The View from Below

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There is a story about the late E.T. Bell, true or not I know not. His son asked, "Daddy, why do they put that plus sign on the top of churches?" Bell also remarked, in one of his popular books, that medieval theologians could be regarded as frustrated mathematicians in an age when that discipline was not fashionable. The joke is convertible. Perhaps mathematicians also are frustrated theologians.

I start this discussion with such jokes because I believe that the sources of mathematical knowledge and invention are in fact rather mysterious. I have some ideas about how such mysteries can be resolved, but I don't think the task is easy.

Let me start with a paradigm case. The Euclidean style of geometry is restricted to compass and straightedge. Working once in a fifth grade class we had introduced these ancient implements, with an initial encouragement to produce any kinds of designs. One of the things which happens, under these constraints, is a multiplicity of circles of the same radius, sometimes secondary circles centered on the circumference of the first one drawn. Two girls had in fact walked the compass around its circle and come back after six steps, almost exactly to their starting point. I happened to be near and caught the question that hovered between them: "How come?" My intervention was honest and it worked. We walked another circle with great care, and the point of the compass landed, by luck, in the hole it had started from. We ended with the hexagon dissected into equilateral triangles, and at one point Suzie said, "This (a radial line segment) is the same as that (a hexagonal edge), so it *has* to be just six."

I regard this as a paradigm case because of the implied transition from the empirical "is" to the emphatic "has to be." I assumed at the time — and still do — that Suzie had shown some flash of understanding of a kind which marks the transition from fact as empirical to fact as mathematical. I have chosen it as a paradigm case because Suzie, like Plato's slave boy,* could be presumed to lack any knowledge of formal geometry. Suzie did give some reason for her apparent insight, she had noticed that since the pencil compass had not been readjusted, it had produced what we would call a cluster of equilateral triangles. At the time I did no lightning-quick analysis of her apparent insight; unlike Socrates I did not proceed to elicit from her the complete steps of a formal proof. I got her to show me that she was looking at half the circle, three triangles. I looked at them later myself, quite hard. I didn't yet wish to lead too strongly, and we went on instead to further partitions of the hexagon. When now I try to imagine what I might have elicited, I am fascinated. The first obvious proof I see depends on the formal Euclidean proposition that the sum of the interior angles of a triangle is a straight angle, or that for any polygon the sum of the supplementary exterior angles, triangle *or* hexagon, is one full rotation. Is there some similar formal proof which would plausibly interpret and support Suzie's apparent mathematical insight, without imputing to her any acquaintance with high-school geometry? Since symmetry is so powerful an idea, yet accessible to visual perception, is there a direct symmetry argument here which would establish her very emphatic conclusion? I leave this, as mathematicians often say, as an "exercise for the reader," though her drawing may be suggestive. (Figure 1)

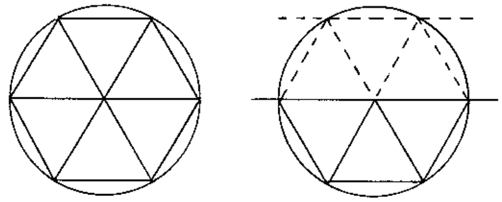

Figure 1

Maybe that particular Suzie, at that particular moment, *was* only guessing. It doesn't matter now. The story itself makes my first point.

If Suzie's insight was valid, she was already in the domain of the mathematician; in a basic sense they could meet as equals; if not, that first step was still to be taken. What then is this transition, so special to mathematics, from the merely empirical fact to the mathematical fact?

In any case Suzie had still a long way to go, even if she was momentarily in the mathematician's domain. There are myriad other facts to be recognized, of the kind I think she saw. When enough of these facts are recognized and held together they are, as a cluster, the substance of classical geometry.

For they are not independent, isolated facts, each true or false independently of the others. They are empirical facts but not what the early Wittgenstein called "atomic" facts, they are internally related to each other. One by itself, or two or three together, will lead to and perhaps require still another. If some can be noticed first as isolated empirical facts, they can still come together by some magnetic affinity, providing guides for further investigation; and the power of the process is multiplicative rather than additive in its potential rate of growth. As this clustering develops, some facts stand out as central, they lead to the recognition of many others.

This organization is what gets formalized as a deductive system. I see that as a later and quite distinctive development. It involves an abstracting, elucidation and definition of what we call idea or concepts. They are not facts, but universal terms, the elements, properties and relations involved in the perception and statement of many facts, all that potential infinity of which can be stated in this common

*Plato, *The Meno*, cf. Wheeler, David, "Teaching for Discovery." *Outlook* 14 (Winter 1974) pp. 38-42.

language. Suzie's construction can be described formally in this language by a few specialized terms: points, lines, congruences, rotations, etc. These same ideas will be implicit also in the perception of many other geometrical facts and will be expressed, at first spontaneously, in stating them. The new interest in analyzing and defining such ideas leads to another sort of clustering and ordering. Some ideas emerge as primitive, central, while others can be defined in terms of them. This development puts the affinities among geometrical facts in a new light. Though endlessly diverse as facts they share a common domain and some can be transformed into others. Taking a few of them as primitives, the rest can all be demonstrated. The perception of mathematics as a deductive *system*, first clearly exemplified in Euclid's *Elements*, has been a paradigm and challenge for all subsequent mathematics, science, and philosophy. Can all knowledge be so organized? Can reason become a substitute for experience? A stubborn empiricist might insist on counting edges, faces and vertices of as many kinds of convex polyhedra as he could lay hands on, and so far notice that the number of faces always happens to be two greater than the difference between the number of edges plus the number of vertices. Euler's theorem says more, it implies that nature is not free to make exceptions, the difference *must* be just two. To the empiricist this is a kind of indignity.

At this point I have outlined two developments. The first, and most primitive, is that some kinds of empirical facts get recognized somehow as facts which must be so; like Suzie we shift them from the state of "is" to "has to be". The second stage is that clusters of such facts, facts which in one way or another seem to require each other, lead to the explication and ordering of a conceptual domain to which all these facts belong. When they are stated uniformly in the terms of this domain it becomes apparent that they are not independent of each other but are linked by bonds of implication in some orderly system. Facts now become theorems. Some theorems are chosen — as primitives sufficient to generate all the others — to be axioms.

If we wish to speak more formally about the relation between ideas and theorems we can describe these as a linkage between two domains. Ideas are related in their own domain by relations of meaning and definition. Some of these can be taken as primitive, and others defined in terms of them. Theorems are related in their own domain by relations of entailment, in which again some are primitive and others derived. But the two domains are also essentially cross-linked in one-many relations. A given idea is involved in stating several theorems, and the statement of a given theorem involves several ideas. Each domain has its own internal connective tissue, of definition in the one case and implication in the other. But the connective tissue in each domain is enriched and elucidated by cross-reference to the other.

Suzie's apparent insight started from a particular drawing which could be described as an affair of circles, lines and points. This drawing was not in her mind an *example* of anything geometrical. It was not a consequence but a starting point. But after we have developed some theorems the drawings *become* examples, in retrospect. Philosophical and pedagogical accounts of mathematics often treat examples as inessential. They are mere starting points, aids to the imaginations, etc.; but they have nothing to do with the essence of mathematics, which is entirely an affair of abstractions detached from their humbler origins, like the modern trigonometry text which has no pictures of right triangles in the unit circle. Lewis Carroll complained about this sort of thing in *Alice in Wonderland*, and he was quite right. In the View from Above examples may seem to have no essential place. In the View from Below, on the other hand, they are of the essence. They are not only vital sources of knowledge but they have a continuing and quite indispensable place all along the way.

Corresponding to any mathematical system of ideas and theorems there is therefore always also a third domain, one-many-related to the other two. It may be called the domain of examples. In the View from Below it were better called the domain of proto-examples, of originals. Any concrete example of a theorem is very likely to turn out to be an example of several more theorems. These may be closely connected in the domain of theorems, but also they may not have been seen to be, and their coincidence may suggest new theorems, or new connections between old theorems. I don't know who first looked at the drawing I reproduced here, and noticed in it the possibility of a new and immediate proof of the old pons asinorum, the Pythagorean theorem. (Figure 2)

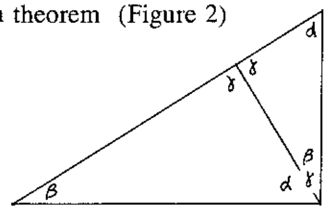

Figure 2

Instead of constructing *squares* on the sides of the right triangle it makes use of the three similar *triangles*, two of which together "have to be" equal in area to the third. The famous theorem is then seen as a direct consequence of the more general fact that the areas of arbitrary similar figures are proportional to the squares of any corresponding linear dimensions — of the flatness of space.

In their domain examples or originals are directly related to each other by similarities and differences, or by part-whole relations, etc. They come also to be indirectly related to each other through the domains of ideas or of theorems. Thus the cube and octahedron are indirectly related by the fact that the description of the one is transformed into the description of the other by simply interchanging the words "faces" and "vertices" and this duality immediately suggests a way of constructing the one from the other; or vice versa.

I should like at this point to acknowledge a substantial debt to the work of Edwina Michener,* from which I have taken the above three-fold partitioning of elements in the structure of mathematics. Her clear definition and discussion of these three domains provides I think a most useful frame for our discussions. She demonstrates this usefulness in several ways, illustrated from several fields of mathematics, elementary and more advanced. A key aspect of her work is to elucidate the nature of mathematical *understanding* Rather however than trying to summarize her discussion I shall continue on my own track, but with that same concern for the meaning and importance of the active verb *to understand*.

I grew up with a somewhat rebellious acceptance of the notion that mathematics was by its nature a highly sequential affair. I gained this impression first from the format of courses and textbooks, and later from some study of formal logic and the foundations. The pecking order of pure and applied math was then the order of the day, and the dominant positivist trend in philosophy, following some traditions of research in mathematics itself, was to declare that what was real was empirical and what was rational was — however elegantly — empty. This emptiness was to be demonstrated by the exhibition of completely formalized mathematical systems bled of all conceptual or factual content, yet with no loss of formal coherence In such a demonstration the theorem domain could be disconnected from those of ideas and examples, by depriving its formulae of any taint of vulgar meaning

This movement can be considered from two points of view. As a special movement *within* logic and mathematics it has been part of a whole new investigation of a metaformal character concerning the logical nature of mathematical system, dealing with problems of consistency, completeness, the relation of axiom systems to models, etc. As a *special* mathematical discipline it clearly fits the general framework of Michener's analysis; it has its own genres of theorems, concepts, its originals being the mathematical systems But the general philosophy I have alluded to, which for its own ends identified the theorem-structure of formulae with the whole of mathematics and left other aspects uncultivated, no longer seems very fresh or intriguing Yet its legacy is still with us, the image of an essential part of mathematics mistaken for the whole

Having been strongly influenced by that philosophical movement, though never embracing it, I tend to be inhibited in any outright opposition to it. Such opposition seems to embrace some allegedly outmoded philosophy of Platonic, Aristotelian or Spinozistic rationalism. Yet the desire to oppose has grown steadily Apart from my own roots in those older philosophical traditions, I have been strengthened in this desire by endless professional curiosity about the thought habits and commitments of physicists, biologists, economists, children, and mathematicians. Having been a student of the last two I have found I frequently came back to mathematics from a new point of view, usually mis-named *applied*. Whenever this happens I find that though the formal academic backtround has helped, I frequently arrive, after some struggle, to a sense that I have not merely *applied* some mathematics, but in the process, and with help, invented or reinvented or extended it along shortened pathways quite different from those I have been taught. Indeed I have often reinvented the wheel, but have come in the process to realize that in such matters — the wheel is a good example — no one has yet quite said the last word on the subject or its implications. These pathways typically start from "applications" but also typically bring close together ideas and theorems which have been treated in quite different parts of the book or in different books. I have begun to get a clear sense that the mathematical territory was not so much like the directed graphs or trees of the theorem-space, but much more like a network in which one could go from A to B by a variety of routes and back again along still others.

Tree structures there are, but they are so interestingly cross-indexed by analogy and reference that no simple metric of distance or closeness, such as that represented by the numbering of theorems in a book, or even the partial ordering of a theorem tree, seems appropriate. One is involved in a network somewhat like a map of airline routes, and like the airlines it brings many things close together. The map is, of course, a high-order abstract. To know the landscape and the culture one must in person get there I have also realized, however, that the image of the net needs major constraints. If every node is connected directly to every other node one is informationally swamped in finding the relevant connections Nodes must be graded by importance, by generality of relevance. Nodes are not all of the same kind, moreover, nor are their connections. What is first seen as a connection, for example the idea of a shared property relating different examples, can thereby itself become a node connecting to other ideas, sometimes by shared examples This sort of duality transformation seemed to blur the image of the net Michener's representation helps; it allows three planes or spaces in each of which the items and relations are of the same kind, but with cross-referencing by specific dual relations to items in the other two I think then the image becomes sharp again.

Michener is concerned primarily to use the net as a framework for improving our understanding of mathematical *understanding* and thus for a description of the personal qualities which we call fluency, resourcefulness, competence. Understanding of course implies knowledge, and knowledge implies subject matter, some independent domain of fact which is just there, incompletely known. The sum $\zeta(2) = \sum_{n=1}^{\infty} {}^1/_{n^2}$ is ${}_{\pi^2}/_6$ I know it, I have even followed and agreed to a proof, but I emphatically do not understand the result; others I hope have a better understanding. Nobody knows what to say about $\zeta(3)$, except that as has recently been shown, it is irrational. I think I understand something about such sums, can transform them, can disentangle connections among them, but the above trigonometric link eludes me. Understanding implies knowledge, though I think it need not imply formal proof Indeed understanding is often the guide to the invention of a proof, or a better one. I think most of us know enough about the counting numbers to see why the unique factorization theorem is true, and perhaps why it won't be true in some otherwise analogous number fields Maybe if I were hard-pressed I could invent a proof.

If the goal of mathematics education were really to work for active understanding on the part of students, what would the consequences be? Michener has experimented success-

*Michener, Edwina Rissland, *The Structure of Mathematics*, M.I.T. Artificial Intelligence Laboratory, A.I. Technical Report No 472, August, 1978 fully with the deliberate and conscious use of her scheme among university math students. No doubt many good teachers do something like that implicitly.

With the emphasis on understanding, on fluency, what does the picture look like for the early years? I cannot give a research report, but only suggest the outcome of a good many years of elementary school work, some with children, some with their teachers. One results is to see a need to modify or amplify the foregoing account. It is one thing to try to extend mathematical understanding among those who already have some formal education, but perhaps quite another to give access to it in the first place? I believe there are two complementary answers. One is from Plato's myth of reminiscence, that children *already* have some mathematical knowledge and understanding, and that we can learn to recognize this and as teachers, resonate with it. The other is that this understanding is typically implicit, context-dependent, and often hardly available for expression in the verbal or notational mode. Henry James says it with characteristic rigor in the preface to *What Maisie Knew*: "Small children have many more perception than they have terms to translate them; their vision is at any moment much richer, their apprehension even constantly stronger, than their prompt, their at all producible, vocabulary" In building bridges for communicating with children one needs therefore to learn (in part by reminiscence?) their characteristic ways of thinking, and this I claim involves a genuine extension of our own mathematical thinking.

This bridge-building also requires, as a result, some qualifications of Michener's scheme.* For children, theorems are not yet theorems, concepts are not yet concepts, examples, above all, are not yet examples. To say this is a less Zenish language, they can and will often employ patterns of recognition and thought, yet be unable to scrutinize them, or communicate about them in anything resembling adult language. What one discovers, through trial and error in teaching, is that their powers of communication are immensely greater when they and the teacher are in the immediate company of the concrete situations, the originals, out of which their understanding has manifested itself. Piaget talks about this phenomena as reflecting a concrete operational stage of thought, which in one sense it does. But this is often taken to mean that children are incapable of having and using high-order abstraction. This is another claim altogether, and I think a false one. Suzie's symmetry argument, as I call it, is rather deep, though she could not at all spell it out. She can only communicate it to me pointing to elements of the diagram we have produced, using gestures and demonstrative pronouns, appealing to *my* visual perceptions. I can't quite partition her insight into theorem, concept, and example, though I can — after some effort — validate it that way.

So for present purposes, as I said earlier, the term *example* is ill-chosen. An example becomes an example only after it is an example of something *previously* named and recognized in the domains of theorem and concept. What is seen is a concrete particular seen *as* something understandable. *That* something is still implicit. For me it is an example, for her it is still a concrete — though somehow pleasing — particular, an intriguing fact. I led her along the pathway of my kind of analysis as far as I dared, hoping to help her build bridges into an adult world.

A mathematician only extends his own understanding by active search, by being personally in charge. He can accept help from talk and print and can follow an argument if he has already shared some of its turf. But he alone develops his understanding. How does this translate into the childhood context? The major change, I believe, is that we must learn to share the childhood turf. A part of this can be at times almost adult, a thing of paper, pencil, of books, even at times I suppose of workbooks. For most children, most of the time, the turf is different. It is the world of concrete experience, presentational rather than linguistically representational. In this world the activity which leads to understanding is not yet separated from overt activity, it is directly perceptual and, a term of Jerome Bruner's, enactive.

In our own work, for such reasons, we have made ample use of the now-commercial concrete math materials, and added others as we were bright enough to think of them; pegboard and golf tees for lattices and graphs, for Mary Boole's curve stitchery; many of several shapes of geometrical tiles for tessellations and growth patterns, various looms for weaving (lit., com-plications), marbles for 3-D patterns, card for making polyhedra, poker chips for graphs and patterns, etc.

An interesting fact about the commercial materials is that often their most appealing uses are those not intended by the designers. Thus the so-called Cuisenaire rods were intended primarily for arithmetic, but I have never seen children first use them for that purpose. The intended use is representational. They were conceived I think as examples of the little number facts of early arithmetic, and their 2-D and 3-D extensions. They were conceived, in short, as new tools for the didactic teaching of arithmetic. What they get spontaneously used for however, is presentational rather than representational; they are fine for building "complicated" and elegant patterns, some of which may happen along the way to raise some very nice questions of arithmetic or geometry. We assign a sort of figure of merit to these commercial materials, the ratio of their unintended usefulness to that intended. In these unintended uses children show you some of the rich turf you have to learn. Natural materials — such as mud-cracks and growth patterns — are often better.

It has been a great help and moral support to me in this work to realize that much of early Greek geometry and arithmetic was developed by the use of Cuisenaire rods. John Trivett and his students found they could represent the sum of successive squares by a rectangle built of rods, but only if they combined three such sequences together. He told me that he then really *understood* why that algebraic formula had a six in the denominator! They had in fact rediscovered a theorem of the Pythagoreans, who also extended the method to find the sum of cubes. I gave these

*to which I hope she would agree, cf. Michener, E., "Understanding Understanding Mathematics," *Cognitive Science* 2, 361-383 (1978), and for comparison: Hawkins, D. "Understanding the Understanding of Children," *American Journal of Diseases of Children*, vol 119 (Nov. 1967), also in *The Informed Vision*, New York: Agathon Press, 1974 problems once to a class of students in an Analysis course, and only four solved them all, using analytical methods I regard such students as under-privileged.

When you are first inventing geometry (or is this number theory?) you don't use standard methods You have to develop your understanding

One of the results of working in this style is that you get into mathematics, not just computational routine The computation comes along, and quite a bit of it could be called practice. In finding tetrahedral numbers, the sum of successive triangular numbers, you can do a lot of sums before you see the pattern And when you encounter those sums again in random walk investigations, also after much numerical calculation you are on the edge of a deeper understanding I emphasize the computational aspect because it is, among other things, of some importance; but also because if you thought someone on the school board would regard your work as time-wasting play, you could point with pride, inter alia, to the number skills.

My purpose has been to try to map a useful and plausible account of the structure of adult mathematics into the childhood milieu. If that doesn't work, something is wrong with the account itself. If it does work, it still may qualify the description of the structure of mathematics and mathematical understanding If my interpretation of the story of Suzie — and we all know other such stories — is correct, we should look more carefully at the ways in which the domain of mathematics, which in some essential sense is discursive, symbolic, digital in its mode of expression, nevertheless is linked to that which is perceptual, presentational, implicit. Such a linkage is required, I suggest, by any account of the historical origins of mathematics *or* of its successful pedagogy. I think it is also needed in any account of later major developments within mathematics itself. If one traces the origins — the originals — of such development, one finds that they very often turn upon some fresh success in discursive explication of the perceptual and intuitive Greek geometry surely depended on such an explication. Its axioms, once explicated, were "evident." The parallel postulate in particular is an explication of the perceptual symmetries of certain lines and angles, themselves defined by symmetries. Archimedes introduced novelty by his derivation of the hidden symmetry of the law of moments from the intuitive symmetry of the equal-arm balance, and used this as a new tool of investigation in geometry. Bernoulli appealed to the intuitive symmetries of gambling devices, and from this derived his famous theorem, probably the first major mathematical step beyond the practical lore of gamblers. Connected to the ideas of groups and invariance, the symmetry principle became the basis for whole new developments in geometry and also in theoretical physics Still more recently it has legitimated the use of probability theory within number theory and in geometry itself.

If one takes such a series of examples of the way in which our understanding can sometimes be mathematized, one is — or can be — tempted to return to those older rationalistic philosophies which I have mentioned earlier. They reserved a place, at least, for the notion that at any given stage in its career the mind possesses some furniture which is in no obvious way simply the outcome of empirical induction, but which it can bring *to* any new experience as a means for reducing that experience to order, of reducing the apparent redundancy of experience. If one does not like the classical rationalism with their appeal to innate ideas, one can try the move initiated by Kant and treated developmentally by Piaget

If one is temperamentally suspicious of all such grand philosophical moves, however, there is another track to try to follow, more modest and in its own way empirical The one example I have suggested is a kind of examination, from historical and contemporary sources, of the ways in which arguments from symmetry have contributed to the mathematizing of otherwise only empirical subject matter. I have mentioned examples from Suzie (and Euclid), from Archimedes, from Bernoulli and those who followed, from the Erlangen program, and — most recent — from the way in which Buffon's ideas of geometrical probability have been taken out of the limbo of mere empiricism by the insight, first apparently voiced by Poincaré, that a mathematically adequate definition of geometrical probability emerges from the choice of that measure which is invariant up to an appropriate group of geometrical transformations The development of this program has contributed to new extensions of geometry itself, as well as to many practical applications in stereometry, etc.

I could mention also the many fascinating examples of symmetry and invariance which have been first postulated, in an apparently high-handed a priori fashion, by theoretical physicists, and often enough (though not in every case!) empirically confirmed Rather recently, it seems, good formal arguments have been developed which derive the classical conservation laws from the symmetries of space I don't understand these arguments yet, but they seem at first sight to be legerdemain at variance with good old-fashioned empiricism At any rate and in the meantime this whole history seems to suggest something important about the process of mathematizing, at least one long and tough thread of continuity between what Suzie knew and the most recent higher development of some parts of mathematics and physics. I don't know how to assess this, though it would have delighted the hearts of the old rationalists. If they are wrong we need to find some adequate account of such matters, one which among other things might be pedagogically important

I think in fact the old rationalists were wrong, though the standard empiricism is wrong too Even along the one line of continuity I have suggested there are incursions of novelty into the development of geometry, modification, extensions of, and attacks upon, preconception The most cherished intuitions, once axiomatized, are open to revision. This is very far from saying they are arbitrary; the intuitive symmetries can be played with, and new pathways explored, the old symmetries subtly modified, as in non-Euclidean geometry Some "firsts" are so important we should learn to be playful about them — but not before we understand their power