Beyond Questions and Answers: Prompting Reflections and Deepening Understandings of Mathematics Using Multiple-Entry Logs

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Nit nit mooy garabam [1]

There is little discussion in mathematics education about methods of effective tutoring; in particular, in so-called office hour or tutorial sessions Nevertheless, these sessions are as much occasions for interactive instruction as classrooms are, and deserve theoretical and pedagogical attention By initiating discussion in this area we invite others to contribute critically to these beginnings. Clearly, discussions in this area will inform and be informed by classroom interactions. For this reason, though we focus on office-hour or tutorial interactions, we do indicate how the pedagogical tool we describe informs classroom practice

During office-hour or tutorial sessions, students customarily set specific goals that, with the tacit agreement of an instructor or tutor, govern the interactions of these sessions These goals typically center on questions or problems for which students seek immediate answers or solutions. Two problematic features often characterize interactions of these sessions. First, matters of mechanics or techniques constrain the mathematical dialogue, and second, in this "dialogue," the instructor or tutor does most of the talking or explaining. Stemming conspicuously from the first feature is the state of affairs that instructors and tutors operate rather like machines, producing worked-out solutions each time students pose questions: "How do you do problem 15 on page 35?" or, on rare occasions, "I was able to solve this problem to here but cannot see how to complete it Would you show me how to continue?" Curricular emphases on content over process and answers over understandings contribute to forming academic cultures in which students respond to mathematics learning in these ways. Moreover the pace and superficiality of many courses inhibit efforts to engage students in justifying techniques, in demonstrating derivations of formulae, or in interrogating and negotiating meanings of mathematical objects and processes Students either ignore these gestures or plead, "Just show me how to do it. After the exam, I may think about why it works."

The second problematic feature of office-hour or tutorial sessions concerns who performs most of the actions, does most of the talking, and therefore, acquires most of the learning Usually, we, instructors and tutors, dominate these sessions with our action and talk Consequently, we, not students, do most of the cognizing: thinking, explaining, solving, and so forth. That is, we appropriate available opportunities to re-experience and re-conceive mathematics. Furthermore, this appropriated time and space offer us the possibility of increasing our insight into a piece of mathematics while students, in awe, mostly listen to our display of knowledge and, when it occurs, witness our mathematical growth. On the whole, then, the benefits of office-hour or tutorial sessions mostly accrue to us

How can instructors and tutors go beyond question-and-answer sessions and provide time and space for students to reflect deeply on, gain insights into, and increase their understandings of mathematics? What pedagogical vehicles can prompt these behaviors and cognitive acts? Learning, contrary to the famous adage, does not occur from experiences alone. For it possibly to occur, learners must reflect on their experiences. They must connect and make sense of their experiences in the context of knowledge they already possess. Instructors and tutors, therefore, need pedagogical vehicles that explicitly engage students in reflecting on mathematics. One such powerful vehicle is writing It can prompt students to reflect critically on their mathematical experiences and respond to mathematical situations and questions that are personal and of their own choosing. [2] Perhaps, we could address both problematic features of office-hour and tutorial sessions if we were to have students write about their reflections on a problem and if we were to use *their* reflections as starting points for office-hour or tutorial discussions. Below, we, an instructor and a student, describe such an approach in the context of an office-hour session

Deepening reflections and understandings

During a real office-hour session, Ramnauth, a precalculus student, enters Powell's office to discuss problems of the course. Ramnauth, in his first semester of his second year, is taking his third mathematics course since entering the university and recently began attending Powell's office for help in precalculus. As one of their first exchanges, Ramnauth hands Powell a sheet of paper divided into two columns on which he has written some thoughts about the composition of functions (see Figure 1a). In the left-hand

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column, Ramnauth has computed the composite function, $(g \circ f)(x)$, of two functions, $f(x)$ and $g(x)$ It is typical of problems in his textbook. In the right-hand column, opposite his text selection, he has written or, as it were, logged his reflections about the text. He declares that the composition operation is new for him. It also appears that he tries to make sense of the textbook explanation of how to calculate the composition of two functions.

Composition of Function	The concept of composition of function is a new operation to me If we have something like $(g \circ f)(x)$ I will ask myself what does this mean. The book says this means $g(f(x))$ We would compose this function like this. We would take $f(x)$ and put it *into* $g(x)$ like I did on the left
Find $(g \circ f)(x) = g(f(x))$ given that	When I was in grade school my teachers introduced me to addition subtraction division and multiplication Here I am in college taking pre-calculus and I am being introduced to a new member of the order of operation family
$f(x) = \frac{x-3}{2}$ and $g(x) = x+2$	
$g(f(x)) = g\left(\frac{x-3}{2}\right)$	
$g(f(x)) = \frac{x-3}{2} + 2$	
$= \frac{x-3}{2} + \frac{4}{2}$	
$= \frac{x-3+4}{2}$	
$= \frac{x+1}{2}$	

Figure 1a
Remnauth's log demonstrating the composition of two functions, $f(x)$ and $g(x)$

Powell reads Ramnauth's log, checking the calculations and trying to interpret the explicit and implicit meanings and messages. The two then discuss their reactions and responses During this exchange, Ramnauth reveals why he chose the text selection: to practice and to understand the mechanics of composing one function with another. Learning this, Powell has him practice with other, increasingly more complex examples, some containing many more than two functions. During this office-hour session, they also examine other ways of combining functions to create new ones. Afterward, at the bottom of Ramnauth's log, Powell responds in writing (see Figure 1b)

Yes, not only is composition a new operation, but also functions are a new set of mathematical objects As you have seen just as with real numbers the operations of addition subtraction, multiplication, division, and exponentiation can be performed on functions. However composition is an operation unique to functions. Is the composition of functions commutative? That is is $f(g(x)) = g(f(x))$?

Figure 1b
Powell's comments and questions on Ramnauth's log

Along with affirming that composition is a new operation, joining functions together, Powell wishes to move Ramnauth beyond the focus of the textbook: the mechanics of computing. Through their interaction, Powell has already assessed that Ramnauth understands the mechanics. By responding with a question, therefore, Powell attempts to "force" Ramnauth to grapple with whether commutativity holds for the composite operation and to become aware of a new mathematical situation He returns Ramnauth's log.

Reading the response, Ramnauth asks a few questions to ensure that he clearly understands the written comments He knows that before their next session he will have to reflect again on his first written thoughts about the text selection and to reply to Powell's response. However, before the present session ends, they discuss other precalculus problems.

A week later, Ramnauth returns with a new log as well as a second written reflection on the log he wrote the week before. In his second reflection (see Figure 1c), he uses the two functions from the text selection that prompted his first reflection to tackle the question of the commutativity of the composition operation.

The composition of functions is not commutative That is $f(g(x)) \neq g(f(x))$ or $(f \circ g)(x) \neq (g \circ f)(x)$ On the other side of the page I looked at $(g \circ f)(x)$ I will now look at $(f \circ g)(x)$

$$(f \circ g)(x) = f(g(x))$$

$$f(x) = \frac{x-3}{2}$$

$$g(x) = x+2$$

$$f(x+2)$$

$$= \frac{x+2-3}{2}$$

$$= \frac{x-1}{2}$$

My answer here is not the same as on the other side of the page This proves that $(f \circ g)(x) \neq (g \circ f)(x)$

Figure 1c
Ramnauth's response or second reflection

The languages, both mathematical and prose, are bolder and more confident than those of his first reflections. He exhibits control, manipulates mathematical symbols meaningfully and, on the basis of his calculations, concludes that composition is not a commutative operation. That is, $(f \circ g)(x)$ is not equal to $(g \circ f)(x)$. Ramnauth generated this conclusion himself and therefore does *not* need to memorize it to retain it. There are, however, a number of questions that we could entertain to structure further discussions. For example, does Ramnauth now believe that the composition of *any* two functions $f(x)$ and $g(x)$ never commutes, or that only the two functions he investigated do not commute? What does it mean to provide *one* counterexample to a statement such as "Under composition, any two functions $f(x)$ and $g(x)$ commute"? And so on. Rarely, therefore, do discussions need to end after two or three written reflections on a piece of mathematics.

Multiple-entry logs

The above exemplifies how writing can prompt learners to reflect on and to deepen their understandings of mathematics and provoke rich mathematical discussions. We call the particular writing vehicle that Ramnauth used (see Figure 1) a multiple-entry log [3] Using writing activities in mathematics instruction, whether as an integral part of a course or as a component of office-hour or tutorial sessions, provides an innovative way to implement an important pedagogical precept: learning is enhanced when students reflect critically on their mathematical experiences and respond to mathematical situations and questions that are personal and of their own choosing. [4]

Satisfying this precept, a multiple-entry log is first a prompt for students to reflect on and form images of a piece of mathematics and, second, a medium for them to record, in prose, multiple and layered versions of their reflections and images. Students create this personal, reflective vehicle by creasing a sheet of loose-leaf paper length-wise into two equal columns. Into the left-hand column, they place a "text" of their own choosing, that particularly interests or strikes them. Broadly interpreted, the word "text" means some combination of mathematical prose or notational expressions selected from a textbook, a lecture, a problem set, a calculator or computer screen, or from any other course material. Students may also excerpt text from a mathematical discussion in which they participate or, as it were, witness. Once they excerpt a text, in the right-hand column students reflect on the text by writing in prose a commentary, an interpretation, an evaluation, a summary, or any other type of elaboration of their thoughts. Finally a crucial and critical aspect of maintaining multiple-entry logs — some time later students reflect on the previous text or reflection entries. In a column on the reverse side of the sheet students revise, reconsider, or refine their earlier reflections or otherwise comment on them. Reflecting on previous reflections can continue and generate new questions, issues, and understandings.

Generative nature of multiple-entry logs

We have used multiple-entry logs as a component of the interaction between a student and an instructor during office-hour sessions. Generally, for each session, students write one or two new multiple-entry logs. An instructor reads, reflects, and discusses with students their logs and writes responses to students' reflections. [5] By responding to specific remarks in students writings an instructor can encourage students to go beyond their initial reflections. Responses therefore often include questions or requests for students to clarify a remark.

The multiple-entry log of Figure 1 contains two reflections of a text. In general, however, reflections generate reflections. Discussions prompted by first and second reflections can stimulate a need to examine or explore further the original text or reflections. On this point, the multiple-entry log of Figure 2 contains three reflections: after the first, each one explores a question in greater depth and with increased clarity.

Using a problem from his precalculus textbook, Ramnauth wrote the first reflection (see Figure 2a) of his multiple-entry log to help him study. While preparing for an examination, he encountered difficulty understanding and remembering how to find the inverse of functions. Writing the steps for finding inverses helped him to remember and acquire a deeper understanding of the process. When Powell first read Ramnauth's log he did not understand why Ramnauth wrote it and only realized his intentions when they discussed Figure 2a. Powell, however, wrote his response before their discussion. His first sentence reveals that he was hard pressed to interpret Ramnauth's intentions. Powell meant his first sentence — "You decided to write out the steps involved in finding the inverse of a function" — to acknowledge what seemed obvious. Nevertheless he then encouraged Ramnauth to consider particular relationships between a function and its inverse. He also attempted to have Ramnauth connect his understanding of the multiplicative inverse of a number with the idea of inverses of functions. The instructor's comments raised a number of questions for Ramnauth and led to a rather involved discussion between him and Powell about mathematical notions of inverses, the actions of inverses, and identity elements of operations. With these questions, as well as in discussions, Powell intended to "force awareness" of particular relationships among mathematical objects. [6]

Text	Reflection 1
Finding the inverse of $f(x) = \frac{x-3}{x}$	I am asked to find the inverse formula $f^{-1}(x)$ for $f(x)$
$f(x) = \frac{x-3}{x}$	a) The first thing I did was to set the function equal to "y"
$y = \frac{x-3}{x}$	b) Then I multiply both sides of the function by the denominator 'x'
$yx = x - 3$	c) Then I subtract 'x' from the right side
$yx - x = -3$	
$x(y-1) = -3$	d) Then I factored out 'x' from the left side
$x = \frac{-3}{y-1}$	e) I then divide by "y-1" and get "x" by itself
$f^{-1}(y) = \frac{-3}{y-1}$	f) I then replace 'x' by the inverse function
$f^{-1}(x) = \frac{-3}{x-1}$	g) Then we replace "y" in the inverse by "x"
	This gives us the inverse: $f^{-1}(x) = \frac{-3}{x-1}$

Instructor's comments:

You decided to write out the steps involved in finding the inverse of a function. How is the inverse of a function related to the original function? Is the inverse of a function the same as the multiplicative inverse of a number?

Figure 2a
The text and first of three reflections of a multiple-entry log on the inverse of functions and instructor's comments

Reflection 2

The inverse function will do the opposite operations of the original function. The inverse of a function is not the same as the multiplicative inverse of a number. For example, the multiplicative inverse of 2 is $\frac{1}{2}$. The inverse function of $f(x) = \frac{x-3}{x}$ is $f^{-1}(x) = \frac{-3}{x-1}$

This will give us what we originally put into the function

Instructor's comments.

In your example, you wanted to point out the distinction between the multiplicative inverse and the inverse of a function. However, you only showed what each is but did not demonstrate explicitly that one is not the other

Figure 2b
The second of three reflections of a multiple-entry log on the inverse of functions and instructor's comments

During the same office-hour session which, as usual, lasted for about forty-five minutes, Ramnauth wrote his second reflection (see Figure 2b). He tried to answer the questions Powell posed. He stated his understanding of the action that inverse functions perform without, though, specifying on what the function operates. Further, he showed the multiplicative inverse of a particular integer and the inverse of a particular function. Curiously, however, he did not explicitly conclude that these inverses are different. It is not clear how Ramnauth relates the inverse of a function with respect to the composition operation, to the multiplicative inverse of a function.

The area under consideration is rich. It contains different and important mathematical ideas. The written record discloses only a portion of Ramnauth's thoughts. For instance, he does not discuss his anxieties about being correct, about ideas he may consider inappropriate, or about other connections that he may have made. Moreover, what he writes is constrained by time, space, and his ability to express his thoughts in writing. Nevertheless, his written reflections provide material for him to consider further and allow him to re-visit, re-conceive, and re-formulate non-ephemeral portions of his reflections. Likewise they provide Powell with some evidence of Ramnauth's mathematical understandings and opportunities to challenge what is displayed and to prompt further reflection. In commenting on the second reflection, Powell intends to force awareness of differences between multiplicative inverses and inverses of functions under the composition operation.

In this way of working Ramnauth did not have to respond to more than one of Powell's written interventions. Each multiple-entry log needs no more than two reflections, even though each has potential for generating many new queries. The topic of the multiple-entry log of Figure 2 intrigued Ramnauth enough for him to discuss Powell's response to his second reflection (Figure 2b) and, for a third time, to reflect in writing (see Figure 2c). In his third reflection, he revises and elaborates his second reflection and, consequently, clarifies his thinking [7]

Reflection 3, given to Powell a week after Reflection 2, differs from the second reflection in important and pedagogically interesting ways. Like the second reflection of the multiple-entry log of Figure 1, in the log of Figure 2 Ramnauth communicates more control, depth of understanding, and confidence in mathematics. Equally as important he states his intense satisfaction with the experience of thinking deeply (see Figure 2c). This entry is longer, more elaborate than the others entries in this log. In this written reflection Ramnauth incorporates ideas from discussions that he and Powell have had on this topic. For instance, for the first time, Ramnauth mentions additive inverses and refers explicitly to and gives examples of the inverses of a function with respect to addition, multiplication, and composition. Using a function and its inverse, he calculates their sum to show that it is not zero and their product to show that it is not one. In doing this he implicitly refers to the identity elements of the operations of addition and multiplication and demonstrates his acquaintance with them. Moreover, an advance on his second reflection, here he states what he wishes to conclude, "...the inverse function is not the same as the multiplicative inverse of a number."

Reflection 3

The inverse function will undo what the original function did. This will give us what we originally put into the function. The inverse of a function is not the same as the multiplicative inverse of a number

For example, the multiplicative inverse of "2" is "$\frac{1}{2}$" The additive inverse of "2" is "-2" The inverse function of

$f(x) = \frac{x-3}{x}$ is $f^{-1}(x) = \frac{-3}{x-1}$

The inverse function is not the same as the multiplicative inverse or the additive inverse. When you multiply

$\frac{x-3}{x} \cdot \frac{-3}{x-1}$ you will get $\frac{-3(x-3)}{x(x-1)}$ Not "1" And when you add

$\frac{x-3}{x} + \frac{-3}{x-1}$ you get

$$\frac{(x-1)(x-3)}{x(x-1)} + \frac{-3x}{x(x-1)}$$

$$= \frac{(x-1)(x-3)-3x}{x(x-1)} = \frac{x^2-4x+3-3x}{x(x-1)}$$

$$= \frac{x^2-7x+3}{x(x-1)} \text{ which is not "0"}$$

The multiplicative inverse of $f(x) = \frac{x-3}{x}$ would be $g(x) = \frac{x}{x-3}$

Therefore, you can clearly see that the inverse function is not the same as the multiplicative inverse of a number

This is deep! It makes you think more deeply into what you are working on in class. It also makes you look back at what you have learned in the past.

Figure 2c

The third of three reflections of a multiple-entry log on the inverse of functions

As Ramnauth successively visited the multiple-entry log of Figure 2, he generated deeper, more elaborated written reflections on meanings of the inverse of a function. Indeed, his discussions move from the mechanics of finding the inverse of a function to one in which he interrogates meanings of the inverse of a function under different operations and whether these inverses are equivalent. Many questions however remain. Clearly, as readers consider Reflection 3, they will generate a host of other interesting mathematical questions for discussion and examination. Indeed, there are questions related to the particular function that Ramnauth chose to use. For example, though it is true that $\frac{-3(x-3)}{x(x-1)} = 1$ is not an identity, are there values of x for which the rational expression is equal to 1? What is the identity function of the composition operation? Having graphed $\frac{x^2-7x+3}{x(x-1)}$, what is the graph of its additive inverse?

The territory is rich; many other mathematical questions and directions are possible. A multiple-entry log provides a powerful vehicle for exploring these questions and directions. Just as with content and cognitive issues, multiple-entry logs raise affective matters, as well. Over the course of his reflections in Figures 1 and 2 the written language and the complexity of the ideas that Ramnauth discusses establish that his confidence grew in relation to the mathematical topics of his logs. Furthermore, in his writing he reveals that he is cognizant of his growing mathematical awareness. In the last paragraph of Reflection 3 he delights in his cognitive acts and cites two benefits of multiple-entry logs: they allow him to think more deeply about course material and to reexamine and connect what he knows to new material

Conclusion

Besides these two benefits, multiple-entry logs can help students in other ways. Ramnauth used multiple-entry logs as a tool for studying. Unlike some students in the course he had not studied precalculus in secondary school, felt as if he did not have an effective means for studying college mathematics, and found himself lost after a few weeks. He sought help and wanted to understand the material. Powell suggested that he use multiple-entry logs, which was Ramnauth's first encounter with them.

The idea of using logs sparked Ramnauth's curiosity and stimulated him to learn more about this tool. Instead of simply doing problem after problem he found that multiple-entry logs made learning more enjoyable: they gave him opportunities to express his own thoughts and feelings and to communicate orally with the reader of his logs, who made comments and posed questions that led him to think more deeply about mathematics. As the log of Figure 2 illustrates, he was also able to look at things that he had learned in the past and to connect that knowledge to the present material.

Ramnauth made other connections with multiple-entry logs: In learning from his textbook, they were a vehicle for him to grapple with and synthesize mathematical ideas. He gained a better understanding of his textbook since writing logs forced him to read actively and critically. At the same time, writing logs gave him an opportunity to spend more time with the course material, and the more time he spent with mathematics the better he understood it.

Aside from using multiple-entry logs for asking questions, obtaining feedback, reading text, and connecting ideas, Ramnauth derived other benefits. First, he usually wrote his logs after finishing his homework. Doing so gave him a chance to reflect and review what he had just completed. Second, he used logs to study for examinations since some contained problems with which he had difficulties. As a result when he went to class for lectures or examinations he felt more confident and at ease and therefore focused more of his attention on mathematics. Third, Ramnauth wrote more logs than the ones on which Powell commented. As a byproduct of maintaining logs, Ramnauth enhanced his writing abilities.

Along with augmenting self-confidence and writing and thinking abilities, multiple-entry logs establish a unique line of communication between students and instructor or tutor. From his first exchanges with Powell, Ramnauth sensed that Powell seriously considered his thoughts as expressed in his logs. These interactions made him feel enthusiastic about mathematics. Consequently he willingly and consciously focused on difficult areas of the course. In general the combined benefits of writing multiple-entry logs gave Ramnauth a deeper understanding, not only of precalculus but also of himself as a learner.

These benefits resulted, at least in part, from the oral and written communications between a student and an instructor. However, though these occurred during office-hour sessions, such discussions can be part of a classroom environment and, more specifically, can occur as students respond to each other's multiple-entry logs. Just as in office-hour or tutorial settings, for each class students generally write a new multiple-entry log and, for an earlier log, write a second reflection in response to a comment. In pairs students exchange, read, and discuss each others logs before writing a comment about some aspect of their classmates' logs.

Initially students may have difficulty responding effectively. To start with they read carefully their classmates' logs, verifying their interpretations with their partners, which potentially initiates rich mathematical dialogue. Then students write a response, commenting on a specific statement or idea that they find striking in their classmates' written reflections. To communicate understanding, students feedback in writing their interpretation of what their partners expressed. Afterward, students write questions to provoke further reflections, or request that their partners clarify, reconsider, or elaborate on a statement or an idea. Instructors can acquaint students with these guidelines by providing models such as in the multiple-entry logs and comments of Figures 1 and 2 and by engaging students in analyzing models.

At the risk of losing nuances and specificity as we aggregate the cognitive and affective benefits students derive from multiple-entry logs, we observe that logs can initiate mathematical dialogue and furnish excuses for sustaining that dialogue. As well, by writing and discussing multiple-entry logs, students *re*-consider and *re*-conceive their initial understandings and engage in socially negotiating and appropriating mathematical ideas. These two broad categories of benefits, as we have seen, are both critical and extensive.

As considerable as the benefits are to students, the use of multiple-entry logs has important implications for instructors or tutors. Some implications relate to pedagogy while others relate to how instructors and tutors view their role and their students. In our use of multiple-entry logs perhaps the most obvious implication is that we reversed the traditional trajectory of questions and answers. Within an office-hour or tutorial setting students have time and space to elaborate and revise profoundly their conceptions of particular mathematical topics. Since the interactions that occur in these setting are dialogues and can be similar to Piagetian clinical interviews, an instructor or tutor can learn much about how students think and about other pedagogical issues [Ginsburg, 1981; Swanson, Schwartz, Ginsburg, & Kossan, 1981]. With such insights an instructor or tutor can more accurately infer what feedback to provide to trigger the awareness of students and thereby help them augment their mathematical knowledge.

Just as important, though possibly not immediately so obvious, is the question of personal and intellectual commitment. Typically, students come to office-hour or tutorial sessions with problems marked in their textbook or problem set that they want to *see* solved. However, since students must choose the text about which to initiate writing multiple-entry logs, they have more of a personal and intellectual, rather than pragmatic, commitment to knowing and understanding the text. By choosing and writing an entry they have committed themselves to a piece of mathematics. This initial commitment often grows into a desire to delve deeply into the topic of the text, into variations of it, or into questions raised by it.

Multiple-entry logs not only engender commitment, the generative nature of logs encourages students to display the richness of their thoughts. As students move from initial to more elaborated reflections on texts they claim the mental space required to engage their minds in quantitatively and qualitatively richer and deeper cognitive and metacognitive acts. As students together with instructors or tutors successively consider the written evidence of students' thoughts, instructors and tutors come to marvel and respect students' cognitive products. Hawkins [1969] stated this idea in the following manner: "A human being is a localized physical body, but you can't see him as a *person* unless you see him in his working relationship with the world around him" [p. 45]. Thus when we, instructors and tutors, listen and, broadly speaking, "read" students as they struggle to make sense of the world outside and beyond themselves, specifically the world of mathematics, we feel more respectful of them as thinking and feeling beings. Moreover, as instructors or tutors communicate their respect for students' cognitive products back to them and demonstrate that these products are worthy of serious, sustained, and critical examination, students gain increased confidence in their ability to do mathematics and, therefore, do more and do it better. This cycle of success also improves the environment in which we labor and the fruits of our efforts.

Using multiple-entry logs supports a critical pedagogical task: helping students to recognize and develop the habit of using the powers of their minds. That is, the value of logs is not the logs themselves; rather, they are valuable to the extent that we, instructors and tutors, can force the awareness of the effectiveness and efficiency of learning through successive, generative reflections. Indeed, this points to a question that we wish to explore further: What educational situations promote students to learn how to employ the power of their minds to reflect spontaneously and autonomously in the service of learning mathematics? What other tools prompt learners to develop as a *habit of the mind* their power of reflection in discursive practices involving mathematics?

The mathematics discourse in classrooms can constrain the writings and reflections that students produce in multiple-entry logs. When the emphasis of a course or textbook focuses on mechanics and finding right answers and on solitary work, then students write multiple-entry logs, at least initially, that reflect these foci. On the other hand, when instruction emphasizes the social generation of knowledge, on finding patterns, describing and analyzing those patterns, and devising conjectures, generalizations, formulae, and rules about how mathematical objects and processes behave, then students reflections in multiple-entry logs take on these features. Over time, instructors and tutors can help students see mathematics in a richer, more meaningful way through the nature of the discussion we have with students, what we ask them to attend to, and the kinds of statements we write in our comments to their reflections.

Finally, there is a challenge in teaching mathematics to so-called developmental or underprepared students. Typically, they have minimal real experiences in doing and writing, discussing and reading about mathematics or have an underdeveloped habit of reflecting on processes. [8] The challenge for developmental educators, therefore, is to make students aware of the power of their minds to reflect; to provide occasions for them to become aware of how reflection supports mathematics learning; to assure them that it is accessible and at the command of their will; and to encourage an understanding of how to control it in the interest of making sense of mathematics. In sum, as educators, our challenge is to teach students *how* to use their power of reflection by making them aware that *they already have* the power to reflect. [9] We have used multiple-entry logs as vehicles for meeting this challenge. They are vehicles for pushing beyond mere questions and answers and, specifically, for moving toward situations in which students "force" their own awareness and in which they are "forced" to reflect on, interpret, analyze, and, generally, understand mathematics more deeply.

Notes

[1] This proverb, uttered by people who speak Wolof, mainly in Senegal and The Gambia, West Africa, translates into English: "The best medicine for people is people."

[2] See Powell and López [1989] for a discussion of this point, including a characterization of writing activities that foster mathematics learning. Other treatments of this assertion are in Mett [1989], Miller [1991], and Rose [1989]. Also, see Powell, Pierre, and Ramos [in press] for a brief literature review and a selected, annotated bibliography on writing to learn mathematics.

[3] Multiple-entry logs [Hoffman & Powell, 1989; Frankenstein & Powell, 1989] are variations, mainly in content and form, but similar in purposes, of what some call double-entry logs [Jones, 1988], divided pages [Tobias, 1989], or dialectical notebooks [Berthoff, 1982, 1987a].

[4] We also speculate that multiple-entry logs enhance learning since, with this writing-to-learn vehicle, learners tend to write in particularly useful ways. Generally, different writing-to-learn activities prompt students to produce different kinds of writings. As Hoffman and Powell [1989] theorize, these writings exist within a matrix of categories: non-personal and non-reflective; non-personal and reflective; personal and non-reflective; and personal and reflective. Within this postulated matrix, they suggest that the category that best supports mathematical thinking is "personal, reflective writing in which the content is mathematics and students' affective responses to it" [1989, p. 132]. Another content that supports learning includes students' reflections on their own learning process. The multiple-entry logs of Figure 1, and as you will see, of Figure 2, illustrate our claim that logs tend to promote personal, reflective writing. The structure of multiple-entry logs appears to prompt learners to enter into a "dialogue" not only with their instructor or classmates but also with the text they selected and, rather than mere summary, encourages interpretation and analysis of the text.

This assessment of the role of multiple-entry logs agrees with and is corroborated by that which Jones [1988] discovered in his investigation, "Double entry logs: prompts for revision and expository comments," conducted in a Communication Skills course, a preparation for placement into a required freshman composition course at Rutgers University, Newark.

[5] For discussions of other writing activities that foster dialogue, among others, see Borasi and Rose [1989], Buerk [1982, 1990] Driscoll and Powell [1992], and Hoffman and Powell [1992, August].

[6] The idea of "forcing awareness" and, more generally, "generating awareness" is due to Caleb Gattegno who, in print and in seminars, elaborated on the psychology and recursive property of awareness. For a general, theoretical treatment of awareness and his assertion that "only awareness is educable," see Gattegno [1987]; and for discussions of educating the awareness of mathematics learners see, for example,

Gattegno [1974, 1988]. Since the pedagogical ideas of awareness and forcing awareness have informed this article, we briefly explain them

Notwithstanding the many objectionable denotations and connotations of the verb "to force," as Gattegno defines the term forcing awareness is pedagogically compelling and useful. He makes clear that learning occurs not as a teacher narrates information but rather as learners employ their wills to access and focus their awareness so that they may generate knowledge. Learners reach their awareness when they attend to the content of their experiences. They educate their awareness as they observe what they are engaged in. In mathematics, the content of experiences, whether internal or external to the self, can be feelings, objects, relations among objects, and dynamics linking different relations [1987, p. 14] Generally, awareness results from "a dialogue of one's mind with one's self" about the content of one's experiences [1987, p 6].

For Gattegno, "forcing awareness" has two meanings: "One is concerned with what we do to ourselves, and the other with what can be done to us so that we become aware of what has escaped us, or might escape us" [1987, p 210]. Therefore, pedagogical interventions such as multiple-entry logs can be used to help learners not escape from acquiring awareness of particular relationships among mathematical objects. As our work with logs indicate, the question of how to structure pedagogical interventions so that they can force awareness leads to interesting researchable issues.

[7] A multiple-entry log, as its name suggests, is a vehicle for re-visiting and re-conceiving Indeed successful writing-to-learn activities incorporate revision. For another illustration of this in mathematics see Gopen and Smith [1990]

[8] This state of affairs, curiously and particularly more prevalent among individuals of dominated groups in our society, is a byproduct of a social, rather than genetic, process of underdevelopment and disempowerment Space does not allow us to comment more substantively on this process. There is, however, an emerging literature that theorizes about and critically analyzes this observed state of affairs and process For a brief discussion, see Frankenstein and Powell [1989]; for elaboration of this issue, see Mellin-Olsen [1987] and the proceedings of the first conference on the Political Dimensions of Mathematics Education [Noss *et al* 1989]

[9] This observation based on epistemological and pedagogical views of Gattegno [1974, 1988] in the area of mathematics is similar to the pedagogical principles of Berthoff [1982. 1987b] in the area of composition theory

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