respect to ratio and proportion: "No evidence in this topic of rules learned and repeated with understanding" (*JRME,* March 1983, p. 124)? Should one teach for meaning, and let automaticity follow—as some proponents of the "new math" advocated. Or should one phrase the issue as William Brownell did in the title of his 1956 *Arithmentic Teacher* article: "Meaning and Skill—Maintaining the Balance." The Hilbertian problem might be posed as follows: *For each skill in the school mathematics curriculum, what level of automaticity is optimal for subsequent use of that skill, and how can the skill be made meaningful without inhibiting automaticity?* Behind this problem is the old joke in which the centipede becomes selfconscious about where he puts his feet and then trips himself up Certain skills need to be brought to consciousness—so they can be understood and controlled more precisely—and then made automatic. We know something about how this might be done in training pilots or coaching athletes; do those principles transfer to the mathematics class?

The second problem concerns the hierarchical view of mathematics learning that many people have adopted—how does it affect mathematics teaching? Jere Brophy, an educational psychologist at Michigan State, was quoted recently as saying that mathematics educators are misguided who believe that, because calculators are so easily available, we can drop from the school mathematics curriculum the multiplication of numbers with more than two digits. Brophy argues that "performance must be perfect on low-level objectives if succcess on higher-level objectives is to be reasonably expected" (*Notes and News,* Institute for Teaching, Michigan State University, 25 February 1983, p. 3). Leaving aside the question of what it might mean to drop a certain kind of multiplication from the curriculum, let us consider Brophy's argument for putting the low-level spinach before the high-level dessert. Certainly, many teachers of mathematics have bought the morality and good sense of this argument. But where is it written that low-level must or should come before high-level? Zoltan Dienes once broached what he called the "deep-end hypothesis"—the idea that learning might be improved if pupils were thrown in at the deep end of a subject, and compelled to sink or swim, rather than being helped along from the shallow end. The Hilbertian problem might be something like: *What are the effects on learning if instruction is aimed at the attainment of certain "higher-level" objectives given imperfect attainment of related "low-level" objectives?* This formulation of the problem begs the question of how one establishes whether and how two objectives are related. It also neglects the issue cited at the beginning of this paragraph—what are the effects on teaching of this low-level/high-level view of objectives?

The third problem concerns transfer. Everyone knows that Thorndike *et al* showed conclusively that the study of mathematics doesn't make pupils better reasoners, yet teachers remain convinced of its power to do so. One way to reinterpret Thorndike's research is to suggest that perhaps his instruments were insensitive to certain changes that studying mathematics makes in how pupils think. If the teachers are right and Thorndike wrong, it might be worthwhile to develop more sensitive instruments for measuring reasoning—and other intellectual—abilities likely to be affected by mathematics learning The Hilbertian problem: *What general intellectual abilities are affected by the study of mathematics, and how are they affected?* Stated this way, the problem is too broad to be addressed reasonably, but pieces of the problem might be amenable to attack

I hope this response to your request for problems hasn't affirmed the old saw that a fool can ask more questions than a wise man can answer Better, perhaps, is James Thurber's observation: It's better to know some of the questions than all of the answers Best wishes to you in orchestrating the responses you get.

Department of Mathematics Education
University of Georgia
Athens, GA 30602
U S A

DICK TAHTA

To try and do for mathematical education what Hilbert did for mathematics in his 1900 Congress address is a magnificent challenge I think this should be a cooperative exercise, because though I find I can list some themes that preoccupy me at the moment (and which I think are important areas to consider) I do not think I can yet identify the right questions to ask about them. This was presumably Hilbert's achievement: to identify problems that could be formulated precisely and which he intuited could be in some respects solved in the following few decades.

Hilbert's authority and reputation meant that many young mathematicians did subsequently take up his challenge so that some of his questions have been answered I am not sure that the situation is similar in mathematical education mainly because it would not be so universally agreed what constituted a good problem—let alone a solution—in this field. Hilbert proposed that a good mathematical problem should be clear and easily comprehended; moreover it should be "difficult in order to entice us, yet not completely inaccessible, lest it should mock our efforts" We could surely demand the same. But whereas he was of course also able to lay down very precise and universally acceptable criteria for what would constitute a solution, this is more difficult in our case. Perhaps the most important 'hilbertian' question for us should be:

> *What are the criteria by which we would recognise a solution to a problem in mathematical education?*

One of the difficulties in dealing with this question is that it could invoke concerns and actions that can be loosely described as political. Problems of mathematical education in the third world or in the inner city school of affluent countries are urgent enough and demand immediate attention. Such problems do not, however, have solutions in the scientific sense—dealing with them is a matter of ordering priorities But independent of our choices in areas affected by politics or economics there remains the vision of a science of education—a discipline which could provide solutions to problems in its own terms so that whether we act as teachers or administrators we have at least some pedagogical or epistemological invariants at our disposal. It is therefore with the possibility of scientific solutions in mind that I list—in no particular order—some problems, framing appropriate questions where I can, although I am aware that a lot more careful work would have to be done on these to make them sufficiently challenging and interesting to others.

I have deliberately tried to frame these questions in my own words but acknowledge that I have been influenced by various questions raised by Gattegno; in particular, the twelve themes he proposed in chapter 5 of his book, *What we owe children.* I hope that any digest of responses that may be printed in the journal could include a reprint of that list—it seemed at the time, and still does, to have an appropriate breadth and depth.

1. Loosely speaking, the primary school classroom in many countries seems to provide an environment that harmoniously reflects the concerns and preoccupations of young boys and girls. The style of classroom organisation, the furniture, the available resources, the mural displays, all seem to symbolise a well-digested understanding of child developmeny. The situation in secondary schools is quite different—both in theory and practice.

> *What are the overwhelming concerns of adolescents?*
> *How can these be reflected (symbolised) by the physical environment in which adolescents learn?*
> *What aspects of mathematics are particularly related to or invoke these concerns?*

If we are able to answer such questions adequately, it may then be possible to deal with some specific ones that currently concern secondary school teachers (at least in the UK): Is there a dichotomy between "individualised learning" and "group teaching"? When is each appropriate? What techniques of classroom management are required in each case? And so on.

2. A lot is known about the history of mathematics and mathematicians are in a sense continually re-writing this history. But the implications of this continual renewal are not always realised in mathematics teaching. There seem to be some important questions here but I cannot formulate the right ones at the moment.

Meanwhile relatively little is known about the genesis of mathematics, whether in the individual or the race An archaeology of mathematics would be concerned with the origins or foundations of mathematical activity Philogenetically this is necessarily a highly speculative enterprise, though there have been some interesting recent contributions in this area But the origins of mathematical activity in babies is accessible and susceptible to investigation

> *What can be learned about the nature of mathematical activity—about the mathematical powers available to all children—from the fact that they have mobilised their powers of perception and action from a very early age?*
> *What mathematical powers are available to one who has learned language?*

3. Mathematics has always been given two complementary stressings: algebraic and geometric There are various consequences for pedagogy, many of them are longstanding problems At a general level the most obstinate are still:

> *What is the place of geometry in mathematical education?*
> *Why is traditional algebra so difficult for a large majority of students?*

It might be more valuable to try and produce more precise questions in this area. For example, if one accepts that geometry is concerned with imagery whereas algebra results from mental dynamics with imagery, then one may ask precise pedagogical questions about working with imagery. In particular, I ask:

> *Are there canonical images (i.e. images which are by tradition or experience found to be overwhelmingly representative)?*
> *If so, how are students at different ages best helped to gain control of such images?*

4. I am not sure whether the following questions can be investigated by a *science* of education They seem to invoke value judgements which might vary strongly within as well as across particular cultures But they seem particularly important in that they concern the inner state of being a teacher as opposed to the didactic techniques a teacher might adopt. One area of possible interest is the psychodynamics of mathematics teaching.

> *What draws people to mathematics—to teaching—to teaching mathematics?*
> *What elements are required in teacher training that will enable people to come to some self knowledge about this?*

More generally and certainly a hardy perennial:

> *How do we identify good teaching?*
> *If we can agree and identify what is good teaching, is it possible to find invariants across the various individual examples of it?*

5. There are surely an enormous number of current questions in the area of the mathematics curriculum. It is perhaps difficult to select from these those that are fundamental and not reflecting very particular circumstances A major challenge is the one proposed by Gattegno:

> *Is it possible to offer a complete mathematics curriculum in terms of awareness?*

It would seem worthwhile trying to extend this question into various particular forms. Meanwhile in another but related direction:

> *What techniques are available for making mathematics "accessible" without "explanation"?*

This is certainly too loosely framed—it has different particular forms whether one is considering preparing "situations" for students that are to be presented directly, through texts, through television or through computer programs (I conjecture that there are some quite precise tools that can be applied to such preparations.)

6 What might be called the pathology of mathematical education might be a fruitful area in the sense that counterexamples can help clarify what is the case. The big question here is the one proposed by Poincaré:

> *Why do people fail at mathematics?*

One could imagine a research programme that might yield something more than opinion about this. More usefully perhaps would be to study why and how people make mistakes in mathematics, both when this is associated with emotional tension and when it is accepted as a matter of course. The important questions about this seem to be about the nature of learning in general and I am not sure if there is a problem specific to mathematical learning. In general, we correct mistakes through some feedback from some sense of the truth of the matter, whether this is generated internally or through some external agency or through "evidence" It seems worth asking:

> *Is there a specific "sense of truth" in mathematics?*
> *If so, in what early experience is it grounded?*

The emotional content of mathematics is a neglected area. I have not a question to pose here yet but offer a quotation from Kandinsky:

"The impact of a triangle on a circle is no less dramatic than that of the hand of Michaelangelo's Adam reaching out to God's "

7 I am sure that there will be better-informed contributions than I could offer in the area of computers One of the big questions here has of course been proposed by Papert. In some ways he has supplied his answer so that we still need to frame a suitably open question albeit in the same area he has indicated.

> *Does the microcomputer present fundamentally new modes of mathematical experience?*

(For example, does it challenge the traditional notion of the continuum?)

And so on.
And so on.

28 Countess Road
London NW5 2XJ