Integrating Theories for Mathematics Education*

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1. An actual historical perspective

> It is not at all true that concepts, even when constructed according to the rules of science, get their authority uniquely from their objective value. It is not enough that they be true to get believed If they are not in harmony with the other beliefs and opinions, or, in a word, with the mass of the other collective representations (the concepts taken for granted by most people in a given time and place), they will be denied; minds will be closed to them; consequently it will be as though they did not exist. [Durkheim, 1912, Engl. translation 1965, p. 486] [1]

In the journal *Educational Researcher*, S I. Miller and M. Fredericks [1991, p. 3] state: "The major concepts of the new philosophy of science are, at best, only marginally relevant to many of the issues studied by educational researchers". Moreover, they lament "the related problem of ambiguity of how these terms are to be applied *exactly* to the field of educational research" [p. 2, my emphasis].

My conviction lies in the opposite position: it appears to be difficult to *over*-estimate the importance of fundamental orientations. This is the case because such basic professional attitudes and convictions as Durkheim had in mind function as part of the researchers' "*epoché* of the natural attitude" or "the method of phenomenological reduction" [Schütz, 1975]. In other words, they are implicitly subsumed, they are taken for granted, and they are very powerful in rejecting any proposed alternative.[2]

For a long time, the mainstream explanations in mathematics education have rested upon cognitive psychology or cognitive science (as it now calls itself), and not solely in the United States Aside from that, there are only a few thin streams of alternately-used theoretical bases. And these have been almost totally neglected, at least by the majority of the mathematics education community — no citations, either of names or of ideas! Now, from completely different backgrounds, a strong ally has appeared on the scene

Approaches which appear under the labels of "parallel distributed processing", "neural networks", or "connectionism" began in principle to form another critical alternative to cognitive science From a thin, isolated stream of research since the 1960s, a positive discussion of the related *connectionist models* of the human mind has rapidly developed, particularly with the two books by McClelland and Rumelhart [1986, 1988] and with Smolensky's article [1988]. In particular, many philosophers of science have somehow become "enthusiastic" in their acceptance of these functionalistic interpretations.[3] In any case, the importance for mathematics education is with the explanatory power and related educational inferences that become possible with these models. The (controversial) details of actual, technical realisations of the models are not of interest here.

It is quite surprising that the growing difficulties with computer simulations of human communication and cognition, combined with the strong actual interest in adequate solutions, have led to the development of new and challenging models. Obviously computer science, education, and philosophical discussions are at present nearer to each other than ever before. Even more interesting and surprising is the relative convergence of these technologically-orientated approaches with a few older and more developed theoretical approaches from different disciplines, where they also have been formed mostly to one side of the mainstreams in their related disciplines, e.g. (pragmatic, linguistics, (radical) constructivism, ethnomethodology, social (or earlier symbolic) interactionism, history and theories of science, and last but not least new perspectives on mathematics itself (see the very detailed overview in Ernest [1991]

2. Two recent analyses

> Explicit rules might play a part in learning to think, but (as suggested by the long history of failure of instruction in logic to improve thinking) a very limited one. [...][The rule-based] family of instructional theories has produced an abundance of technology on an illusory psychological foundation. [Bereiter, 1991, p 14]

Limiting himself to "the contributing fields to the Science and Technology of Cognition", Francesco Varela has recently described this development as comprising three successive waves in the main [1990, p 26/8][4], moving:

- from *representation* and *symbol processing*, where symbol processing is both based upon *sequential rules* and is *located* within the system — what he calls the "two widely acknowledged deficiencies of cognitivism" since "the loss or malfunction of a part of the symbols or rules of the system results in a serious malfunction" [p. 56/27], i.e. COGNITIVIST PARADIGM [p. 27/9];
- towards "*emergence*: alternatives to symbol manipulation" [p. 27/9], where meaning resides in the function of the whole state of a network, rather than being localisable in certain symbols or areas, i.e. the CONNECTIONIST PARADIGM [9. 79/41];

and further on:

• towards "*enaction*: alternatives to representation" [p. 27/9]. where the phenomenon of interpretation is "understood as the circular activity linking action and knowledge, knower and known in an indissociable circle. ..with the dominance of usage, instead of representations", i.e. ENACTIVE APPROACH [p. 91/49].

Varela has organised his overview in a "polarized map" [p. 119/66] in which each subsequent wave includes the preceding one like a set of "Chinese boxes".

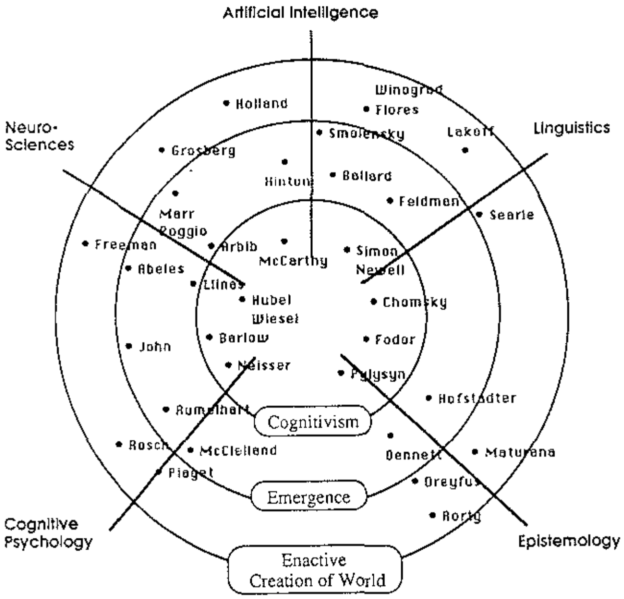

Figure 1
The "polarized map of Science and Technology of Cognition", from F. J. Varela, *Cognitive science, a cartography of current ideas* 1988

He writes: "The centrifugal direction is a progressive bracketing of what seems stable and regular", and "one can go from enaction to a standard connectionist view by assuming given regularities of the domain where the system operates". Whilst in the centripetal direction, "one goes from emergence to symbolic by bracketing the base from which symbols emerge, and working with symbols at face value". Varela insists the notions in the table "should not be seen as logical (or dialectical) opposites. They represent more the particular and the general, the local and the more encompassing category" [p. 120/66]

From a philosophical perspective, Richard Rorty [1991] has recently pointed to the drastic change which the idea of language has undergone: from our treating language as a limited object and taking words as carriers of meaning, towards a pragmatic "bottomless" stance, a viewpoint from which the following have to be abandoned

• The concept of *meaning*, since language cannot serve as an instrument for objective reductions: "the death of the concept of meaning" [Hacking, 1975] is "the end of the attempt to make language a transcendental topic" [Rorty 1991, p. 50]. There is an unlimited specificity of "meaning" in both the speaker's and the listener's perception of the actual situation.

• The distinction between "the formal and the material", for "a plurality of forms of experience or forms of consciousness looks much like a plurality of actualities" [Rorty, 1991, p. 53]. Or as Varela has described it, the separation between "form" and "meaning": "the master stroke that created the cognitivist approach" [1990, p. 78/40]

On the other hand, Rorty accepts the following

• In accordance with the late Wittengstein: "whether a sentence had sense did indeed depend upon whether another sentence was true — a sentence about the social practice of people who used the marks and noises" [1991, p. 57]. In other words, language is "just a set of indefinitely expansible practices", rather than "a bounded whole whose periphery might be "shown"" [Rorty, 1991, p. 57]: related to language use, he claims "social practice" to be "the presupposition of the demand for exactness" [p. 60]

• In accordance with the young Heidegger: "the primordiality of the *Zuhanden*", which means that "the present-at-hand was only available in the context of pre-existent relations with the ready-to-hand" [Rorty, 1991, p. 60], a conviction which the late Wittgenstein shared,[5] and which Roland Barthes has expressed as a situation where "the only given is the way of taking" [6]

3. An attempt at integration

If the kingpin of cognition is its capacity for bringing forth meaning, then information is not pre-established as a given order, but it amounts to regularities that emerge from the cognitive activities themselves. [Varela, 1990, p. 121/66]

The outlined positions are very near to the radical constructivist principle [von Glasersfeld, 1991], as well as to fundamental pragmatic linguists' or social interactionists' theses [Mehan and Wood, 1975; Walkerdine, 1988; Coulter, 1990], to fundamentals of discourse analysis [Cazden 1986], and to certain perspectives of systems theory approaches [Luhmann, 1990; Maturana and Varela, 1986] One cannot expect to identify clear boundaries for the region of convergence at this level of abstractness. But it appears to be possible to enlist a few shared core convictions in this area (The descriptors used will present a mixture, just because it is impossible to describe the deficient parts of an approach with the specific "language game" of the very same approach.)

3.1 *Learning* is a process of personal life forming, a process of an interactive adapting to a culture through active participation (which in parallel also produces and develops the culture itself), rather than a transmission of norms, knowledge and objectifed items.

3.2 *Meaning* lies with the use of words, sentences, or signs and symbols rather than in the related sounds, signs or pictures, or even in a related set of such items.

3.3 *Languaging* (the French term *parole* as distinct from that of *langage*) is a social practice, serving in communication for pointing at shared experiences and for orientation in the same culture, rather than as an instrument for the direct transportation of sense or as a carrier of attached meanings.

3.4 *Knowing or remembering something* denotes an actual activation of options from experienced actions rather than a storable, treatable, and retrievable object-like item, called *knowledge*, from a loft, called *memory*.

3.5 *Mathematising* is a practice based on social conventions rather than the applying of an universally applicable set of eternal truths; according to Davis and Hersh [1980], this holds for mathematics itself.

3.6 (Internal) *Representations* are individual constructs, emerging through social interaction as a viable balance between the person's actual interests and her realised constraints, rather than an internal one-to-one mapping of something pre-given or a fitting re-construction of "the" world.

3.7 *Using visualisations and embodiments* with the related intention of using them as didactical means depends on taken-as-shared social conventions in classroom practice rather than on a plain reading or the discovering of inherent or inbuilt mathematical structures and meanings.

3.8 *Teaching* is the attempt to organise an interactive and reflexive process, with the teacher engaging in a constantly continuing and mutual differentiating and actualising of activities with the students, and thus the establishing and maintaining of a classroom "culture", rather than the transmission, introduction, or even re-discovery of pre-given and objectively codified knowledge

The notion of an "integrating perspective" in the following will refer to this set of core convictions It is quite challenging to extend such an integrated perspective into possible didactical considerations — no inferences, clearly, since what we have mostly refers to single theories, if anything at all. At least in the interest of students and teachers, such an attempt appears to be as necessary as the bustling discussion of compatibilities, of the drawing of boundaries, and of attempting to decide the relative dominance of one model over another.

4. The culture of a mathematics classroom

> [Even] smell, as is the case of color, reveals itself not as a passive mapping of external traits, but as the creative dimensioning of meaning on the basis of history. [Varela, 1990, p. 109/59]

It is neither by chance nor solely an act of preserving neutrality that the NCTM [7] did not produce more detailed criteria for the recommended actions of teachers than: "...in ways that facilitate students' learning" and "... providing a context that encourages..." and "... necessary to explore sound mathematics". But how is one to decide about the facilitating of learning without a pragmatic theoretical model for it? What is an *encouraging* context? And what is necessary to be able to *explore* mathematics? Among mathematics educators, I think, there is an increasing awareness of the need for more developed *theoretical* bases for the teaching and learning of mathematics. And this is an international phenomenon The thematic orientation of many conferences over the last few years speaks in favour of this assumption. In the following I shall try to draw some general inferences from the integrating perspective for mathematics education. Clearly the outcomes cannot represent more than tacit and preliminary working hypotheses.

The radical constructive principle says, in essence, that every cognitive construction is not passively received but (a) a person's individual construction, and (b) results from adapting to constraints and acting upon challenges [von Glasersfeld, 1987 and 1991]. Accordingly, as a prerequisite of schools' adequate functioning, the student will have to find her-/himself in a situation which inspires and promotes engaged personal activities, which enacts an effective — and not necessarily overt — interactive control over the students' adaptive processes (as well as, reflexively, over the teacher's inventions), and which potentially goes beyond what is already available In particular, the student will need ample opportunities to develop, present, and discuss her/his personal constructions and ideas, and to check their convergence; that is what the expression "negotiation of meaning" stands for.

From a social interactionist or ethnomethodological perspective [e.g. Mehan and Wood, 1975; Erickson, 1986; Cobb, 1990] teaching/learning processes in the classroom appear to be mainly social processes. Only the active participation in the related language game and the permanent negotiation of meaning can lead to the constitution or accomplishment of taken-as-shared meanings for actions, symbols, and objects. Regulations and norms will emerge from the permanent interaction between teacher and students, as well as among the students themselves, rather than from an explicit talking about something without living it. (I do not want to go deeper here into consequences from other related fields — see, for example, Bauersfeld [1990] and [1991].)

What connectionist views can add to these perspectives is the more detailed modelling of the *personal* processes of the emergence of acceptable actions — from simple skills to any "higher order" thinking and meta-structures of mental organisation — through the formation of fluently functioning and weighted connections in the brain's large network. Connectionist models can do best:

> what people do best — recognise patterns and similarities. They work in the messy, bottom-up way that nature seems bound to They approximate rather than embody rationality. In a very natural way, they model the gradual transition from vagueness to clarity, from uncertainty to decision, that characterizes much of human thought and understanding Whereas rule-based systems tend to be helpless when presented with situations where their rules do not fit. [Bereiter, 1991, p. 13]

Of particular interest for the organisation of the mathematics classroom is the question of *how* rationality develops in the student. From a connectionist viewpoint there is no principled difference between the forming of scientific rationality versus everyday rationalities.

> That private thought conforms to public standards of rationality is conventionally conceived of as internalising a set of rules. From a connectionist viewpoint, this concern errs on both sides — in assuming that public rationality is based on rules and that individual cognition is as well. The development of personal rationality is better conceived of as the tuning of a massive network so that its outputs achieve an increasingly fine fit to what is publicly justifiable [Bereiter, 1991, p. 14].

From this it should be clear that single lessons on the objectivised and isolatedly thematic issues — like strategies, reflection, metacognition, etc. — can hardly provide the necessary support for related learning. What opportunities do students have to develop abilities in arguing, making inferences or coming to adequate decisions in mathematics, as well as occasions for self-reflection, if such issues are not an *integrated part* of *regular* classroom *processes*?

The metaphor of "tuning" used in the Bereiter quotation again points to the crucial role of the person's relation to something outside of the person, what cognitivists may term "environment", "context", etc., thus objectivising forces which again function mentally only via the person's interpretations and ascriptions, and which therefore can be described more adequately from a sociological perspective. In ethnomethodology and social interactionism concepts such as "social interaction" [8], "negotiation of meaning", the "constitution of common activities", the "emergence of regulations", the "accomplishment of taken-as-shared norms", and the "reflexivity" and "indexicality" of the interactive processes in the classroom are used [see, for instance, Mehan and Wood, 1975; Erickson, 1986; Cobb, 1990] and, last but not least, the "language game" which is more or less specific to each classroom, even in mathematics.

The general possibility of taking the whole affective domain into account, as well as the experience of one's own body, marks another advantage of connectionism and provides a better basis for integrated theories. "The ability of connectionist models to incorporate feeling into cognition may eventually prove to be decisive in their competition with rule-based models." [Bereiter, 1991, p. 13] This is indeed an essential extension if learning is to be understood as a holistic process: in any situation, all of the senses are involved. There is no deliberate switching off (apart from rare exceptions). One never learns solely cognitively, or auditively, or visually, or physically. The cognitivists' growing interest in "embodied cognition" [Johnson, 1987] indicates an increased realisation of deficits in this direction.

With all this we are very near to forming an analogy between classroom realities and the functioning of a (sub-) culture. Both concern the person as a whole. Both are permanently changing and developing microworlds, intimately interrelated and intertwined with the change and the mutual development of their participants. Both undergo the impact of more powerful societal forces, and both are limited in time. Therefore, I like to speak of the *culture of a mathematics classroom*. This concept of "culture" is very close to the one described by Michelle Rosaldo, a student of the anthropologist Clifford Geertz: "A matter less of artifacts and propositions, rules, schematic programs, or beliefs, than of associative chains and images that what can be reasonably linked up with what. [...] Its truth resides not in explicit formulations of the rituals of daily life, but in the daily practices of persons who in acting take for granted an account of who they are and how to understand their fellows' moves" [cited in Bruner and Haste, 1987, p. 90].

I prefer the notion of culture for the processes under discussion, not least because of the connotatively-related dimensions of time and history. Cultures are permanently developing, reproducing and renewing themselves conjointly. One can become a member of a culture through active participation only; it involves a process of adaption and co-operation. Most of what is learned in terms of acceptability, validity, norms, languaging, and even personal identity is learned by the way, implicitly; it emerges in the interaction. the ever-historical result on the person's side is something like a *habitus* (a structuring, "structure-generating mechanism"). Particularising Bourdieu's notion [1990], I speak of the *school mathematical habitus* of a student. And the result on the social side is the practice of a living culture, the structures and regulations which a member lives but rarely reflects upon, and which only an informed observer can model and describe.

The crucial point with the functioning of the classroom culture seems to be the nature of the related teacher-student interactions. How seriously does the teacher her-/himself take the subject matter that is to be taught? To what extent does the teacher "live" the mathematising required and the interrelated social virtues like being a model for any other serious member of the (school) mathematics society? Students, I am very sure about this, have a very sensitive perception of the teacher's concerns and thoroughness. It depends upon the teacher's aptitudes and the whole person's engagement to the degree to which this "as if" microworld becomes a culture of prime importance for the mathematical development of the student.

5. Characteristics of alternative classroom cultures

> I am fully convinced that a mere mechanical facility in manipulating figures, sufficient though it may be for the calculation necessary in everyday life, is in no way conducive to a healthy development of the reasoning faculty. [Chakvavarti, 1890, p. 1, preface to 1st edition][9]

Are there possible particular and more concrete orientations which could be related to the outlined fundamental changes towards a more integrated theoretical basis for mathematics education? The characteristics offered in the following are not at all new. But their combination, perhaps, may mark another design of what mathematics education — actually, in this historical situation — can be and what alternative approaches might look like. In keeping with the main area of my own empirical work, I shall limit the examples to the early school years.

5.1 *Fundamental attitudes*
If mental development rests upon the student's repeated and engaged personal activities on the one hand, and on the potential power and richness of the culture of the mathematical classroom on the other, then the permanent support of an *attitude of curiosity*, of *inquisitiveness*, of searching for pattern and regularities, of expecting to find surprising issues, appear to be markedly helpful. The English "investigations" approach [see, for example, Banwell *et al*, 1986; Burton, 1984; Mason, Burton and Stacey, 1982; Mottershead, 1978 and 1985] and the many contributions of the Freudenthal Institute in Utrecht [see, for instance, the journal *Willem Bartjens*], using open-ended tasks and providing for (relatively) self-organised processes of searching for solutions, often in small groups of students, offer promising realisations.

Let me give an example of how to challenge such attitudes. My class [10] is accustomed to beginning each mathematics lesson with mental arithmetic and geometry. One day, early in third grade, I started with a series of "number shacks" with two given numbers on the first floor of the first shack. Their sum is to appear on the ground floor and their difference in the roof. And these two computed results then fill the first floor of the next shack, and so on.

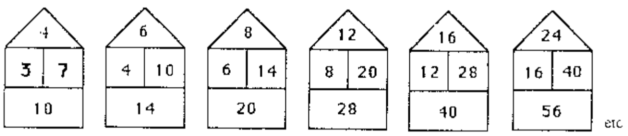

Figure 2
Sum- and difference-shacks for mental arithmetic
[see Wittmann and Müller 1990, pp. 102-103]

The students soon caught on to this simple procedure and began to fill the houses, and a few even tried other starting numbers. On one occasion the teacher intervened and asked for predictions about the numbers for the next shack's roof and ground floor — *without* doing the required calculations. Students swiftly came to see the 4, 6, 8 sequence in the roof and expected 10 to be the next result. The surprise of "12" led to different assumptions, in particular, when the bottom numbers were also taken into consideration later. In the following, the doubling pattern in each next-but-one shack became obvious as more of the shacks were completed. Stephen Brown and Marion Walter's excellent book [1983] on the variation of problems made me ask for other possible types of completion.

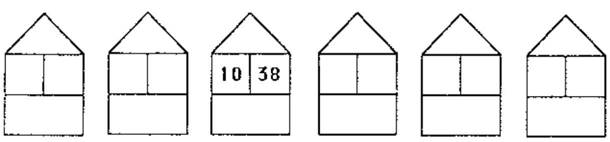

Figure 3
Can you complete this sequence of houses
forwards and backwards?

The students took this idea up and tried other patterns to start with. They did a lot of experimenting and calculating with these shacks, at home as well, coming up with different ideas in the next lessons also: "I've got a new one!". Over a week or so they were keen to find new patterns, especially in the quite different situation of routinely solving the rather boring sets of calculation tasks in their textbook. From this, with each subsequent piece of mental arithmetic (but soon with geometry also), the students searched for similar phenomena and patterns. The quicker students were the first ones to try their developing aptitude in other situations as well, which we appreciated and encouraged as creative insertions.

What is different when compared with earlier "discovery" approaches? It is the shift from rare and inserted exceptional situations towards a permanent attitude and an integrated part of life in the mathematics classroom.

> *Discovery approach*
> In explicitly defined situations, the student "researcher" starts off from an introduction to working on prepared material, and finally ends up discussing and sharing the findings in a whole-class session.
>
> *Integrated (culture) approach*
> In *every* classroom situation, the students are expected to search for patterns, to assume regularities, and to relate developing or contrasting ideas, as well as to give reasons and arguments for the issue under discussion.

Aside from this, there are fundamental doubts, which are not discussed here in detail, about what the notion of "discover" can describe at all from a constructivist perspective [see Bauersfeld, 1991, for details].

5.2 *Language, languaging and the teacher*
In a narrow interchange with the described attitudes, views about language will also have to undergo change. "Learning how to use language involves both learning the culture and learning how to express intentions in congruence with the culture" [Bruner and Haste, 1987, p. 89]. And "one has to conclude that the subtle and systematic basis upon which linguistic reference itself rests must reflect a natural organization of mind, one into which we *grow* through experience rather than one we achieve by learning" [ibid., p. 88; emphasis in the original].

For many teachers the strength and the generalisability of mathematics is inseparable from the strictness and the precision of the related verbal or other symbolic representations. Similar to priests who celebrate the esoteric language game of their caste, many mathematics teachers permanently insist on saying things as precisely as possible. An observer may find the teacher insists on this technical language. For the students, the emphasis functions as a requirement to say it exactly "as she/he said it".

One may wonder whether many teachers "have it" at all in any other way. That is to say, they know how to talk about "it" in the terminology of the accepted language game but there seems to be not much more beyond this, as the limited availability in other "contexts", the difficulties and shakiness of use in other situations suggests, and is also indicated by an inability to find adequate metaphors for the issues.

To be fair, nobody has trained them in the initial phases to speak about the intended subject matter in everyday language, to "point at" similar issues, etc (Cognitivists may prefer descriptions like: they cannot "translate", or "say it in other words", they cannot "embed" or "visualise", or "refer it to", thus treating what is meant as an object rather than as something emerging from the actually situated processes.) In consequence, many mathematics teachers are quite rigid in their *verbal* aspirations and their related evaluations of students' utterances. But they are quite permissive in the *social* organisation of their class. Under the integrating perspective the opposite way round appears to be a more promising one: to accept and encourage students' mathematical utterances within very wide limits with respect to how it is said, as long as a serious background (reason, argument, etc.) can be identified. But to be rigid about keeping the social regulations, namely, insisting on listening to others' inventions and explanations, keeping turn-taking order, taking seriously the others' serious contributions, etc. [11]

Analysing many videotapes has convinced me of the all-too-general poverty of classroom communication with respect to this view (in many countries, by the way). If the culture the students inhabit in the classroom is poor in languaging and in presenting models of what is wanted, if it is lacking incentives and challenges, if it is more a non-transparent celebration of technical language rather than a participation in a scaffolding [12] culture, and if it is neither providing resistance to the critical mind nor further orientation for the keen-minded, what then are we to expect from our schools?

A counter-example [13] may demonstrate what I am talking about here. Many years ago, during teaching practice with my intending teachers, I observed a lesson in which a young teacher tried to introduce sixth graders to the characteristics of reflections: in particular, the relations between original and image elements. He had followed recommendations for the use of a vertically-fixed glass pane in a dark room and a lighted candle placed in front of it, asking two students for support: one to move a second lit candle behind the pane, following the directions of the other student who was placed in front of the arrangement. The latter had to try to make the second candle move to a position where both the image of the first candle and the actual second candle would come into coincidence.

Unfortunately this teacher talked about the arrangement but had not prepared for actually doing it. The students ran into difficulties. They would not believe that one could see three candles: the two originals and the image of the candle in front. "This is impossible! Either you can *look through* the glass, and then there is no candle behind!" The poor teacher ended the situation, shrugging his shoulders in desperation and saying: "OK, mathematicians use these words in order to visualise the relations!"

5.3 *Problems as developing processes*

Teachers usually treat mathematical tasks and problems as objects, as carriers of a more or less well-defined enigma to be solved. Mostly tasks are "given" tasks (with the exception of the few problems the students get a chance to define on their own. The students are expected to understand the text problem, transform it into a mathematically-tractable form and solve it. What happens in many cases is that in an obscure and poorly-controlled process, *prima facie* associations lead directly to calculations and results, both by following frequently-used paths of related activities and by applying associated procedural skills — a perfect example for a connectionist model! Consequently, the students *learn* (= covertly develop fluent connections) to treat the tasks as "given" ones, expecting everything one needs to know to be "in it". Tasks fall into two classes: "known" and "unknown" problems. It becomes a case of activating ready-at-hand methods rather than a case of an active and explicit construction of meaning, of a reflected-upon selection among possible alternatives, and *after* those, of calculation and checking.

If the individual student's adaptation to the approach favoured by the teacher occurs only across the frequent "right" or "wrong" evaluations and remains restricted merely to a discussion of technical solution procedures (operations, order, writing schemes, etc.), then students will have no chance to develop better strategies, a more sophisticated self-awareness and self-control over their fundamental processes of ascribing and formatting mathematical meanings. The technical solution procedures dismiss the vulnerable, tacit production of helpful ideas and, in the end, they replace them by the drilled fluency of current solution techniques. But these techniques:

- are bound to narrow classes of "problems" and to the specificities of the "presentation" of the tasks;
- get lost, if not trained permanently;
- resist extensions and generalisations.

The often-bemoaned mathematical "inferiority syndrome" (or complex) — "I wasn't any good at maths!" (nobody would admit this for her or his mother tongue!) — may have part of its origin here. The process-oriented, integrated view realises problems as *always problem forming* [14] "Precisely the greatest ability of all living cognition is ... to *pose* the relevant issues to be addressed at each moment of our life. They are not pre-given, but *enacted* or *brought forth* from a background, and what counts as relevant is what our common-sense sanctions as such." [Varela, 1990, p. 90, emphases in the original]. According to the radical constructivist principle, the student develops her/his own sense related to the symbols, texts, or pictures offered by the teacher or the textbook during the solution process. And every step and every decision taken in the process of dealing with "the problem" changes the issue. What in the end the problem has been for the individual is open to an interpretative reconstruction from step to step in retrospect only. It is a kind of a biography of this "problem" related to this specific "solver".

These tacit and obscure processes of creating and selecting are developed across — or, better, *emerge* from — related classroom interactions. During the first years at school, students' participation in the classroom culture already leads to the emergence of a typical school mathematical *habitus*, which enables them somehow to produce acceptable solutions. But obviously, in regular classrooms, these covert processes are very rarely touched on and reshaped explicitly.

An experience from "my class" may illustrate some of the needs involved. In third grade, the teacher writes on the blackboard:

> Sandra, Peggy and Martin go to town by tram. They want to buy a birthday present for Jan.

and asks: "Anything else you want to know?"
— "How much is a ticket?"
The teacher writes: "One ticket costs 2 DM."
Suddenly, a vivid discussion arises.
— "That's too much!"
— "We go for 1.20 DM!"
— "No, they cost 1.60 DM!"
— "Nonsense, you buy a card with six, then it's cheaper!"
and so on.

The students are obviously well informed about tram prices. The teacher interrupts: "Let's calculate with 2 DM per ticket". The next surprise is the children say: "Aha, then, Sandra, Peggy and Martin buy their own tickets...", and that is it. For many students, there is no need for addition. You just pay, surely, everyone for her/himself. This is what everyone does in the tram. They never add up their payments!

The teacher again reads the task, but with a slight change in the wording: "... They want to buy a birthday present for *their brother* Jan." (This was to be the planned, original text of the task, but the teacher has omitted the key words when writing the problem on the blackboard.) This, indeed, changes the whole problem for the children: "Aha, the children are siblings!" "Oh, and then mother will clearly have to give them money for the tram!" "That is 6 DM" (A few arrive at 8 DM, probably including Jan.) Then, most students find they have done it.
— "Next task."
— "Now, how do they get home from town once they bought the present?!"
— "Aha!"
And, after an additional discussion of the possibility of a cheaper return ticket, we all finally arrive at 12 DM for the adventure in town.

The example demonstrates to what extent minor variations can change the individual understanding of a text. Moreover, and more importantly, it demonstrates how necessary discussions are for an extended clearing-up of individual understandings and for the negotiation of the actually-constructed prerequisites, methods, and strategies for solution. An adequate use of non-standard problems and of open tasks can help to start related discussions, and to encourage students to unveil their ascriptions and interpretations.

It may be useful to pay more attention to these tacit and usually covert processes. They should undergo more overt demonstration in the classroom, and — as far as possible — discussion and negotiation, thus commencing and developing another language game, rich in metaphors and open to manifold supportive associations and analogies. This is to avoid misunderstandings and to resist the easy reproduction of the usual instructional methods applied to new content only (I do not speak of "teaching" such ideas and "talking about" alternative constructions here). The *enacting* of principles and decisions, the *living* of a culture, appear to be helpful alternative models. Every communication functions only across a shared practice. Likewise, effective enculturation in the classroom functions across active participation, doing it yourself, realising others do "it", and communicating about the "it" as well as the "doing", negotiating ideas and what one thinks one has just learned. Insight and understanding will emerge only from a practice.

Let me give another example from my class. Late in first grade the textbooks begin to present pictures from everyday scenes as story problems. They are treated as the first steps of an introduction to the solution of text problems. Usually such "picture problems" appear right after an elaborated section on addition or subtraction. And teachers invest much effort in making students "read" the expected number sentence into such pictures. In a mathematics lesson, all students around the world, I am sure, encountering a picture with three birds sitting on a roof and two others approaching them in flight will react by saying "3 + 2 = 5". Or, in the case where the two birds are flying away, the answer will be: "5 - 2 = 3".

That this one-to-one relation between a pictorial presentation and a number sentence is merely a social convention (and *not* an objective truth) becomes clear when, before such "introductions" come to deform the minds, students get the chance to comment on the pictures [for more details, see Bauersfeld, 199]. With my class I have tried to organise some training in the creation of more serious and reflective mathematical interpretations — "mathematisings" — of a picture. And once the matter was opened to the floor, a rich variety of number sentences and related reasons (no acceptance without a reason being provided!) came about for the same picture.

Most helpfully, the students interactively varied each other's interpretations. They competed with new relations (number sentences) and new arguments. It became very clear that each mathematisation of a picture, of a text, etc., depends upon the analysing person's actual interests: What do you want to do or to know? This, I find, has helped a lot with the later interpretation of text problems, when students, on the way to producing acceptable solutions, also discussed the sense of exotic interpretations and the quality of others' arguments and ideas. I am happy to realise that my students have recently begun to turn my perpetual "How did you come to think that?!" against myself.

5.4 *The horizon of criteria: deficit repair versus constructive orientation*

Inevitably teachers realise a student's mathematically-inadequate action is "wrong" since their professionally-developed mathematical expectation functions as a criterion. Consequently, but didactically counter-functionally in most cases, their reaction aims to reduce the difference between actual performance and perfect fulfillment to zero. Often the questioning goes on and on — the "funnel pattern" of interaction is just around the corner [Bauersfeld, 1978] — until the expected answer is heard, either from the student or from the teacher her/himself

Using connectionist models we can understand the student's mathematically "wrong" interpretation as the activation of a ready-at-hand option for action. Under the actual pressures of the situation the abandoning of the activated option and the activation of another option (i e. slipping into another fluent connection in the network) takes explicit effort: in cognitivist terms, "transfer" or "creative production". This becomes less and less probable the more the teacher forces the student into a repair of the activated option. Moreover, the expected change requires a degree of differentiation and explanation at this end of the domain which the student never had to produce before (since the option just functioned!). In the end the related network connections are left in a mixed-up state, laden with negative emotions and unstable changes.

Why not replace the *deficit criterion* by a *constructive orientation*? Alternatively, the teacher would identify the positive or convergent elements in the student's actual production and change the actual task accordingly so that the student finds a chance to extend the use of the positive elements (reinforce the positively working partial connections). Adapting the actual tasks towards a possible extension of what the student can already do more or less successfully (Vygotsky's "zone of proximal development") will surely function less frustratingly and more encouragingly than the usual forcing into contradictions and the continued "squeezing" questioning. Furthermore it carries the potential for constituting "better" connections by succeeding at accepted activities.

5.5 *Taboos, theory and rules*

Classroom taboos are among the least discussed issues in mathematics education. But they comprise some of the most effective forces in classroom realities "Never tell a student what he can find out by himself" is anchored as a guiding principle in many German syllabuses, the state regulations for mathematics instruction at school. In-service also teacher training institutions disseminate this principle; presumably in North America as well There is a clear relation to the suspect issue of "discovery" [see section 3 1, and Bauersfeld, 1991].

A related issue is the conviction that students' mathematical errors are mostly caused by a strategy or a rule. (As linguists say: "in human communication nothing occurs by chance".) The usual attempt at repair is to replace the faulty rule by an adequate one From a connectionist perspective "the computational algorithms, the things that generate thought, are not anything like rules of logic. They are, rather, algorithms for constraint satisfaction." [Bereiter, 1991, p 14] In other words, if a student acts in a specific situation as if he had followed a rule, then this will be an indicator of a developed network connection functioning towards the fluency of repeating a successful action (successful before, perhaps, in slightly different contexts), rather than an outcome of the conscious knowing, selecting, and applying of a rule. Since there is no explicit rule in the game, and since there is no chance of a direct adoption or "internationalisation" of another rule, the idea of repair — even if the teacher's invention seems to end up successfully — appears to be an illusion.

Bereiter points to several examples from research on instruction "indicating that the rules students learn are not the same as the rules they are taught" and he asks consistently: "If rules are useful in teaching but are not what students actually learn, how are we to make sense of their function?" [Bereiter, 1991, p. 14]. Referring to experiments by Magdalene Lampert [1988], his answer is very near to the notion of classroom culture used here: "Instead of concentrating on getting rules into the minds of the students, the teacher uses *rules as a way of representing* and talking about mathematics and *encourages the students to do likewise*" [Bereiter, 1991, p. 15; my emphases].

Judged by his earlier writings Carl Bereiter has apparently changed his own position quite radically: "The intelligence of the system, which achieves the sense making, is a property of the network and not of the rule-based mechanisms that subserve it" [ibid., p. 15]. Moreover, "The classical rule-based view of rationality enjoys such prestige that when we think of actual thought as an approximation, we tend to assume it is an inferior approximation. Although this is surely true on some counts, the opposite may be true on the whole." [p 14] He demonstrates the case through an analogy, using the relation between recipes for cooking and actual cooking performance: "With a novice cook, actual performance is an inferior approximation to the recipe; with an expert cook, the recipe (even if written by the expert) is an inferior approximation to actual performance" [p. 14].

6. A final remark

As easy as it sometimes is to interpret a scene from a connectionist point of view it will take time before teachers make effective use of it, and even longer to exploit it in an integrated perspective But there is an incredible power available from such attempts. Fluency, the ease of activating certain connections, only develops across activities. And both teachers and researchers will profit from being able to extrapolate what might result from certain activities in the classroom in the future, as well as the retrospective possibility of tracing back the origins of actually activated options in order to identify their relations to certain earlier classroom activities

Notes

[1] This quotation is taken from Leary [1990] p. 359

[2] With rare exceptions, admittedly: in his book *The structure of scientific revolutions* [1970], Thomas Kuhn has pointed to the fact that whenever the first doubts come up, the natural attitude as a matter of course is already broken and the revolution has begun

[3] See E. Scheerer's call for "more scepticism" against the "philosophical enthusiasm for connectionism", the title of his 1989 article, and his attempts at identifying the fundamentals of both cognitive science and connectionism [1991]

[4] Though the original is written in English, with the title "Cognitive Science: A Cartography of Current Ideas", the text obviously has not been published in English yet My quotations, e g. p 26/8, refer both to the German translation [p. 26] and to the English manuscript [p 8] In the English manuscript, Varela refers to the publication as *Les sciences cognitives. Tendencies et perspective actuelles*, Editions du Seuil, Paris.

[5] Rorty discusses a striking similarity between the pragmatism of the young Heidegger (*Time and being*) and the late Wittgenstein (*Philosophical investigations*) — "there are no final analyses of and through language" — and, the other way round, between the early Wittgenstein's and the late Heidegger's (over-)estimation of language, where again their related ways of theoretical development have taken opposite directions and have crossed half-way in between

[6] "La seule donnée est la façon de prendre" in the French original, which Ernst von Glasersfeld has passed to me

[7] See p. 7 in the "Executive Summary" of NCIM's *Professional standards for teaching mathematics*, March 1991

[8] The notion of "interaction" is used here in the sense of Blumer [1969] and Mehan and Wood [1975], but not as in computer science for man-machine relationships, nor as in recent "multi-media" approaches

[9] I owe this quotation to Ernst von Glasersfeld

[10] If I speak of "my class", the children with whom I am working are meant, in their mathematics lessons at an elementary school in a project since August 1988 We started together on the first day of grade one; they are fourth graders now The final responsibility lies with an experienced female teacher (an excellent specialist in arts and crafts); we "share" the teaching of the mathematics lessons The first eighteen months have been videotaped and analysed [see Voigt, 1991]

[11] The "Executive Summary' (see footnote 7) explicitly recommends the "encouraging [of] students to take intellectual risks by formulating conjectures'. But there is no mention of the intellectual risks which the teacher has to take with her/his languaging in classroom communication and consequently, has to be trained for

[12] See Jerome Bruner's use of the concept of 'scaffolding' related to language learning in early childhood [Bruner, 1983] What are possible analogies in terms of minor communicative games and other activities, which children can take over and which could give children the chance to take an active part in mathematical discourse, not only to join it but also to actively contribute to it?

[13] For the role of case studies, and of stories in particular, see Schön [1991] and his discussion of the concept of "reflective turn"

[14] The notion is taken from Hugh Mehan's claim: 'Forms of life are always forms of life forming. Realities are always realities becoming" [Mehan and Wood, 1975 p 205]

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