Towards New Customs in the Classroom

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Diagnosis of the problem

The mathematical productions of many students at the beginning of their first year in the University often seem to mimic the writing of the teacher: the control of meaning does not appear to be a primary purpose of the texts Syntactic characteristics often seem to prevail over semantic characteristics

Meanings are not used as a *means of controlling* the results of algorithms.

Another observation is that the students have no interest in proof as a functional tool. Proof is only a formal exercise to be done *for the teacher,* there is no deep necessity for it It is true, of course, that the problems which the teacher's proofs resolve have not usually been appropriated by the students.

These observations of students beginning scientific studies at the University show that mathematics is not seen by them, in the way that physics is, as a scientific subject that has been developed in order to solve problems and to give an understanding of reality. The students' epistemology, where it exists, is inadequate For instance, in response to the question

What does mathematics mean to you?

the students frequently give answers of the following kinds

"Mathematics is used in physics and chemistry "
"You *must* know mathematics in order to get a degree "
"Mathematics can be fun sometimes."

To sum up: a lack of concern for meaning, a weak grasp of meaning as a means of controlling mathematical actions, a lack of appreciation of proof as a functional tool, an inadequate learning epistemology, are characteristics found in many students The problem we have set ourselves is to go beyond these observations and try to imagine an alternative way of teaching mathematics in the University.

Confronted with this problem we have designed an experimental teaching method, set in a particular theoretical framework, and applied it to teaching mathematics in the first year at the University.

The framework is based on the following general cognitive and didactical hypotheses.

— First, contructivism as a theory of knowledge acquisition [PME, 1987] and its epistemological interpretation in the "théorie des situations didactiques" [Brousseau, 1986]. Students construct their own knowledge through interactions, conflicts and re-equilibrations in which mathematical knowledge, other students, and problems are involved (the learning "milieu," in short). These interactions are managed by the teacher who makes the fundamental choices.

— This knowledge is all the more solid when it has been constituted and applied in more than one conceptual setting. [Douady, 1986]

— To develop a sense of the need for proof we use the work of Balacheff [1982] that emphasizes the role of contradictions, how they can be made sharper and more explicit, and how they can be resolved.

— More generally, and following many other researchers [e.g. Bishop, 1985; Balacheff & Laborde, 1985], we consider the role of the group of students to be very important, especially in the construction of meaning.

— Meta-mathematical factors, such as systems of representation in mathematics, and how mathematics is learned, are very important influences on the learning process, especially when students are solving problems, as Schoenfeld [1983] shows Moreover we think that, at university level at least, these factors can be worked on explicitly in a way that emphasizes, for the students, the points just made. [Robert,]

— Lastly, we believe that true mathematics learning, in its full scientific sense, must include the constitution of a "learner's epistemology" — by which we mean the set of problems, of situations, that, in the personal experience of the student, have become associated with the introduction and the progressive constitution of a concept and which therefore give this concept a particular meaning for this particular student

General observation shows that the default value of this epistemology can often be summarized as follows: "The teacher introduces a new subject because it is part of the curriculum; only stereotyped exercises are proposed because there is some overriding reason why only such exercises need to be solved." A learner's epistemology based on some real problems in mathematics or physics, for instance, for which the mathematical concept was developed can help students to a deeper understanding of mathematics.

The experiment

We now describe more precisely the background to the experiment using the idea of didactical "customs", i.e the implicit or explicit rules that drive the working of the system constituted by the classroom, and especially what the actors (students, teacher) expect of each other. [Balacheff, 1986]

The experiment has been undertaken with about 100 students at a time (95 last year, 130 this year) in the first year of the so-called Deug A program (students are in their first year and have courses in mathematics, physics and chemistry) at the University 1 in Grenoble. The experiment continues throughout the academic year.

In the light of our diagnosis of the problem and of the general theoretical considerations, we have laid down the following as constituting the new "customs" in the classroom

(1) A large place must be left for uncertainty in the learning process. Uncertainty in relation to mathematical knowledge is institutionalised in the notion of conjecture, the validation of which, and even the production of which, is devolved onto the community of students. The conjectures concern those parts of the mathematics curriculum that students must learn during the year. We believe that the necessity, the functionality, of proof can only surface in situations in which the students meet uncertainty about the truth of mathematical propositions

> GENERATING SCIENTIFIC DEBATE
> How it happens in the classroom
>
> *First step.* The teacher initiates and organises the production of scientific statements by the students. These are written on the blackboard without any immediate evaluation of their validity
>
> *Second step.* The statements are put to the students for consideration and discussion. They must come to decisions about their validity by taking a vote; each opinion must be supported in some way, by scientific argument, by proof, by refutation, by counter-example, ...
>
> *Third step.* The statements that can be validated by a full demonstration become theorems; those found to be incorrect are preserved as "false-statements" associated with appropriate counter-examples. The students' lecture notes are observed to contain these two kinds of statements.

Example

In 1986, at the end of the course on the integral, the students were asked as part of one of their biweekly homeworks to write down what they had said in their notes about this course. What follows shows the kind of answers they gave.

F is always differentiable on I. *False.*
Counter-example: $f(t)=I$ if ..... (here followed an explicit function)

If f is Lipschitzian then F is differentiable. *False.*
Counter-example the preceding function F.

This is an example of a conjecture formulated by a student during a debate as a tool to support another conjecture

> EXAMPLE OF A SCIENTIFIC DEBATE
>
> "If I is an interval on the reals, a a fixed element of I, and x an element of I, we set, for f integrable over I,
>
> $F(X) = \int_a^x f(t)\, dt$"
>
> The teacher asks the question, "Can you make some conjectures of the form
>
> if f ... then F ... ?"
>
> About twenty more or less complicated statements have been produced by the students. One session begins with an examination of one of these.
>
> "If f is increasing then F is increasing too" (which is false)
>
> The variation of F is an item belonging to the curriculum that the students must learn. The following steps were observed during this lesson
>
> (a) *Counter-example:* a student produced an example contradicting the statement. So the class concluded that the statement was false
> (b) *Modification of statement.* A student proposed "If f is monotonic then F is monotonic too." (Of course this is also a false statement.)
> (c) *Counter-example:* the same function as before, but defined on a different interval, contradicted the modified statement
> (d) *Observation.* considering the counter-example, it seems that if $f > 0$ then F is increasing.
> (e) *New conjecture:* A student now proposed that "If $f \geq 0$ then F is increasing." (The majority of the students thought this statement was false, though it is of course true.)
> (f) *An argument* the student produced the explanation
>
> $F(x') - F(x) = \int_x^{x'} f(t)\, dt \geq 0$
>
> if $f \geq 0$ and $x > x$.
>
> Many of the students did not believe that this was always true ... In fact, this point in the debate revealed to the teacher that some of the results and definitions, discussed and settled previously, had been misunderstood by many of the students. In particular they had not grasped the convention that $\int_x^{x'} f\, dt$ is the Riemann integral on the segment $[x \quad x]$ if $x > x$, and the opposite of this integral if $x < x$. We can analyse this phenomenon as the reappearance of old, stable, knowledge about the integral learned in previous years
> (g) During the debate the students reached *a validation* of (f) with the argument
>
> $\int_x^{x'} f(t)\, dt \geq (x - x) \inf(f)$,
>
> in this case
>
> The debate took almost the whole of a two-hour class

more closely related to the course material. The debate gave this statement — false, but apparently accepted by some students — the opportunity to come to light and to be refuted.

> If f is continuous then F is differentiable on I. *True. Theorem.* (Here followed the classical theorem and the demonstration given in the course.)
>
> *Remark:* At a point where f is not continuous one can't say anything about the differentiability of F.

Demonstrations are produced through the interactions of the students and, when necessary, the teacher after the students have been confronted with the particular problem in the course of a debate.

(2) In this form of "scientific debate" the proof arguments made by a student are not addressed to the teacher but to the other students. We distinguish between "proofs to convince," in which arguments are produced to convince someone (such as another student) of something that is not already a part of his institutionalised knowledge and "proofs to show" where the aim is to show someone (such as the teacher) that we have reached some knowledge that he already possesses.

One of the main hypotheses of research is that the activity involved in the first process is fundamentally different from that involved in the second, and that it is able to produce a deepening of knowledge and its meaning.

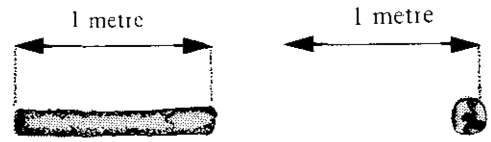
The theoretical description of such a teaching system, which we have been using for four years, is based on this hypothesis. We call a "codidactic situation" one in which a student tries to convince others, and himself at the same time, of the truth of a conjecture that has been formulated (by himself or by another student) in answer to some problem that the whole group of students is trying to solve. The students all know that the conjecture is not necessarily true, and in particular that it is not yet established as an item of institutionalised knowledge.

In such a situation, interactions and conflicts between the students' conceptions will appear, resulting in a clarification of contradictions and an emerging need for proof. It also gives rise to a clarification of the background to the conceptualisations of different students, which are not all the same. Because of the necessary decentring that accompanies the desire to convince, the process gives a functionality to the different conceptualisations in their various settings.

In this way the main properties of concepts are constructed through debates of particular conjectures.

(3) The introduction of some new mathematical tools (the integral, for instance) is organized in such a way that they appear necessary for the solution of some complex problems, often chosen from the physical sciences. The necessity is made evident by the impossibility of solving some concrete "problem situation," over which the students can nevertheless have some control.

AN EXAMPLE OF A PROBLEM SITUATION



a homogenous stick weighing 1000 g

a marble weighing 100 g

What gravitational attraction is exerted by the stick on the marble?
(Introduction to the Reimann integral)

- Faced with this problem, students are able to give some answers, generally false, based on their knowledge of the gravitational attraction between two points.
- Using the notion of centre of gravity, they can give some upper or lower bound to the result.
- However, in order to find the right answer they must enter into a procedure for cutting the stick. Our observations show that they usually find this way.
- But to arrive at a conclusion and give the final result they do not have the right mathematical tool yet, viz. the Riemann integral.
- This is a way to introduce this powerful tool by means of a problem that takes into account the true epistemology of the notion. [See Grenier, Legrand & Richard, 1985, for further details.]

The complexity of the problems becomes an element that subsequently justifies the important theoretical effort that is needed to establish the main properties of the new mathematical tool.

We think that through such constructions students are able to constitute for themselves a "learner's epistemology" for the particular concept. In particular, for some concepts, we deliberately postpone the introduction of very powerful algorithms (here, for instance, the calculus of primitives, or partial derivatives in the case of the differential) to allow deeper reflection about the concept, trying to give a functionality to its meaning through the resolution of the chosen problems.

(4) We think that, at this level at least, the students' awareness that they have to construct their own knowledge by a process of reflection is a positive factor. Some reflections of a metacognitive nature about the knowledge being taught

and about learning processes are developed during the teaching session

Example
The statement quoted above, "If f increases then F increases too," had been discussed in another form in connection with functions constant over an interval. After the conjecture had been discarded the teacher reminded the students of this and emphasized how misleading a "simple" statement can be.

Teacher Well this conjecture . I remind you that we have studied this conjecture before . Do you remember?

Student

Teacher I think we have already looked at this conjecture in connection with functions constant over an interval. So you must be aware that there is something behind this conjecture . Let's see. There are 35 of you who think at first glance that it's true It means that in practice many of you will be tempted to use this conjecture as a fact You need to be aware that it is the kind of property that one likes to use because it has the right ring to it "f increases so F increases." there's a symmetry, it's pleesent . it looks like $(a+b)^2 = a^2 + b^2$; it's so pleasant to say that one wants it to be true

The setting for this experimental system necessarily supposes a radical change in the didactical "customs" of the teaching system which are widely responsible for the facts noted at the beginning of this article We organize new customs which are constitutive of what we call the "codidactical system:" for instance, as we noted earlier, as a component of the new system it is necessary to give an institutional status to statements whose veracity is not yet sure by naming them conjectures The rules of the "scientific debate" have to be explained to, and their functionality accepted by, the students.

The methodology of the study
We now indicate briefly how we study the teaching system and the constitution of the new "customs," and also the consequences with respect to our original definition of the problem

Sessions for scientific debate are prepared by the research team by a *prior* analysis of the problem-situation or the mathematical field in which conjectures are to be proposed. We try to estimate, for instance, the potential of a problem-situation to elicit the processes we want to encourage, what means might be called on if the session runs into an unforeseen block Generally we try to anticipate the main choices that the teacher may have to make, and especially the critical points when he must not interfere, in potential codidactical situations, for instance

Sessions are recorded and observed by members of the team and then analyzed, comparing predictions and realizations with respect to the new "customs," trying to interpret observations within the theoretical framework we have outlined

Analysis allows us to see what deep changes in the teacher's behaviour are necessary. It forms the basis for the study of this new teaching system: the characterisation of what we mean by "scientific debate" in the classroom, of various types of debate, of ways of setting up a debate during a session, of techniques that may be transmitted to other teachers .. and, more generally, of the new classroom customs themselves. A research report discusses these matters [Alibert, Grenier, Legrand & Richard, 1986]

For our study of the changes in the relationships between the students and mathematical knowledge or mathematical proof in this new setting, in the light of the problem as we have defined it, we use several other observations:

— How many students participate in the debate, the level of the arguments they give, the kind of mathematical knowledge they produce; in particular, what part of the group's knowledge has been elaborated by the community of students and what part by the teacher alone.
— We analyse the students' answers to the following questionnaire, administered several months later:

(1) *In general, when you are faced with a conjecture, do you feel strange, at a loss, without any ideas. directly concerned, deeply involved, passionately favourable to . ?*

(2) *After the event does it seem to you that you understand a concept better*
  (a) *if its is introduced through a conjecture?*
  (b) *if it is introduced in a clear and well-organized lecture?*

(3) *What does mathematics mean to you?*

The analysis takes up four points:

1) What are the students' evaluations of the changes in classroom customs?
(2) What are their conceptions of learning mathematics?
(3) What are their conceptions of mathematical knowledge?
(4) What are their attitudes towards uncertainty, in particular regarding the necessity of proof?

In this article we will not give details about these observations. We can say that they show an evolution in the students towards greater care for the meaning of arguments, not only for their form

First results, perspectives
As regards the students' participation in debates, we frequently observed that about a third of the students spoke during a session, often after consulting with their neighbours. We conclude that the collectivity is strongly implicated in this activity: the stakes are felt to be high enough to enourage the students to enter into this reflective process.

Propositions that are discussed during these sessions are far from trivial: they allow students to approach real problems about the concepts involved

Some of the statements *elaborated by the students* and then discussed during the course on the Riemann integral:

* If f, g are integrable functions on a set Ω, and A B are measurable subsets of Ω. the following properties hold:

$$\int_A f + \int_B f = \int_{A \cup B} f + \int_{A \cap B} f$$

$$\int_A (f + g) = \int_A f + \int_A g$$

** If I is an interval on the reals, a a fixed element of I and x an element of I and we set. for f integrable over I. $F(x) = \int_x^{x'} f(t)\, dt$, then:

If $f = 0$ then $F = 0$, or F is constant, or F is linear
If f is continuous then F is continuous
If f is increasing then F is increasing.
If f is positive then F is increasing
F is differentiable . . .

Many of these statements, of course, are false Nevertheless they are very important: first, because they show what some students really think about the integral at a particular stage in the course, and second, because during the debate about their proposals these students can become really convinced of the falsity of their conjectures, and possibly become aware of deeper misunderstandings about some key concepts In this way they enter into a producer's relationship with their knowledge, not a consumer's or a visitor's.

From the students' answers to the questionnaire we derive some significant facts

(1) About 60% of the students answered the question about the comparison of teaching methods. 75% of these prefer the method incorporating debates but they do not exclude the traditional lecture. In fact many of them think that the debate is interesting when new concepts or new properties are first encountered but that it is necessary for the teacher to give a clear summary lecture in order to institutionalise the knowledge worked on in the debates.

> "It compels us to reflect more on the question. One often listens to a lecture without reflecting deeply."
> "A concept introduced through some conjecture makes the problem that the concept poses much clearer than in a lecture."
> "It allows us to have several views, to eliminate some intuitions that are wrong."
> "A well-organised lecture is sometimes welcome afterwards to put a little more order into the course."

(2) Regarding the effect of the debates, many students emphasise that they allow them to understand what problems the new mathematical knowledge is intended to solve, and also what are some of the errors that may be made.

(3) Their answers clearly show reflection about the learning of mathematics, taking into account some epistemological issues, and an appreciation that errors are no longer a fault but play a useful part in the construction of knowledge. Some students note that the only way they can assimilate a concept is to investigate it first

> "A path is meaningless (in maths) if one has not made it oneself."

(4) Some students, about 10% of the total, reject the method. They feel it is inaccessible, not sufficiently ordered. They do not feel interested in a conjecture unless they know where the investigation may lead

> "I understand a concept better if it is studied in a well-planned course; debates have a tendency to confuse me, to lose me"

(5) Many students stress how difficult it is to study a conjecture: to enter the problem, construct a proof, formulate ideas about the truth of the proposition within a climate of uncertainty. These answers show that the students are really facing the special learning style that this system is aiming to produce

> "I often have difficulty forming an opinion, following it up with a proof"
> "For me the hardest thing is finding counterexamples when I think the conjecture is false"

Despite the difficulties, they feel involved and interested. They cite their increased curiosity, their discussions with other students In this they show a new relation with their "milieu"

Their conceptions of mathematics are interesting — and important for their learning. A large majority of the students answer the third question at an epistemological level; their "school" epistemology has almost disappeared The answers are divided between "mathematics as a tool for the other sciences (physics, chemistry, economics)" (36%) and "mathematics as a method, a science with a particular relationship to reality". (44%) The more interesting evolution is the reference to reality in relation to mathematics.

> "It's to contribute to the understanding of the external world."
> "Generally speaking I have the feeling that I increasingly use the mathematical method when faced with a current life situation."

Finally we indicate briefly the consequences for the teacher if these new customs are adopted The implications are important

The teacher's relationship to the to-be-taught knowledge changes The fundamental didactical choices are still his responsibility (for instance, the choice of problems, the management of time), but he must learn to no longer be the one that on every occasion gives the answers. On the contrary he must delegate this responsibility to the community

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