ADDRESSING PRIORITIES FOR ELEMENTARY SCHOOL MATHEMATICS

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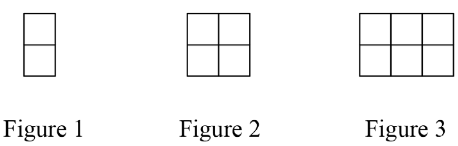
What experiences in early mathematics are needed for young children to develop competencies and dispositions to engage in sophisticated and complex mathematics? Consider, for example, the assessment problem from the Grade 4 TIMSS 2011 released items shown in Figure 1. The international average of students who answered correctly was 58% for part A and 45% for part B.

Bill is arranging squares in the following way:



A. Draw Figure 5.

B. How many squares would Bill need to make Figure 16?

Figure 1. Assessment item based on the 2011 TIMSS Grade 4 Released Items: Mathematics, item ID number M041115A, in TIMSS (2013) [1].

The design of the problem requires students to do more than merely count the number of squares in each figure. Part B requires reasoning about the relationship between two variables: the figure number and the number of squares needed to construct the figure. What experiences help students recognize the opportunity to use a more efficient, non-counting approach? What curriculum do students need in order to develop skills for solving numeric problems that demand more than counting? In this article, we highlight the importance of elementary mathematics in setting a foundation for student success with algebraic concepts and skills, and suggest an approach that takes into account work from Davydov (1966, 1975a, 1975b) to promote access to higher and more complex mathematics. Our aim is to offer a basis with which to clarify educational priorities in elementary mathematics to support algebraic understandings. In the first part of the article, we describe three major lines of thought that formed the basis for Davydov's work and briefly discuss their educational implications. Based on this work, and evidence from our research in the US, we argue that the concepts of quantitative relationships (equality and inequality) and unit should receive particular attention in considering an alternative approach to a mathematics curriculum.

A Russian perspective on mathematics instruction

V. V. Davydov led a group of mathematicians, mathematics educators, and psychologists in the development of a scientifically grounded mathematics curriculum (Davydov, 1975a, 1975b; Minskaya, 1975). Differences between that curriculum and curricula in countries such as the US, especially with regard to the earliest years of schooling, are striking. Its objectives aimed to foster mathematics achievement in students who were not particularly outstanding academically.

Davydov and his colleagues began with the premise that the starting point for mathematics education should reflect what is conceptually fundamental or basic in the structure of mathematics. That is, since children instinctively make sense of their world in a way that is inherent to mathematics, school instruction of mathematics should build from such everyday concepts. For example, as children interact with their environment they investigate how common quantities compare: "Do I have less milk than my brother?", "Is my hand as large as my mother's hand?", and "Is that ball heavier than this one?" are types of relational comparison that can be translated into mathematical statements such as $A > B$ or $A \neq B$. Davydov's approach builds on Vygotskian theory with the belief that instruction supports knowledge development by building from the child's environment. In early instruction, this means using activities that children do naturally to scaffold to a mathematical interpretation of those experiences. Students' natural language is linked to mathematical language and symbolic representations to capture and promote reasoning inherent in the way mathematics is communicated.

An important element of Davydov's analysis, therefore, was an examination of the interrelations among mathematical concepts, and particularly an inquiry into whether number is the most basic concept, as most contemporary mathematics curricula assume. This mathematical analysis was complemented by two other kinds of considerations: psychological factors and pedagogical issues. We consider each of these components of Davydov's analyses of the proper foundations for mathematics education in turn.

Psychological considerations

Davydov (1975a, 2008) fused Piagetian ideas about the nature of logico-mathematical development with a Vygotskian perspective on the profound role of instruction in developmental processes. As Davydov noted, Piaget traced the origins of mathematical thinking to children's earliest

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interactions with objects. The sensorimotor schemes that infants develop in their interactions with objects embody transformations of inversion (undoing or negating a previous action) and reciprocity (compensating for an initial action via a subsequent, different, action that restores the original relation between two objects). According to Piaget, these transformations become properties of the intellectual structures that develop later in childhood; and from those intellectual structures, in turn, mathematical structures are abstracted.

Davydov (1975a) took issue with this account primarily in that he argued that it fails to consider the impact of instructional practices on the rate of logico-mathematical development. He shared this perspective with Vygotsky (1978), and posited that while Piaget found that mathematical structures did not emerge until relatively late in development, instruction designed to support the discovery of mathematical relationships could enable them to develop much earlier:

> answers [...] to questions about the so-called developmental characteristics of the child's thinking [...] can be only *relative*, dependent on the way the development of thinking is "actually organized" in the instruction process (p. 94, emphasis in original).

Davydov's reference to the instructional process extends beyond the sequencing of teaching acts. Mathematical concepts are formed and developed through deliberate instructional activities.

Davydov (1975a) argued that Piaget's failure to recognize the importance of instruction for development derived from his emphasis on the process of abstraction, which he viewed as the basis for the relationship between intellectual structures and mathematical ones, rather than on the properties of objects to which those structures correspond. Davydov took the position that mathematical relationships are objective "relationships among things that really exist" (p. 90) and that children "encounter these relationships very early" (p. 90). Specifically, he maintained that classification, seriation, and reversibility (key aspects of the intellectual structures Piaget described) develop in the context of children's manipulation of objects and exploration of the "relationships among definite quantities of objects" (p. 90). Thus he argued that classification and seriation are not "logical structures" as Piaget maintained, but "practical methods of distinguishing and designating certain mathematical relationships" (p. 90). The implication of this analysis is that introducing children early in instruction to sound mathematical methods for investigating relations among quantities could enable them to develop these structures much earlier.

As evidence for this position, Davydov cited Russian work in which a preschool mathematics curriculum was presented that familiarized children with ways of measuring quantities (Gal'perin & Georgiev, 1961; as cited in Davydov, 1975a, 2008). Children who received this curriculum no longer made the kinds of perceptually-based errors in reasoning about quantities that Piaget had described as characteristic of young children.

Davydov suggested that certain errors in the way first-grade children count are similarly based on a reliance on perceptual attributes of aggregates of objects and can likewise be eliminated through different methods of teaching children about number. Again, he believed that what children needed was to be given specific means for performing mathematical operations so that they did not have to base their judgments on perceptual impressions, which lead to errors. In particular, he proposed that children needed to be taught, "an operation defining the relationship of a whole and a part" (Davydov, 1975a, p. 93). The word *operation* refers to the activity of measuring (counting), comparing, and identifying quantities and their relationships. Combining or separating quantities results in new outcomes that provide opportunities to elucidate the relationship of whole to part.

The meaning of this idea is clarified by looking at the first-grade curriculum that Davydov and his colleagues developed (*e.g.,* Davydov, 1975b). In this curriculum, number is first introduced in a measurement context, that is, as a way of describing the relationship between two continuous quantities, such as length, that were measured using direct comparison (see Clements & Stephan, 2004). Children characterize the relationship between the two quantities by determining the number of times the smaller one needs to be iterated to generate the larger one. For instance, it takes four successive iterations of a 3-unit strip to generate a length equal to that of one 12-unit strip.

An important feature of this method of instruction is that numbers are always relational, not absolute. The longer strip in the above example has length of 4 if our unit is a 3-unit strip, but it could equally well be described as having a length of 6, given a unit strip that is 2 units long, or some other length, given another unit. Thus, children are taught to identify the unit to be used before assigning a numerical value to a quantity. Using this approach to teach number focuses on the impact unit has on the count or measure and is a direct result of using measurement as the basis of number. Consequently, children identify and understand the need for precision in knowing the unit to quantify and compare.

Mathematical considerations

A theoretical underpinning of the Russian work was that instruction focuses on generalized mathematical properties prior to specific applications or examples of the concepts (*e.g.,* computation work with numbers). Davydov (1966) believed that the concepts selected to begin the study of school mathematics laid the foundation for the entire academic subject. Furthermore, he argued that general mathematics concepts have been introduced through school courses in the wrong sequence and that the concepts that students learn do not correspond to the concepts that constitute mathematical structures (Davydov, 1966). For example, Davydov argues that the study of natural numbers together with fractions is problematic because natural numbers have to do with counting objects and fractions have to do with measuring quantities.

Davydov (1975a) argued that numerical systems are defined on the basis of a chain of other concepts, which are therefore more fundamental than number. For instance the concept of equivalence, which has relevance throughout the study of mathematics, can form the basis for completing a numerical statement such as $5 + \_\_\_ = 4 + 7$, where a student would think that 5 is one more than 4 so the missing addend must be one less than 7. The development of this idea supports children's creation of a framework for understanding mathematics beyond a numerical example. Students work with the quantities, comparing them and joining them without being distracted with the counting up of the items. This allows them to focus on the relational definition of the equal sign as opposed to the "solve this now" definition.

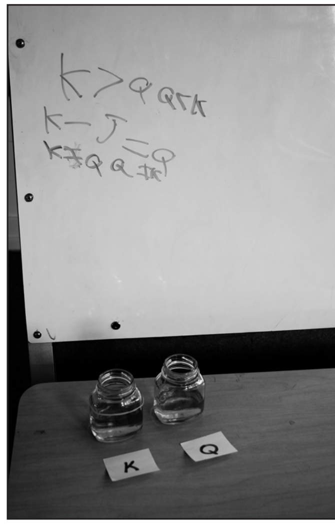
Drabkina (1962, cited in Davydov, 1975b) and Minskaya (1975) called attention to the importance of the concept of unit for understanding the seemingly simple act of counting. Counting, Drabkina noted, is actually a measurement of a quantity that consists of a defined unit of measurement used a whole number of times. Thus, counting involves more than merely assigning successive number terms to the objects comprising a collection to be enumerated. Fundamentally, it involves selecting a unit (what part of the material to be enumerated will count as "one") and by iterating that unit determining the relationship between the total quantity and the selected unit.

If understanding units is a prerequisite of counting, what then should constitute the introduction to elementary mathematics? Davydov (1975b) advocated that instruction begin with comparisons of physical attributes of objects and collections. Consideration for this approach, as mentioned above, attends to the way in which children make sense of their everyday world. These comparisons can be described without using numbers—shorter, longer, heavier, lighter, more than, less than, and equal to—and represented with relational statements, like $G > L$, that use letters to represent the quantities being compared. Fundamentally, students realize that if $G > L$, then it is also the case that $L < G$.

First-graders can write these relational statements and understand what they mean, because the statements describe the results of physical actions that the children themselves have performed in the comparison process. Figure 2, for example, shows student recordings from a class activity. From these beginnings, Davydov proposed that instruction should proceed to modeling addition and subtraction as operations that can change the relation between two quantities. To make two initially unequal quantities equal, two actions can be performed—the greater quantity can be decreased (subtraction) or the lesser quantity can be increased (addition). The amount that must be added or subtracted to make the two quantities equal is the difference between the two quantities (see Figure 3).

Children in grade one also determine that relations can be maintained when the quantities are altered. Specifically, if two quantities are equal, the equality relationship is maintained if equal amounts are added (or subtracted) from each quantity. Similarly, if two quantities are unequal, they remain unequal if equal amounts are added (or subtracted) from each quantity. Learning activities are structured to develop children's flexibility while preserving mathematical properties. Unlike mathematics curricula that introduce topics in isolation and hold fragile truths, the mathematics suggested by Davydov and his colleagues is robust and maintains rules developed in the primary grades.

Number is introduced only when children are well acquainted with the notions of equality and inequality asso-



"If it's an inequality, then you can write four statements. Look, you can say that volume *K* and volume *Q* are unequal—like *K* not equal to *Q* and *Q* not equal to *K*. But you can also say that *K* is greater than *Q* and *Q* is less than *K*. See, four statements, just like I said."

Student "J", first grader

Figure 2. *Mathematical statements recorded by students to represent the relationship between the quantities and transcription of the student's explanation from the Measure Up project, 2003.*

ciated with different types of quantities (*i.e.,* length, area, mass, and volume), and with the use of addition and subtraction to transform relations of inequality to relations of equality and vice versa. Generalized continuous quantities provide the context for developing mathematical relationships, which are basic and primary to the discipline of mathematics. The question, "How much larger is the quantity?" motivates the introduction of a unit of measure. As a means of representing the relation between a chosen unit and a larger quantity, the concept of number is carefully introduced. Thus children learn that they must identify the unit before they can assign a numerical value to any quantity. To compare quantities, the unit used to measure them must be the same. At the same time, children can determine the relationship between alternative units of measurement if a quantity is measured first with one unit and then with a different one, yielding a different numerical outcome. For example, consider volume *T* measured with volume-unit *B* and volume-unit *F*. The resulting statements, $T/B = 5$ and $T/F = 8$, indicate that volume-unit *B* is larger than volume-unit *F* because it took fewer of the *B* units to measure the quantity.

This instructional progression reflects the notion that mathematical structures, not numbers, constitute the foundation for mathematical knowledge. Mathematical structure at an early age can be thought of as relations among aggregate quantities. By beginning with these relations, children can explore and define generalized structures related to algebraic properties such as associativity, commutativity, and inverseness. The use of objects or groups of objects that can be manipulated to represent these properties allows for thinking beyond specific cases to examples that build the abstract notion first, with applications to the real number system later.

For example consider two lumps of clay, one on each side of a balance scale, with equal masses. The clay on one side is yellow and is named mass *Y* and the clay on the other is

Using two unequal areas of paper, the papers can be stacked such that the area of the larger piece that is not covered by the smaller piece can be cut off. The piece that is removed is defined as the difference. Similarly, beginning with the unequal areas of paper, by taping the precise amount of area to the smaller area to create a combined area equal to the larger area, defines the difference.

Given quantity $B > $ quantity T.
If $B - C = T$ and $B = T + C$,
then $B = T$ by C.

The last statement is read, "Quantity B is equal to quantity T by the difference, quantity C."

Figure 3. Concurrent representation used to model change from a statement of inequality to a statement of equality.

blue and named mass *B*. Children watch the pointer on the scale as the two masses exchange places. They determine that the masses remain equal (*i.e.,* are commutative), neither quantity increased or decreased. Then, three pinches of yellow clay are removed from mass *Y*. The children determine that regardless of the order with which the pinches are rejoined (*i.e.,* associativity) mass *Y* will again be equal to mass *B*. Finally, say the masses are reset such that they are no longer equal. When asked what one could do to make them equal, two strategies are raised, increase the smaller quantity or decrease the larger one. This measurement task creates an opportunity to develop the concept of inverseness.

Davydov (1975b) wrote, "there is nothing about the intellectual capabilities of primary schoolchildren to hinder the algebraization of elementary mathematics. In fact, such an approach helps to bring out and to increase these very capabilities children have for learning mathematics" (p. 202). This initial work with general quantities enhances children's abilities to apply those concepts to specific examples that use numbers—at first whole numbers, but in time also rational numbers and real numbers.

Instructional considerations

Perhaps the most general assumption underlying traditional curriculum sequencing is that instructional material should progress from the particular to the abstract (*e.g.,* from number examples to generalizations). This seems to be a sensible approach and enables teachers to use children's everyday experiences with concrete things as a basis to assimilate school curriculum. Davydov (2008) agreed that using real-life experiences is useful, but he challenged the shortcomings of this approach when it fails to build the theoretical knowledge of the discipline. In this research, theoretical knowledge, characterized as abstract, general, and verbal, is contrasted with concrete, empirical knowledge, which is attained through sensory experiences.

Davydov asserted that the relationship between the particular and abstract knowledge is not unidirectional: "The general not only follows from the particular [...] but also changes and restructures the whole appearance and arrangement of the particular knowledge which has given rise to it" (Davydov, 1975a, p. 98). For instance, through the comparison of unequal quantities of volume such that the larger volume, volume *S*, overflows from a container that precisely held the smaller volume, volume *M*, the abstraction, volume $S >$ volume M, is concretized. That is, the action itself dramatically illustrates the relationship between the volumes. The demonstration is carefully orchestrated to accomplish several things. First, the students who are still developing the notion of conservation can visualize the relationship. Second, the order of pouring from the larger into the smaller creates the impact and the containers used in the comparison inevitably become gauges for considering capacities of other containers. Furthermore, the momentous display of inequality contributes to one's knowledge of the composition of a quantity. Scrutinizing pertinent features of the containers (*i.e.,* height, area of the base, material used to create the containers) contributes to a notion of proof and the ability to predict relationships between other quantities. Comparing unequal quantities in other attributes (*e.g.,* length, area, mass) develops the flexibility of the concept of inequality, and introduces children to the robust beauty of a mathematical property.

An important pedagogical principle that is implicit in this argument warrants special emphasis. A sound pedagogy for the early years is one that effectively prepares children for the whole course of mathematics learning on which they are embarking, not just one that produces immediate learning. Davydov (1975b) and Minskaya (1975) both applied this idea to instruction in number. Davydov criticized traditional curricula for perpetuating a great divide between counting numbers and real numbers. While traditional instruction may seem to be satisfactory in the early years, when children are working exclusively with whole numbers, the limitations of this approach become apparent in later years in the difficulties children have with rational numbers and then real numbers. Minskaya (1975) specifically addressed the importance of the concept of unit for avoiding such pitfalls:

> many first-graders who are "good" at counting (by the ordinary standards) still identify a number (a set of units) with an actual aggregate. They make no distinction between what they are counting and the method of recording the result and [...] they do not understand that number depends on the base which is chosen. As a result, these children do not acquire a full-fledged concept of number, and this has a negative effect on all their subsequent study of arithmetic. (p. 211)

For instance, learning about place value without an understanding of the base used to generate the number often leads to an absence in recognizing the significance between a one-digit and a multi-digit number. That is, one could know that the number "10" follows "9" and overlook the creation of a new place value in the recording of a "1" and a "0." However, if the child is given a unit of length and is asked to find a quick way to measure the length of the classroom using base three, and compare her result with her classmate who is given the same unit but measures in base nine, the concept of place value becomes incredibly vivid. Treating base as fundamental to the development of number can affect the way students interpret mathematical relations, even in the absence of a reference to base (see Figure 4).

An important point made by Davydov (1975b) and Minskaya (1975) is that the impact of how mathematics is presented in early instruction becomes evident as children move forward in their mathematics learning. While a straightforward approach to number may give the impression that children are progressing well in the early elementary grades, its limitations will emerge in later years. The work with counting numbers limits children's understanding to that particular subgroup of numbers, resulting in generalizations children inappropriately transfer to work with rational and real numbers. Indeed, that is precisely what emerged for the US in international comparisons of mathematics achievement (*e.g.,* National Center for Education Statistics, 2012): students appear to be progressing well in elementary school, but they fare less well in comparison to students of other countries by high school.

"Which is larger, 3 or 8?" asked Mrs. W.

"It could be 8," said Caitlin. "But it could be 3. It could be a small 8 or a big 3. See, if you have 3 really, really, really big units, then 3 could be greater than 8. Or you could have 8 really, really, really, really small units. Then 8 would be less than 3. So it's hard to tell if you don't know the unit," said Caitlin.

"What if it's on a number line?" asked Mrs. W.

Caitlin thought and then said, "Oh, then that's uh, different. You know that 8 has to be larger than 3 because the units are the same."

Figure 4. Excerpts from interview dialogue, Education Laboratory School, February 2002. Reported by Dougherty (2008, p. 400).

Ama caught *k* fishes and Chris caught *e* fishes less. How many fishes did Chris catch?
Dusty had *b* fishes in a bucket. When Anthony added his fish, there were *g* fishes in the bucket. How many fishes did Anthony catch?
Sarah caught *p* fishes, and Tara caught *c* fishes. How many more fish did Sarah catch?

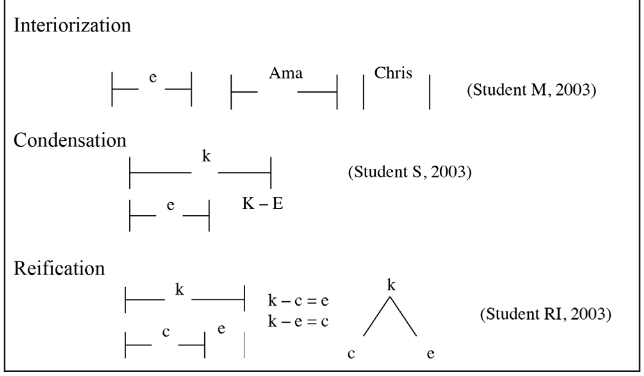


Figure 5. Sample problem and student solutions from Dougherty and Slovin (2004).

Measure Up curriculum research and development

In 2001, the El'konin-Davydov Russian curriculum (Davydov, Gorbov, Mukulina, Savelyeva & Tabachnikova, 1999) was translated into English and adapted for use in modern US classrooms by the Measure Up project of the Curriculum Research & Development Group of the University of Hawai'i. The mathematics of Measure Up follows the Russian development, such that a measurement context is used to develop mathematical properties. The project continues to explore the potential for developing comprehensive Grades 1–5 curricula, as well as to research the effects on children's learning. Findings from this research include the following.

- Dougherty and Slovin (2004) found that children in their third year of the Measure Up program (8–9 years old) were capable of using algebraic symbols and generalized diagrams to solve problems. Regardless of the child's achievement level, they used multiple representations to symbolize the problem and associated actions. Even though the problems did not use numbers (see Figure 5), the representations supported the students as they made sense of and solved the problems. Children were able to solve problems with concurrent representations: (a) a physical model (*e.g.,* two amounts of liquid), (b) an intermediate model (*e.g.,* paper strips or a line segment diagram to represent the comparison of the quantities), and (c) symbolization (*e.g.,* algebraic-looking statements of equality or inequality).
- Measure Up students were better prepared for algebra than their non-Measure Up peers (Slovin & Venenciano, 2008; Venenciano, Dougherty & Slovin, 2012). Interviews with individual Measure Up students about how they solved the assessment item "$2a + 5b$" revealed how students considered the letter representations as units. One fifth grade student stated, "… you have to find *b*, then you can add that unit to the *a* unit. You have to find out how many *b*'s go into *a* or *a*'s go into *b*." Another student stated, "If $2a = 1b$, then you could say it was $12a$ if *b* was twice the size of *a*. Without that information, you don't know how to change it." Another student, one with identified special needs, wrote "7" on his test and in a follow-up interview he responded that he first thought the answer should be $7b$ because $a + b$ is *b*. But then he noted the use of the different letters, so he left it as "7" not realizing an appropriate response could be, "not possible." By responding with "7," this student recognized that expressions could be simplified, but he was confused by the combination of different letters. Had he dismissed the significance of the use of the letters his response may have been to combine the terms into something like "$7ab$." However, even upon further reflection, that was not what this student thought to do.
- Venenciano, Dougherty and Slovin (2012) examined the effects of logical reasoning capabilities.

Findings from this study showed that Measure Up experience correlated positively with logical reasoning and algebra preparedness, statistically significant only with algebra. Logical reasoning correlated positively and significantly with algebra preparedness and completely mediated the effects of age and prior achievement. The Measure Up subgroup had weaker prior achievement scores than their non-Measure Up counterpart, but yet they were better prepared for algebra.

Implications

It is helpful to begin our analysis of the implications of these ideas by considering carefully what Davydov and his colleagues had to say about number. Their rejection of number as the proper starting point for mathematics instruction will surely strike many educators as an astonishing position, given the heavy emphasis on number in most elementary school curricula. A careful review of the ideas they presented makes it clear that they did not view number as unimportant. It is important to distinguish here between "primary" and "basic." Davydov maintained that number should not come first in mathematics instruction; in this sense, it is not primary. It is, however, basic, in the sense that learning about number *in an appropriately general way* is thought to lay a foundation for subsequent mathematics learning.

The ideas Davydov and his colleagues expressed about how number should be taught help to clarify the implications of their views for identifying what is important for elementary school students to learn about mathematics. Instruction in number, they maintained, should be preceded by comparisons among physical quantities and the concepts of equality and inequality that emerge from those comparisons. Those concepts, then, are both primary and basic: they are the starting points, and they are also the foundation for further learning. When number is introduced, Davydov and his colleagues stated, it should be in the context of measurement—as a means of representing relationships among quantities. Even a single whole number is relational in the sense that it describes the relation between a unit and a larger quantity. The concept of a unit of measure is critical to the kind of understanding of number Davydov and his colleagues deemed essential, and this too is basic in their analysis.

Let us return to the question we posed at the start of this paper: what is important in early mathematics for young children to develop competencies, dispositions, and motivations to engage in sophisticated and complex mathematics? Beginning with children's experiences with quantities to build intuitive notions about relationships and comparisons of quantities allows for an organic development of generalization. The primary focus of elementary mathematics should be placed on the instruction of fundamental properties, as opposed to counting and rote memorizing numerical results. Numbers are introduced and used only when quantification is necessary, as in the case of using units in the measurement required to specify exact relationships and comparisons. In the TIMSS Grade 4 problem at the start of this article, for example, with a counting approach one may look at individual squares to determine how to draw the later figures. From a measurement approach, however, one may view the unit of measure as a composite of the two squares, that which is iterated with each successive figure. This second approach enables one to apply the notion of defining a unit and consider a scale factor to solve the problem. Inherent in this approach is the property of equality and the opportunity to avoid an excessive counting of squares, a strategy that is characteristic of a counting approach.

Following a Vygotskian learning-leads-development perspective means that if instruction begins with a focus on the structure of mathematical properties, students are given the opportunity to learn arithmetic, algebra, and other mathematics as a connected discipline. The schema for mathematics begins with the experiences young children have with everyday, accessible, non-technical objects. Teaching is not reduced to solely modeling computational examples for students to mimic, but instead focuses on the properties that structure mathematics and scaffolds the learning of arithmetic and beyond. Consequently, students expect mathematics to stretch beyond a collection of facts. They develop a disposition for sense-making and reasoning in the doing of mathematics.

Conclusions

Our goal in this article is to encourage the couching of decisions about educational priorities in principled considerations about the mathematics we want students to learn and the psychological factors that shape their learning. The triad of concepts—equality, inequality, and unit—is arguably the most fundamental mathematical content to be taught in the elementary years and the foundation for effective mathematics learning throughout the school years. The brevity of this list is deceptive, in that each of these concepts encompasses a great deal of knowledge. For instance, even in the pre-numerical portion of the Russian curriculum developed by Davydov and his colleagues, work on equality includes consideration of the properties of reflexivity, symmetry, and transitivity, without which the identification of a relation of equality would have little significance. Nevertheless, in examining the relative merits of different mathematics curricula for providing children with a solid foundation for the study of algebra and other mathematics, one would do well to look closely at how they treat the concepts of equality, inequality, and unit. At the core of establishing a dynamic, effective, and cohesive elementary mathematics program is the very conception of the discipline of mathematics.

Notes

[1] Retrieved from timss.bc.edu/timss2011/international-released-items.html.

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A journal is a product: a product—or perhaps a by-product—of people talking to people. Like face-to-face talk, printed talk takes many forms and serves many functions. It can entertain, inform, reassure, tell lies, beat about the bush, play on feelings, inflame, and do any of the other things that people use words for. Like face-to-face talk, too, printed talk is received in context, and the context may determine that the words tell much more or much less than they say.

Words, spoken words and printed words, are so much a part of our social furniture, so pervasive, so everyday, that we don't take them seriously most of the time. A professional writer knows that words are hard to master: they insist on being opaque when he wants to be clear, and blunt when he wants to be subtle; they are—worst of all—glued to the page, stuck at a particular point of time. He can rarely expect from his readers a comparable effort of re-animation, of re-creation.

Words can be offered seriously and can also be taken seriously. They can be worked at by a reader, re-read and thought about, until they yield up meanings that may have escaped a first scrutiny. At this level the production of a journal may also be an educational enterprise.

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