In Calypso's Arms

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One day five hundred years ago — on the 14th of December 1486 to be precise — the mathematician Luca Pacioli met a Florentine merchant, Onofrio Dini, at the warehouse of the mutual friend. during the conversation, Dini posed a problem about the disposing of an estate in certain proportions. This problem was given by Pacioli in his textbook, *Summa de arithmetica,* published in 1494. In concerns a man who on his death-bed wished to make a will that made provision for his wife and his as-yet-unborn child. His estate amounted to 600 ducats. If the child was a girl she was to get 200 ducats, but a boy was to get 400 ducats In each case, the mother was to get the remainder. Shortly after making this will the man died. But then the widow gave birth to twins, a boy and a girl. How should the estate be divided according to the wishes of the deceased?

Problems of this sort did not arise spontaneously in casual conversations between friends. In fact, this particular problem about a widow turns up in an earlier Arabic text and similar problems occur in Hindu collections. The reason why they were taken up assiduously in fifteenth-century Italy is, of course, due to developing capitalist mercantile practice. It becomes important to get the calculation right when the time comes for investors in a partnership to share profits.

Various other proportion problems were important in commerce: gauging, brokerage, interest rates, discount, and so on. But perhaps the most important, and at the same time the most complicated, was currency exchange. It is easy nowadays to forget the complications there must have been when every large town had its own coinage. A typical problem in a 1489 German textbook concerned a man trying to change 30 Nuremberg pennies into Viennese currency. The moneychanger has to calculate the rate from a long sequence of official local exchanges: 7 Vienna are worth 9 Linz, and 8 Linz are worth 11 Passau, and 12 Passau are worth 13 Vilshofen and....

The Rule of Three became a crucial arithmetical technique and it continued to play a large role in arithmetic textbooks. As a young child in the thirties, I spent almost all my primary school arithmetic lessons employing the rule of three in what seemed to be an interminable series of costing exercises: if this many of such and such costs so much, how much does that many cost? Every now and again an example would appear that, conditioned as we soon were, we would all get wrong to the amusement of our teachers: if it takes 3 minutes to boil and egg, how long will it take to boil 10 eggs?; if Henry VIII had 6 wives, how many wives had Henry IV? It seems clear now that this curriculum choice was a relic of Victorian education for future clerks; but it was certainly not questioned by teachers or parents in a Manchester just emerging from a slump and still maintaining its "middle-man" white-collar hold on the Lancashire cotton industry.

Nowadays it seems that textbooks do not refer to the rule of three as such; it also seems more common to calculate holiday expenses rather than shopping bills. But most secondary school mathematics courses still seem to circle round the issue in some guise or another: speed, percentages, conversion graphs, scale drawings, pi, trigonometric ratios, equivalent fractions, enlargement and so on Does this ubiquitous linearity all stem originally from commercial practice?

Put that baldly, the question is easily deflected — of course the development of mathematical practice is more complicated and more subtle. But any attempt to discuss this development must necessarily be a construction (rather than a reconstruction) At a time when there are an increasing variety of such constructions being offered, it becomes particularly important to "deconstruct" hidden ideologies. There was a time, for instance, when historians of mathematics would very confidently assert that mathematics began in the needs of highly organised social systems to calculate taxes and keep inventories. In a less confident economic climate, there has begun to be some cautious speculation about other origins. It remains difficult to know how to judge the issue except by allowing oneself to be swayed by the rhetoric of one, or the other, side.

Moreover, at the end of the twentieth century, the role of mathematics in a school curriculum is still discussed in terms of a polarisation — loosely speaking — between the needs of the market-place and the personal development of students But the participants in such debates rarely stand back from their pre-conceived positions to explore the often-hidden assumptions with which they start There seems to be an enormous amount of work that needs to be done here. I cannot pretend to be able to do more than scratch the surface in this exploratory article which — though I am ultimately more concerned with the problem about curriculum choice — will keep to historical issues.

To return to the Renaissance interest in proportion problems, it should be noted that the full title of Pacioli's textbook was *Summa de arithmetica, geometria, proportioni et proportionalita.* This reminds us that an inherited tradition explored *proportion* as an aspect of cosmology or morphology that was distinct from arithmetic or geometry. Mediaeval writers had used the word *proportio* for the "ratio" (Greek *logos*) between two magnitudes and *proportionalitas* (Greek *analogia*) for the equality of two ratios. (The word *ratio* was not introduced until the 17th century; it has never really passed into popular usage.) Architects had consciously invoked the Greek theory of ratio, especially of incommensurables, in order to achieve the right "commodulatio" in their designs. When emphasising Luca Paciola's close contacts with commercial practice, it is also worth recalling that he is said to have been a student of the painter Piero della Francesca, whose work is centrally concerned with harmonious proportions, and that he also wrote a book on the regular solids, *De divina proportione,* that was heavily influenced by Piero.

The 2nd century writer Nicomachus seems to have translated the theory of ratio between magnitudes into algebraic terms by elaborating a theory of fractions, each of which, in the absence of an adequate notation, had to be separately defined and named; the ratio 9:76, for example, could be still called *suboctupla subsuper quadripartiens nonas* in the fifteenth century. Vieta continued the development: "every equation is a solution of a proportion ... every proportion is the construction of an equation." Algebra helped to free ratio from the tyranny of magnitudes; people began to identify ratios and fractions and some writers on mathematical education, like Lebesgue, have been particularly scathing about the continuing retention of "ratio and proportion" in the school curriculum. Others have continued to press for a distinction between fractions and ratios, and distribution problems of pythagorean purity still remain popular with examiners and textbook writers.

The original Pythagoreans are said to have encountered whole-number ratios in the course of their investigation of musical scales. These were certainly invoked in the early mediaeval development of music. By the fifteenth century, people were using musical metaphors to describe some preferred proportions (i.e. ratios); according to the architect Alberti, if any part of a building plan is altered, "si discorda tutta quella musica". Some ratios described by Alberti were, for example, *diapason* which was an octave (12:6::2:1), *diapente,* a fifth, i.e. the ratio known as sesquialtra (12:8::9:6), and *diatesseron,* a fourth or sesquitertia (12:9::8:6). The latter ratio specifically identifies Pythagoras as in Raphael's picture, *The school of Athens.*

Some of these ratios were also found by the Pythagoreans in making combinatorial comparisons of the regular solids; these are given cosmological significance in Plato's *Timaeus.* Now, according to Plato, "to the man who pursues his studies in the proper way, all geometric constructions, all systems of numbers, all duly constituted melodic progressions, the single ordered scheme of all celestial revolutions, should disclose themselves .... (by) the revelation of a single bond of natural interconnection." One particular interconnection that is very difficult to trace is the appearance of the musical intervals of a fourth (3:4) and a third (4:5) in the simplest Pythagorean triple (3,4,5). It is known that such triples were known before Pythagoras; for example, a famous Babylonian tablet lists fifteen.

But why would anyone have wanted to do this? What sort of explanation is acceptable? How can we decide between conflicting ones?

Oscar Neugebauer, who first deciphered the incomplete Babylonian tablet referred to by its catalogue number Plimpton 322, is careful not to speculate about *why* the particular list should have been made. But, in general, he emphasises the algebraic/arithmetical aspects of Babylonian mathematics and claims that geometric interest and achievement was insignificant: for him, the triples seem to be about squares of numbers rather than numbers of squares. The development of this sort of algebra might then be seen as "internally" generated by an elite body of scribes who developed their number system as far as they could. Another scholar, Derek de Solla Price, has suggested that the Plimpton list continued with further entries; he claims that the motivation is the expression of the area of the square on the diagonal (i.e. the hypoteneuse) in terms of the area of the square on the (even) side; for example, in the case of the (3,4,5) triangle the area would be 25/16. This suggests a mensuration problem but again it is not clear whether this has any particular application or source.

Taking a quite different point of view, Ernest McClaim, has suggested that the table contains musical ratios within the interval of a perfect fifth. This suggestion is related to a general interpretation of some cosmological myths — in particular, the *Rig Veda* — in terms of the problem of making a musical scale. Instruments are tuned from a reference pitch from which the tones of a scale are *generated* one at a time according to an established set of rules. McClaim suggests that many myths invoke this generation by sexual metaphors. He develops a detailed interpretation of the *Rig Veda* that leads him to assert that the square root of two, which is the crucial factor in the generation of musical scales, is being referred to in a passage such as: "Between the wide-spread world halves is/ the birthplace: the Father laid the/Daughter's germ within it".

Many people find this sort of thing far-fetched. It belongs to "fringe" speculations which, to some rationalist minds, easily pass into crazy, undisciplined notions like ley-lines, extra-terrestrial visitations and so on. But there are some interesting questions about fringe historical speculation. What prompts them? What orthodoxy are they reacting against? What makes a particular speculation suddenly become respectable and widely discussed? What makes another disappear forever? What ideologies are concealed within the passions with which they are urged or rejected? There are various examples that are less easy to dismiss than UFOs and the like. For instance, Giorgio de Santillana is a widely respected expert on Galileo and his time, a period when there is plenty of documented evidence about what people thought; what are we to make of his inversion of traditional accounts of how constellations are named in various cultures after gods, so that myths are read as being *about* the constellations?

Another example concerns a reaction to the notion, which has been referred to already, that mathematics began with the keeping of records, whether astronomical, military or mercantile. The traditional view tended to go with a pedagogical emphasis on cardinal number and was supported by mathematicians immersed in set theory and by psychologists who borrowed notions like one-to-one correspondence when investigating the development of number in infants. An older emphasis on ordinality was invoked by Abraham Seidenberg in his 1962 paper on the ritual origins of counting. "One, two, buckle my shoe/ Three, four, knock at the door." The gods are ushered into the ceremonial; the stars majestically ride across the sky; men create numbers "in rite order"

This was speculative stuff indeed when it first appeared; but it was taken seriously for it was part of a continuing discussion about whether new ideas or artifacts arise independently in various places or whether they are diffused from some single source. On the whole, those who prefer explanations of independent invention — and this included most historians of mathematics at the time — tend to find the origins of counting in terms of mastering the environment, e.g. counting animals, measuring fields, and so on. On the other hand, diffusionists tend to seek explanations in terms of rituals and myths. Why should one plump for one rather than another? What would be convincing evidence either way? Are there further implications in either case?

The diffusionist case was given some academic status, at least in the UK, by references, albeit cautious, in the Open University's course on the history of mathematics, first published in 1975. This influential course was prepared by a team that included the mathematical historian, Bartel van der Waerden. It is, therefore, particularly interesting to turn to his recent [1983] book on geometry and algebra in ancient civilisations [1], which is firmly diffusionist. The opening chapter is on Pythagorean triples! The Plimpton tablet is discussed and related to work on triples in Chinese, Indian as well as Greek, mathematics. For this author, the motivation for listing triples lies in the classroom; he claims that there was a tradition of well-chosen didactic sequences of problems with solutions that spread across Babylonia, Greece and China.

As in the work of the authors mentioned above, it is not emphasised — perhaps being taken as read — that the Plimpton list is really rather surprising: it is ordered in increasing size of an acute angle of the corresponding right-angled triangle; the opening line refers to the triple (119,120,169) giving a triangle with one angle just over 45 degrees; the simple triple (5,12,13) is not listed and though some authors have conjectured a continuation that would have included it (with an angle of nearly 67.4 degrees), such a list would not have included the equally simple and well-known (15,8,17). Moreover, the extant columns list values of z^2/y^2, z, x for each triple (x,y,z) and the acute angles change in small jumps. The whole thing has the feel of a set of tables. It is still not obvious what problems these tables might have been designed for. The ordering is clearly geometric but the intention seems to be algebraic; for example, the table could have been used to approximate square roots or perhaps the musical ratios McClaim suggests

Van der Waerden also notes some simple triples in the designs for megalithic stone circles that have been proposed by Alexander Thom and others. This confirms his diffusionist stance; and the centre of diffusion is placed firmly in Western Europe. There are some obvious speculations that can be made about the attraction for Europeans in shifting historical attention from Mesopotamia to the EEC countries. But what is particularly remarkable is the respectability now conferred on the work of Thom, whose early articles in the sixties were — and still are in some quarters — treated with some disdain by academic archaeologists. Thom's speculations about Stonehenge and other neolithic sites were enthusiastically taken up in the late sixties by the young who had other axes to grind. It seemed as if there could be a renewal of values in lost neolithic wisdoms. The more this was pursued, the more cautious and doubting were the academics. The support of an astronomer, Fred Hoyle, did not help as he had lost the argument about the creation of the universe and was being considered by the scientific establishment as an untrustworthy enthusiast for fringe ideas (he is currently proposing that certain germs came to us from early visitors from outer space).

Neolithic artifacts have also been discussed recently by an architect turned speculative historian, Keith Critchlow: in particular, some unusually carved stone balls, mainly found at sites in Scotland. Surviving specimens include, he claims, symmetrically carved examples of all the five regular solids. (Some photographs of these were reproduced in the first issue of this journal.) He has suggested that these indicate a level of sophisticated mathematical knowledge well before the Greeks. Such work is indeed very speculative and can hardly be backed up with the wealth of statistical argument that for example has made Thom's thesis acceptable to many. But it does seem relevant to anyone wanting to get a sense of what might have led people to become aware of a limited number of regular solids, especially as this awareness is often cited as a major achievement of classical Greek mathematics.

Critchlow has been interested in various other forms of "lost knowledge"; for example, he has studied the famous floor maze of Chartres Cathedral, claiming to have found in it, as in the related characteristic pattern of hopscotch, echos of pre-Christian cosmology. Throughout his work is a yearning for "traditional understanding of the wholeness of the human condition, body and psyche". It is a regularly recurring intellectual habit to glorify ancient wisdom; the associated rhetoric is often easily dismissed. But in considering such diverse examples as the attempts to reconcile Eastern mysticism and modern physics, or the revelation by scholars of a hermetic tradition in Renaissance thought, what seems to be most significant is *why* a particular speculation is accepted and another is ignored; it does not always seem to be a matter of detached appraisal of carefully presented evidence.

As far as the history of mathematics is concerned perhaps the most compelling and most influential, yet the most disputed, speculations have been about the nature, origins and purposes of Greek mathematics. For the purist there is almost nothing that can be said about the early classical period with any certainty. We know the names of a handful of individual mathematicians over a period of about three hundred years. Our accounts derive from secondary sources, so much so that some scholars have even doubted that there was an actual historical person called Pythagoras let alone the arithmetic tradition that is mainly interpreted from commentaries written *several centuries* later. Yet there are some powerful and compelling strands in the general account which have dominated the development of mathematics since; for example, the discovery of irrationals, the tactile geometric approach, the development of deductive proof.

Such aspects have been mythologised to such an extent that it hardly seems relevant to question whether they describe what was the case. This is, however, to accept a view that "narrative" truth, or myth, is — in some situations — more important than historical truth; it is to accept willingly that myth grows by accretion, so that, for example, what people have thought about Greek mathematics may become part of the history of Greek mathematics. Many would strenuously disagree with this view and are particularly scathing about the liberties that are taken by the makers of myths. In the last few decades, for instance, a growing academic discipline in the history of mathematics has tended to be very scornful of the popularising work of Eric Bell. My own view is again that it is more important in such a case to consider what uses a particular myth has and why it might be retained or discarded. For all Bell's unfashionable cult of personality and sense of history as heroic individual achievement, his work did emphasise that mathematical notions were human constructs, generated through personal awareness. I found this emphasis useful when I taught adolescents; it still offers, I think, *a good story*.

Currently there is some critical revision of the notion that the discovery of irrationals was accompanied by a supposed *horror infinens*, a retreat into finite mathematics. It has been suggested, by Wilbur Knorr and others, that Book II of Euclid's *Elements* contains a pre-Eudoxian theory of proportion that was a controlled and confident response to the discovery of irrationals. David Fowler has claimed further [2] that the procedure of anthyphairesis (repeated subtraction as in the so-called Euclidean algorithm) is in fact a definition of *logos* (i.e. ratio), which is notoriously weakly defined in Book V. This, according to Fowler, enabled mathematicians to handle irrationals in a way that was a geometric counterpart to the modern algebraic technique of continued fractions. This proposal implies a fairly radical recasting of part of the standard interpretation of classical Greek mathematics. It has an attractive mathematical coherence and interest — indeed, this is presented as part of the argument supporting the admittedly conjectural reconstruction.

Who will decide whether such reconstructions are correct? Or appropriate? Much seems to depend on the agreement of a handful of acknowledged authorities. Failing the unlikely discovery of some further documentary evidence, no-one can prove authoritatively what was or was not the case. So, to some extent, all of us are in charge of the myth. We may note that some scholarly revisions still require the notion of a lost or hidden wisdom. Moreover, we may also note that many reconstructions are offered by mathematicians who seem to dislike on ideological grounds the notion of "crisis". There is, for instance, a contemporary revision being offered of the standard account of a late nineteenth-century "foundations" crisis. Perhaps gripped by that other powerful myth about Greek mathematics, its purity and freedom from the market-place, mathematicians also tend to ignore any accompanying social factors. But the discovery of irrationals and the tale of the fate of Hippasus of Metapontum does aptly symbolise a culture in crisis. It is not irrelevant, I suggest, that between the time of Thales and Theatetus there was a steady increase in the number of slaves in the Greek world, and increasing tension between the city-states. The purity and ontological security of mathematics is exaggerated for ideological reasons, both then and now.

Purity is also associated in mathematics with proof. Pedagogues have prized Euclid's *Elements*, until very recently, for its supposed training in deductive logic and this version of Greek mathematics dominated education for many centuries. Modern commentators have been interested in looking at Euclid with axiomatic spectacles. Indeed, Ian Mueller, in his definitive study [3] of the deductive structure of the *Elements*, specifically sets out to compare Euclid with Hilbert. He starts by suggesting that Hilbert's formalism can be variously interpreted. Hilbert characterised axioms as expressions of "fundamental facts of our intuition", but it is not clear whether these are constructs or insights into reality. Furthermore, the constructs might be purely formal — which is the usual interpretation — or they might be multivalently related to the various unspoken shared images and meanings. Thus it may not be that "arithmetisation" has now replaced geometry, but rather that geometry is "simply an interpretation of certain parts of modern algebra".

The historians continue to debate the extent to which (Greek) geometry is an interpretation of certain parts of (Babylonian) algebra. Mueller tends to support the orthodox account of Book II as "geometric algebra" but his particular interest in structure does lead him to less concrete interpretations than are customary. For example, he asserts that the most appropriate interpretation of magnitudes is as "abstractions for geometrical objects ... quite like numbers", and that, whereas fractions are objects, ratios are not. This seems to be a reading from hindsight. It might be more useful to invoke Oswald Spengler's dictum: "There is no such thing as number as such — only several number worlds."

One of the attractions of the anthyphairetic point of view is that it does offer a very concrete, tactile image for *logos*. This is quite apart from the recent suggestions that the mathematical usage of the word derived from the concrete image of a musical interval or consonance. But in any case, it is from various points of view always worth emphasising the human subject who mathematises. Mathematics has to be taught, and when working with others it does seem as if *some* sort of imagery is invoked and manipulated, whatever the written record says. I think it is possible that experience of teaching mathematics can, and should, be used to contribute to the development of the myth about mathematics.

Robert Brumbaugh, commenting on the difficult problems of interpreting the mathematical passages in Plato's writings, points out that we easily forget that the technical device of accompanying text with diagrams had not occurred to anyone when Plato wrote. Moreover, mathematics had not then become divorced from music or astronomy; number was not yet dissociated from, say, sex or justice It is an open question what images contemporary readers carried in their heads, but the texts amply suggest that some were expected. The situation might be contrasted with the story (which is often misinterpreted) about the geometer Jacob Steiner not drawing figures — indeed, drawing the blinds during his lectures — so that his students could more easily construct and work on their own images.

The possibly misleading specificity of a particular image was of course one of the issues in the nineteenth-century development of analysis. But it occurs right at the heart of any mathematical proof, as Mueller subtly indicates, for proofs in mathematics all involve "the setting out of an apparently particular case and arguing (a general case) on the basis of it". Euclid followed a general statement of a theorem (the *protasis*) by a specific statement of what was to be proved for a particular figure (the *diorismos*); the theorem was then re-stated at the end (the *sumperasma* — Q.E.D.). This three-fold repetition indicates Greek awareness of the problem about imagery.

So far I have been implying that various historians of early mathematics (and I — and perhaps, dear reader, you) have particular interests at heart in their necessarily speculative re-constructions. To suggest this and to look at some of the interests involved is not to denigrate the points of view that are arrived at. But, as I hope I have made clear, I think the question of which versions become acceptable cannot be settled "objectively". It is, I claim, a question of preferred myth. Those concerned with pedagogy may need to work hard to preserve mythopoeic elements that they find powerful and helpful as well as discarding ones that seem constraining and elitist.

Consider, for example, the detailed study of the philosophy of nineteenth century geometry by Roberto Torretti [4]. This is a mandarin, often highly technical, account which will surely be of absorbing interest to mathematicians and influential in its contribution to the ongoing interpretation of the contemporary mathematical enterprise. It openly takes a twentieth-century view of nineteenth-century mathematics. For example: "it can be proved that Exp_p maps a neighbourhood of 0 in $T_p(M)$ diffeomorphically onto a neighbourhood of P contained in V; we shall see that an essential step in Riemann's investigation rests on this result, which he, with his incredible flair for mathematical truth, assumes without proof".

But it also betrays unacknowledged assumptions on almost every page. Thus, Poincaré, whose conventionalism is sympathetically presented, is called a great mathematician "seduced by his philosophical colleagues into believing their psychological fantasies". It is claimed that mathematical misconceptions are not uncommon among "*soi-disant* scientific philosophers" and philologists earn particular scorn. In discussing Felix Klein's statement that he found it necessary to refer constantly to a figure, Torretti confesses that he cannot see "how an admittedly imprecise image can be of any help in the actual proof of a statement concerning the unambiguous ideal entities determined by the axioms"; his "pedestrian imagination" is not always able to follow someone else's "in its sometimes frenzied flight"; and there are lots of assertions like "it is not difficult to see ..." or "we see at once that ..."

It is intriguing to note his comment on the problem about particular images: "An intelligent look is not overwhelmed by the rich fullness of its object, but pays attention only to some of its features". This begs a lot of pedagogic questions! But it is in a longstanding tradition. For the fifth-century pedant, Proclus, the problem of seeing the general in the particular was that of finding the One in the Many; the aim was to lead geometry "out of Calypso's arms ... to more perfect intellectual insight, emancipating it from the pictures projected in imagination". Disdaining Calypso causes a lot of problems in classrooms.

Torretti realises that there may be a problem about whom he is writing for: "I fear, however, that many a reader will look down upon (this) as just another piece of algebraic abracadabra, incapable of giving any insight into the origin of geometry". His account of the historical background starts from the Greeks and concentrates on those aspects that will lead him quickly to the nineteenth-century metaphysics of space. Before "space" became a structured point-set, it was thought of as a medium in which the points and lines of geometry were somehow embedded. The problem was whether this medium was a mental construct or part of "reality". Torretti asserts that the Greeks had quite different notions and criticises previous writers who had claimed to find the modern notion in ideas about the Void held by Democritus and others.

Commitment to contemporary solutions of age-old metaphysical problems may help explain why there has been so little general discussion of an interesting and relevant interpretation of Parmenides who, in reaction to the Pythagoreans, criticised the concept of the Void out of which non-Void, or Being, was held to be continuously created. De Santillana interprets "being" as precisely the modern notion of space. In quoting the relevant fragments, I use a standard translation [5], rather than the one given by de Santillana, in order not to beg the issue; this uses the phrase "what is", rather than "being".

> "Yet look at things which, though far off, are firmly present to thy mind; for thou shall not cut off what is from clinging to what is, neither scattering itself everywhere in order, nor crowding together ..."
>
> "Nor is it divisible, since it is all alike; nore is there more here and less there, which would prevent it from cleaving together, but it is all full of what it is. So it is all continuous; for what is clings close to what is ..."
>
> "... for it needs must not be somewhat more here or somewhat less there. For neither is there that which is not, which might stop it from meeting its like, nor can what is be more here and less there than what is, since it is all inviolate ..."

This Space — if that is indeed what is being described — is certainly continuous, homogeneous and isotropic, as Poincaré required It seems tangible and contained, consonant with the idea of some sort of withdrawal from the open and unlimited. It is not clear whether it is bounded rather than finite: "since there is a furthest limit, it is bounded on every side, like the bulk of a well-rounded sphere . . ." In any case, the important point here is not whether the Greeks did or did not have our concept of space, but that viable interpretations of the writing of a pre-socratic philosopher might be worth incorporating into the myth about the nature of Greek mathematics. De Santillana's thesis is exciting and carries a certain inspirational charge. I confess that I am moved by it and that I want to incorporate it into my myth about what happened after Pythagoras.

My interest in the construction of myths about early mathematics — other people's as well as my own — partly stems from my continuing attempts to digest the remarkable seminar on the history of mathematics held by Caleb Gattegno at Bristol in the autumn of 1984 Can the history of mathematics be recast in terms of awareness? Gattegno's question was difficult; but it turned out to be particularly fruitful in that it offered a wide range of insights, exposed various shades of opinion amongst members of the seminar and yielded many more questions.

I can indicate here only briefly my own conclusions — which are still tentative and idiosyncratic. The main thing *I* took from the seminar — or, indeed, imposed upon my memory of it — was a heightened sense of the importance of the history of mathematics for teachers as opposed to scholars In a sense, the past only exists in the present, so that its echoes have necessarily to be sifted in order to preserve only that which is deemed to be of any significance for the future The continuing reflexive generation of the account mathematics gives of its own history is too important to be left solely to historians — or indeed mathematicians. Teaching is part of the mathematical enterprise and teachers can help decide what is to be considered significant at any one time. They do already unconsciously do this, but the challenge is to do it knowingly, from an understanding of the acts of mathematisation that lie behind mathematical content, from — to return to the seminar — a recasting of the historical record in terms of awareness.

One of the threads that ran throughout the seminar was the notion that though things could look alike from our present point of view they should not therefore be supposed to be the same. According to Gattegno (and I quote, here and later, from a tape-transcript), "One of the things that we shall have to dispel... is this notion that because Newton and Leibniz were concerned with the calculus at almost the same time, their ideas were not at all (like) what people did afterwards. . .their inspiration and their relationship to it . . .were totally unrelated." The suggestion seems to be that, for example, we should be careful to distinguish how Pythagoras worked from the Pythagorean "tradition" inherited through others. The inheritance that matters is not the history but that which contributes to our own evolution: "we have to be concerned with mathematisation not mathematics"

How can we become aware of past acts of mathematisation? This was an obvious problem for many of us in the seminar. Gattegno gave a hint or two about what the answer was for him: "I have made myself vulnerable to the workings of the mind. When I read something, I read it at two levels; I am concerned with what there is to learn and I am concerned with what the person who is there is saying about how things are done. . . It is not always clear, but it is often there and not picked up because we are concerned with knowledge. I am concerned with how they worked."

Intuitive claims of this sort are not easily accepted by others as a way of knowing about the past For example, not many scholars were prepared to recognise Robert Graves' claim to be able to interpret myths, not only by the necessary textual study but also by his poetic imagination which enabled him, he claimed, to enter more closely into their meaning. On the other hand many historians of mathematics coming from a mathematical background have been very critical of, say, philologists, who have made certain claims about Greek mathematics without — their critics imply — an appropriate mathematical awareness. Was Augustine a better saint for having been a sinner? Perhaps each example has to be judged on its merits. The relevant question to consider here is whether teachers can claim to some special awareness about mathematics through their experience as teachers, as distinct from — though no doubt complemented by — their experience as mathematicians. To be "concerned with what the person who is there is saying about how things are done" is indeed to be able to surrender oneself in a way that might be universally recognised as being the essence of good teaching.

One example, among many that were offered in the seminar, of a relatively unorthodox approach was a juxtaposition of two famous proofs in set-theory. Cantor had shown in 1873, a year after he had met Dedekind, that the rational "may be corresponded to" the natural numbers whereas the reals could not. Textbooks now give the proofs that he offered much later The method of enumerating the rationals, tantamount to labelling the points of a square lattice which are then threaded along the diagonals, must have been worked out by 1886 at least, for it was given in a letter of that year to a Berlin schoolteacher, Franz Goldscheider (Cantor did not give a diagram, his ordering always traverses the diagonals in the same direction; textbook writers usually modify this to one that zig-zags up and down.) As for the famous diagonalisation method of proving that the reals cannot be enumerated, this was formulated for a lecture in 1891. Had the *image* for one proof influenced the development of the other? Is a suggestive juxtaposition — one that is almost a pun — a legitimate way of getting at an insight into how things happened?

Another example — which will also disturb specialist historians but may delight teachers — illustrated how a myth can be developed and used creatively in the classroom. The life of Thales is almost totally legendary; the myth is of a traveller, merchant, and engineer, in binary contrast perhaps with Pythagoras, the vegetarian and ashram guru. Although British writers sometimes give his name to the theorem about the angle in a semi-circle, Thales' Theorem has always meant in Europe the midpoint theorem or its generalisation. What is this all about and how do you get anyone to be involved in it? Well, you might start by checking that people can use ruler and compasses to divide a line into two parts. You can propose dividing it into four parts, eight parts, and so on. But can it then be divided into three? and the story follows:

> "Now wouldn't that question come to a Greek who had been working on these things and can say to himself: since three comes after two, can't I ask the question how do I divide into three parts? So the Greek fellow went to bed and fell asleep thinking: can I divide?"

> And people start offering ways of doing it ..which do not work any better than those the Greek thought of as he went to sleep:

> "And in his sleep he thought: well, I want three, I want three.. And the next day he went to the beach as he used to everyday; and on the beach he started walking, asking himself how to divide the line into three. And then he shifted his awareness from the problem to the fact that he was walking and he became aware that he was walking. Can you imagine that? And he said: Oh, well, if I stop after three steps, one, two, three, and I put this on the line, then I have divided it into three parts. Then a critic said: but what about if my line is like this. Well, he says, it is quite easy..."

Then Gattegno asked the students — for he had been describing a lesson given to a class whose teacher, Maurice Laurent, was a member of the seminar — what the man did after he had shown how to divide into three parts. They suggested that he then did it in five, in seven, in any number of parts. Thales became "someone with problems, who pondered on problems, found himself caught in tradition, broke away from tradition, tried something and found something new..."

According to Gattegno there are two ways of working in mathematics: "One is to make chains of arguments that support each other, that are consistent with each other.. But there is another movement which is what we do in the beginning when we provide definitions. This is the creative moment in mathematics, it is when we introduce a new entity.... Definitions are new starts." Theorems and definitions capture and enshrine awareness. A "narrative" sensibility may be used to unlock these, in order to offer students access to insights that they may develop and apply themselves. This is, of course, an enterprise that would only be of concern to teachers. The underlying question for them is how mythopoeic they wish, or are prepared, to be.

In the end there is still *logos*. This has many meanings other than the mathematical one, including word, spirit, reason, truth. Arab and mediaeval writers translated from this general cluster to get *rationem* = reason and *surdus* = deaf, i.e. without word. The mathematicians then used the words *ratio* and *surd*, but there was then the double meaning — in popular and mathematical usage — of *irrational* and the echoing link between *surd* and *absurd*. It is an aspect of myth that it carries associations that are not logically or chronologically relevant. If irrationals no longer produce horror, they do still — unreasonably — produce a frisson. But not in Calypso's arms

References

I do not want to burden this article with a lengthy book-list; some of the authors cited without further reference are mentioned at the end of my article on geometry in the first issue of this journal.

[1] B.L. van der Waerden: *Geometry and algebra in ancient civilisations*, Springer-Verlag, 1983. This is a rather haphazard collection of essays — some of them carelessly edited. Apart from the material on neolithic mathematics, there are some interesting accounts of Chinese and Hindu mathematics, the work of Diophantus and its subsequent development by others, and Greek "geometric algebra"

[2] D.H. Fowler: 'Ratio in early Greek mathematics', *Bull. Amer. Math. Soc.* 1 (1979) 807-846. The author's thesis has been extended and developed by him in various articles and in a forthcoming book, *The mathematics of Plato's Academy* (O.U.P., 1986?), which is likely to re-orientate our whole thinking about classical Greek mathematics

[3] I. Mueller: *Philosophy of mathematics and deductive structure in Euclid's Elements*, M.I.T., 1981. This detailed and thorough work complements rather than replaces the standard historical account by Heath. The emphasis is very much as the title suggestes, but inevitably the author is drawn into historical comment. A feature of the book is the use of modern symbolic notation to elucidate Euclid's logic; though this does tend to introduce a certain bias in interpretation, it does help enormously in following the argument

[4] R. Torretti: *Philosophy of geometry from Riemann to Poincaré*, Reidel, 1978. A rewarding study of some philosophical issues, past and present. The historical aspects I have commented on are not a very important part of the book which, apart from a curious use of sophisticated mathematical jargon and notation, does give a very readable account of the apriorist, empiricist and conventionalist positions. There is a discussion of non-Euclidean geometries with some welcome detail about Riemann's ideas.

[5] G.S. Kirk & J.E. Raven: *The pre-socratic philosophers*, C.U.P., 1957. The three quotations are from paragraphs 349, 348, and 351, respectively. De Santillana's interpretation appears in a lecture, "Prologue to Parmenides", reprinted in his *Reflection on men and ideas* (M.I.T. 1968)