A Journey through Geometry: Sketches and Reflections on Learning

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When I was 11, my mathematics teacher gave me a copy of *Prelude to Mathematics* by W. W. Sawyer Much of it I did not understand, but I became fascinated by a chapter on weird geometries in which people got smaller as they moved about, and infinity was the circumference of a circle. 'Non-Euclidean geometry' did not mean very much to me, but the images of 'straight lines' being curved and the angles in triangles not adding up to 180° appealed to my schoolboy sense of anarchy. Breaking the rules of geometry seemed to produce new rules that had their own coherence and logic.

Nearly a decade later, I became deeply interested in differential geometry Until then, mathematics seemed to consist of a series of discrete and disconnected topics and techniques. Differential geometry changed all that. I began to understand what mathematicians called *elegance* - economy, coherence and structures that linked together what appeared to be unrelated areas. It was also powerful, providing a language to describe the large-scale structure of the universe, and exotic objects such as black holes. Differential geometry drew together my intellect, imagination and interest, allowing me to speculate in a precise way about "life, the universe and everything"

Turtle geometry entered the picture when I got my first computer. It liberated me from BASIC programming and opened out the possibility of an experimental and exploratory approach to mathematics Reading *Turtle Geometry* by Abelson and diSessa (1980), which starts with drawing polygons and finishes with a simulator for General Relativity, fired my interest in finding a turtle version of the models that I had met as a schoolboy. Through a series of sketches, I am going to describe the journey that brought together these three geometries in creating a turtle-based exploratory tool for non-Euclidean geometry My aim is to reflect on some aspects of learning mathematics that emerged along the way

Exploring the terrain

Non-Euclidean geometry developed in the nineteenth century from attempts by mathematicians to investigate the status of Euclid's so-called 'Fifth postulate' - in one of its later formulations, given any line and a point not lying on the line, there is one and only one line through the point that does not meet the original line. Not only did mathematicians find that the postulate was independent of the others in Euclidean geometry, but came to the startling and counter-intuitive conclusion that there was a collection of logically consistent geometries to be obtained by negating the 'fifth postulate' in different ways. Perhaps the two best known are spherical geometry, in which there are no parallel lines, and hyperbolic geometry, in which there are uncountably many lines parallel to a given line.

The results were so surprising that, for many years, mathematicians, including Gauss, investigated the geometries in secret (Gray, 1989). It is not difficult to see why he - and many others - were so reluctant to publish their findings Non-Euclidean geometries seem alien to our everyday experience, and they run counter to the universally accepted results and methods of Euclid. Towards the end of the nineteenth century, however, non-Euclidean geometries became accepted as valid elements of mathematical knowledge. Euclidean models were developed for spherical geometry by Klein in 1871, and for hyperbolic geometry by Poincaré in 1881, to aid mathematicians' investigations and illustrate their results. I will describe the models briefly before describing some of the issues that arose in designing a turtle-based version.

Both models make use of circular arcs and ordinary 'straight lines' to represent the straight lines (geodesics) of spherical and hyperbolic geometry Figure 1(a) shows a triangle in the model for spherical geometry, made up of straight Euclidean lines from the centre (OP and OQ in Figure 1(a)) or the arcs of circles which intersect the unit circle at opposite ends of the same diameter (arc PQ in Figure 1(a)) Note that the angle sum of this (and every) triangle is greater than 180°

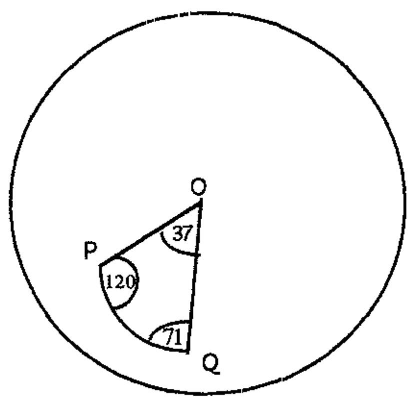

Figure 1(a) Hot-plate universe

Gray (1989, p. 215) describes this model as the *hot-plate universe*, in which one imagines that the circle is a plate

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whose temperature increases as one moves out radially from its centre. Using a metal rule to measure the distance along an arc or a straight line out from the centre, one would be unaware of the ruler's expansion due to the increase in the plate temperature. Only by comparison with a ruler 'at room temperature' would the change in length and the increase in the 'unit step' of measurement be revealed.

In hyperbolic geometry, 'straight lines' are either Euclidean straight lines from the centre (OP or OQ in Figure 1(b)) or arcs of circles which lie within and are orthogonal to the unit circle (arc PQ in Figure 1(b)). The angle sum of this (and every) triangle is less than 180°. This is the *cold-plate universe* (Gray, *ibid.*), in which the temperature decreases as one moves radially towards the circumference of the circle. A metal ruler used to mark out distances along the arcs of circles contracts as it is moved out from the centre, and in censequence the 'unit length' decreases. However, living in the surface one is unaware of the variation in length, and again, it only becomes apparent if one compares the ruler with one in a 'constant temperature' Euclidean world.

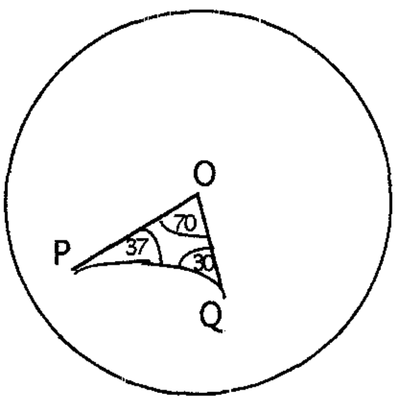

Figure 1(b) Cold-plate universe

The variation of distance-measure according to position in the models is a key perceptual feature, and runs contrary to our usual sense of congruence in which distance-measures remain constant irrespective of spatial position. This contrast makes the models difficult to interpret, but, given their success over the past century, not impossible understand or use. I became interested in how we can learn non-Euclidean geometry using the models, and what difference it would make if the models were computer-based. In particular, I wanted to examine the interplay between learning non-Euclidean geometry using the models implemented in turtle geometry, and the design of a medium in which that learning could take place (Stevenson, 1996).

My starting point in trying to create a version of these models for turtle geometry was to use Cartesian geometry in calculating the turtle's path on the screen. This was a relatively easy calculation using the turtle's screen position and heading $m = \tan a$, where a is the heading of the turtle relative to the positive x-axis, to find the equation of the circle or straight line that it had to move on. A more critical issue was how the turtle should move on the circles or straight lines once their equations had been found, and involved solving two problems: orientation and step size. I would like to describe this problem in some detail, since it had a direct bearing on my subsequent course of action.

Orientation is concerned with whether the turtle should move in a clockwise or anti-clockwise sense around a given circle, for a particular position and heading. The problem was that in calculating the co-ordinates of the circle-centres which formed the turtle's path, I used $m = \tan a$ which was obtained from the turtle's heading. This meant that a heading of either a° or $(180 - a)$° would give the same value of m, and it was necessary to develop an algorithm for deciding whether the turtle would move in a clockwise or anti-clockwise sense given a particular quadrant of the screen co-ordinates in each model. *Step-size* refers to the how much the turtle moves on screen around the circle, corresponding to one 'ordinary' turtle step.

As Figure 1 shows, both models are position-sensitive, since the size of step taken on the screen varies according to the turtle's position. In the hot-plate universe, this means that the turtle's screen step should increase in size as it moves away from the centre of the screen, and decrease as it heads towards the centre of the model. In the cold-plate universe for hyperbolic geometry, the reverse happens, and the turtle cannot pass beyond the 'infinity' of the unit circle. A significant practical problem was how to manage the relation between the two since the curvature of circular paths - given by the ratio of turn to step - had also to be kept constant for a given circle.

After several months of trying to implement the models in this way, I had developed some algorithms for deciding the turtle's motion depending on position, but the 'devil was in the detail'. It became a non-trivial task trying to account for every possible case that the turtle could adopt in terms of the quadrant and orientation, and I grew more dissatisfied with the approach, having an intuition that there was a more elegant solution.

While I was working on the approach with Cartesian geometry, I began to re-read some aspects of differential geometry [1], and started to investigate whether there was a solution using it. Starting with the metrics for the two models that I came across in Thorpe (1979), I was able to find the equations of motion for the turtle using standard techniques, solve them numerically and plot the results on a spreadsheet - all within the space of about eight hours. To see the models reproduced on the screen in a way that was easily adaptable to turtle geometry was a real *eureka* moment!

Why is this significant? The episode illustrates some important issues about the process of learning, I think. In effect, the approach that I had developed used Cartesian geometry and a collection of 'fixes' to try and match two essentially different kinds of geometry. Cartesian geometry is global and extrinsic. The equations used to describe the two models rely on having a reference frame that is external to the shapes being defined - after all, Cartesian geometry is specified relative to 'Cartesian axes'. The equations define the lines and arcs of the models 'all at once', in the sense that they provide a means of finding every point on a curve.

By contrast, turtle geometry is local and intrinsic (Abelson and diSessa, 1980). Shapes are defined relative to the turtle's heading and direction with no external frame of reference, and locally, since the turtle draws them 'piece by piece'. Differential geometry and the initial spreadsheet version of the models gave local and intrinsic descriptions of the models' features which could be easily adapted to turtle geometry They contained, implicitly, all the information about orientation and step that had caused so much difficulty, and were in a numerical format that could be implemented in turtle geometry. In short, I learnt that Cartesian geometry was the wrong sort of geometry for what I wanted to do, and I began to appreciate some of the deep connections between turtle and differential geometry.

From an epistemological point of view, the failure of Cartesian geometry to provide an effective tool for the task that I was engaged in made me simultaneously aware of its nature and its limitations Learning began here, not with conflict, but in a period of disruption that provoked renewed reflection on the task at hand, and set me on a search for new tools to complete it.

Two things fuelled this search: a need to find a solution and a sense of the form that I wanted the solution to take The first factor provided the dynamic for the search and the second formed the criterion that guided my efforts and formed the basis for judging its success. Finding a solution was an iterative process that involved drawing out the possibilities that I saw in differential geometry for making a working tool. In so doing, I became more discriminating and precise about those aspects of differential geometry that I needed and gained a clearer insight into some of its theoretical structures.

The 'tool' that emerged from this process was a synthesis of pragmatic and analytic factors, derived from the interplay between mathematics and my intentions (Stevenson, in press) The spreadsheet's role in this process was crucial, since it provided me with a medium in which to turn the 'static' equations of the turtle's motion into a 'dynamic' solution - something that I will return to later

After the euphoria had subsided, I began to think about the solution that I had obtained, and how it was related to the original three-dimensional surfaces I wanted to find a mathematical basis for the elegant computational solution that I had found, and I would now like to describe and discuss the process of justifying the metrics that I used in the computer-based models.

Making a map

To say that I had no justification for the metrics would not be correct. The model for spherical geometry is the result of projecting the surface stereographically onto a flat surface, shown in Figure 2. Drawing a straight line from A, at the 'north pole' of a sphere, through a point B on the surface, projects its image B′ onto the equatorial plane. The hot-plate model is viewed in the equatorial plane

It is possible to show that the 'straight lines' of the sphere - so called 'great circles' - are projected to Euclidean straight lines or arcs of circles, and that induces the necessary metric in the viewing plane (Stevenson, 1996). Figure 1(a) shows the projected image of the triangle OPQ, formed by three great circles in Figure 2 My problem was trying to account for the way in which the metric for the Cold-Plate Universe could be induced by projection. Clearly there had to be some connection with the stereographic projection, but how and what remained open questions. A resolution of the issue came about in an unexpected way.

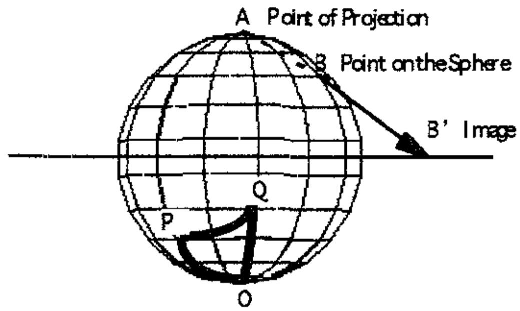

Figure 2 Stereographic projection of a sphere onto the x-y plant. The image of triangle OPQ is shown in Figure 1(a).

I was reading Thorpe (1979) and came across an exercise to show that the hyperbolic metric was induced in the *x-y* plane by projecting the positive sheet of the hyperboloid $Z^2 - X^2 - Y^2 = 1$ from the point (0, 0, -1) - the arrangement is shown in Figure 3.

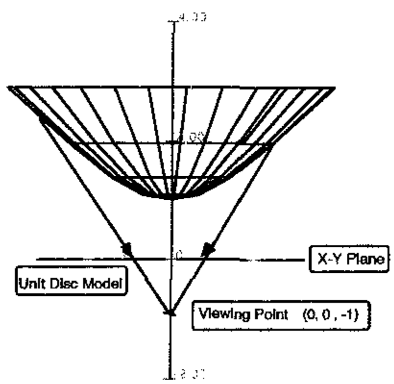

Figure 3 The projection of $Z^2 - X^2 - Y^2 = 1$ from the point (0, 0, -1) to give the cold-plate universe

It was relatively easy to show that the projective map

$$(X, Y, Z) \longrightarrow \left(\frac{AX}{A-Z}, \frac{AY}{A-Z}\right)$$

induced the metric for both the spherical and hyperbolic models in two dimensions, where (X, Y, Z) is any point in the 3-D surface, with $A = 1$ for spherical geometry and $A = -1$ for the hyperbolic case. I could use this map to show how the images of 'straight lines' on the sphere and hyperboloid induced either Euclidean straight lines or arcs of circles expressed in a Cartesian form mentioned in the previous section. [2]

Finding and developing the projection for the hyperboloid was a surprising discovery, but not of course for Thorpe (1979)! It allowed me to create a coherent framework that I could use to develop arguments which both explained and demonstrated various disparate results that I had found (Hanna and Jahnke, 1996) I hesitate to call this a systematic proof, but it did raise in my mind a question

On the one hand, stereographic projection is an arbitrary process in the sense that there are different ways of projecting a three-dimensional surface - Kreyszig (1991), for example, describes those projections of a sphere which preserve area. On the other hand, once the method was chosen there seemed to be a number of 'consequences' which followed from the choice, some of which I came across I was interested to know where this 'necessity' came from: was it an *aleqia* - an uncovering of a pre-existent structure - or a *techne* - a construction - or something else?

The 'something else' option might be to argue that it is both Choosing a particular projection was a genuine choice and not forced upon me because it 'reflected' some kind of underlying mathematical reality that justified it Rather, the justification seemed to lie in making the connections which were expressed in the equations and which served to elaborate the semantics of the process using the 'rules of algebra' The resulting sets of mathematical equations seemed to constitute what it meant to 'project the sphere using stereographic projection', and were not consequences of the projection process.

As I worked through the equations it occurred to me that there was an interdependence between the network of equations that I was generating and the objects that I was using - spheres, circles and straight lines. Elaborating the equations seemed to clarify, in detail, the properties of the objects that I was interested in. It is the phrase 'objects that I was interested in' that is crucial here, because it connects the question of ontology to intentionality in the context of an algebraically expressed rule-network.

Mathematicians do take what might be called 'ontological attitudes' towards objects in their specific discipline - concrete relationships, as Uri Wilensky (1991) describes them. However, that is not the same as making ontological commitments in the sense of saying that the objects of interest exist independently of the interested party. It implied a provisionality in my commitments and a clarification of my interests by identifying a specific entity as the 'object' of my attention In this sense, the connections that I was working on were both creations and discoveries, expressing the relationships between meaning, structure and objects of the projection

Plotting a path

In this section, I would like to show how the various strands that I have described came together in creating the final piece of software. My central aim was to develop a version of the Euclidean models for spherical and hyperbolic geometry using turtle geometry. I chose Object Logo - an object-oriented dialect of Logo - as the programming medium (Drescher, 1987).

I wanted to explore the object-oriented approach because it offered the possibility of creating turtles which 'lived' in their own geometric world and responded in their own way to 'ordinary' Logo commands. For example, a turtle could be created so that it moved in a way appropriate to either of the hot- or cold-plate universes when FORWARD was typed. This capacity to 'shadow' commands such as FORWARD was significant because learners could type Logo commands in the usual way, but obtain screen behaviour which is based on the models described. The power of the object-oriented paradigm emerged when I combined it with the differential geometric approach [3] By changing the value of a single parameter, which is owned only by the newly-created turtle, one of three different geometries can be selected for the same turtle.

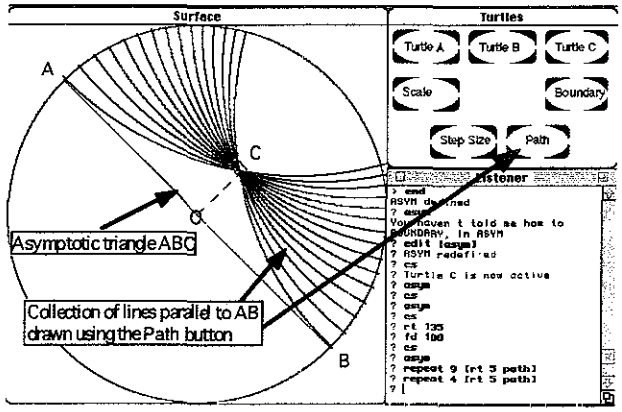

Figure 4 A screen-shot of the cold-plate universe. It shows an asymptotic triangle ABC, and a collection of 'straight lines' through C parallel to AB.

Figure 4 shows a screen-shot of the cold-plate universe, illustrating several features of the software and the geometry. Triangle ABC is asymptotic, formed by two 'straight' lines (AC and CB) meeting AB at 'infinity' represented by the circumference of the circle. A collection of lines through C parallel to the line AB are also shown, indicating that, in this geometry, there are many parallel lines which can be drawn through a given point to a given line - AC and CB are the 'limiting' parallels for the point C.

Figure 4 also illustrates two features of the software that emerged during the development of the models: dashing of the turtle track OC to show variation of congruence in the model, and a Path button to give the large-scale behaviour of the turtle, given its state at any position (Stevenson and Noss, 1999).

As the software was nearing its final form, I was struck by how much mathematics of differing varieties were combined to create what was, initially, a visual illusion. On the one hand, there were the large number of discrete calculations made by the software in order to produce the apparently continuous motion of the screen object. On the other, this could only be achieved using a computational medium that transformed the static algebra into a dynamic object.

Perceptually, however, this intersection of different kinds of mathematics that made up the 'software' did gave me a real sense that I was looking at a surface (the screen), and through it at another surface (the hyperboloid). I experienced a sort of duck-rabbit gestalt that enabled me to move between the domains of the surface and the screen.

Sometimes, it seemed that I was using a computer and thinking about the screen image, and at other times it felt like I was actually working on the physical surface. It was akin to the sense of simultaneous engagement with and separation from the geometric domain that Ackermann (1991) refers to when she describes making and using models.

This sense was highlighted by the curious lack of control that I had over what the turtle actually did. Once I had given the turtle its instructions, I no longer controlled how it behaved and could only watch what happened. As I grew more experienced with the software, I could begin to predict the turtle's macro behaviour, but there still remained that underlying sense of uncertainty about its precise actions, despite having constructed the 'turtle's rules' myself. It was this uncertainty that provided the impetus for my investigations. On the one hand, I wanted to confirm what I knew about non-Euclidean geometries, and on the other, I wanted to explore new possibilities that emerged unexpectedly in the process of these confirmations. What had started out as a 'model' gradually became transformed into an autonomous domain with its own practices and language which I had to learn.

Epilogue

And the end of all our exploring
Will be to arrive where we started
And know the place for the first time
(*Little Giddings* by T. S. Eliot)

'Learning mathematics' can have many meanings, and I would like to conclude these reflections by drawing out what I think are their contributions in helping to make sense of the term. My interest is in exploring the epistemological dimensions of my attempts to find a way of building a version of turtle geometry for the hot- and cold-plate universes, not to give a psychological account of it. In so doing, I want to draw out what I think are critical points in the evolution of what I came to understand about the mathematical domains that I was exploring.

My journey began in a transition from one kind of geometry to another. Transitions are interesting because they often reveal unspoken and sometimes unconscious assumptions and beliefs that shape our actions. In this case, the transition was marked simultaneously by disruption and disclosure, which structured the process and provided the dynamism for it (Heidegger, 1962). Disclosure of a possible solution emerged dynamically from the disruption that was engendered by the failure of a mathematical tool. Recognising that Cartesian geometry was the 'wrong' kind of geometry for my purposes was an integral part of the process which led towards a resolution, since it began to suggest what a 'right' solution might be like.

Turning a mathematical resource - knowledge of differential geometry - into a tool relied both on my fundamental engagement with the problem, and a recognition - albeit ill-defined at first - that it fitted the task at hand. Seeing possibilities in resources so that they can become tools is a dynamic process which has both practical and cognitive dimensions. It draws on a network of existing knowledge which identifies the problem as a 'problem', but also helps to frame possible solutions in relation to the needs, interests and intentions of the 'problem solver'.

Realising those possibilities - turning a potentiality into a working implement - is a long and often frustrating process, built on experimentation and partial justifications relative to the desired end. Learning mathematics in this context is concerned with expanding networks of 'necessary' connections between concepts and objects that the learner thinks are significant, while at the same time grounding that necessity in socially mediated practices and standards (Wittgenstein, 1953).

The computational context that I am describing emerged at the intersection of three types of geometry with an elaboration of a projective technique and an application of numerical analysis. Each of these component parts can rightly claim to be called *mathematical*, and yet they each have different ontologies and modes of justification. Together they provided, arguably, a new type of geometry - non-Euclidean turtle geometry - which has an inherited axiomatic structure and a drawing tool that is adapted to variable congruence. Engagement with the 'new' turtles created by the software implied learning how to manipulate them and interpret their behaviour - interpretation, not as a kind of inference but as a way of seeing and working with a new ontology (Mulhall, 1990).

Mathematics is not a monolithic concept, on this reading, but a series of overlapping discourses each with their own meanings, expressed through rules, practices, and ontologies. 'Learning' has the sense of charting and navigating the connections and discontinuities between these discourses - often necessitating difficult transitions and complex elaborations of pathways - but which generate new syntheses at their intersections.

Notes

[1] Differential geometry models space as a pair (M, g), where M is a differential manifold and g is a metric function which maps vectors in M to the real numbers, and carries all the geometric information. The metric at any point (x, y), for both models is $\frac{4\,ds^2}{(1 + k(x^2 + y^2))^2}$, where $ds^2 = dx^2 + dy^2$ is the metric on two-dimensional Euclidean space. When $k = 1$, the metric gives the model for spherical geometry, while $k = -1$ gives the model for hyperbolic geometry and $k = 0$ gives Euclidean geometry.

[2] For example, straight lines on the hyperboloid can be obtained by taking plane sections of the through the origin - planes with equation $aX + bY + cZ = 0$ - and $Z^2 - X^2 - Y^2 = 1$. Using the projection equations gave

$$\frac{x^2 + y^2 + 1}{1 - (x^2 + y^2)} = k\,\frac{2x}{1 - (x^2 + y^2)} + m\,\frac{2y}{1 - (x^2 + y^2)}$$

where (x, y) lie in the X-Y plane, and this simplifies to $(x - k)^2 + (y - m)^2 = k^2 + m^2 - 1$, with $k = -a/c$, $m = -b/c$ and $c \neq 0$. This is the equation of circles orthogonal to the unit circle in the cold-plate model for hyperbolic geometry (see Figure 1(b)). A very similar argument gives the circles of the hot-plate universe for spherical geometry.

[3] Expressing the turtle's position in the model as a complex number $z(s) = x(s) + iy(s)$, so that its conjugate is $\bar{z}(s) = x(s) - iy(s)$, where s is the 'turtle step', and applying the standard equations of differential geometry to the equations governing the turtle's motion in both models are given by $\frac{d^2z}{ds^2} = \frac{2kz}{(1+kz^2)}\left(\frac{dz}{ds}\right)$. Using Runge-Kutta order 4, and initial values for position (z) and heading (dz), it was possible to solve these equations to give the motion of the turtle in a step-by-step manner in either model.

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