APOS THEORY: CONNECTING RESEARCH AND TEACHING

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Theoretical frameworks in mathematics education may differ in aspects related to their objects of study, aims pursued, kinds of results obtained, contexts in which they can be applied, research questions that they might help answering, among others. Gascón and Nicolás (2017) question some of these aspects, in relation to the specific ends of a theory which can consist in understanding a didactic phenomenon or in a teaching related aim, possibly linked to normativity. In particular, they would like to bring to the fore and interrogate the undisputed or implicit assumptions of these approaches with the purpose of initiating a dialogue among different perspectives; they invite the community to participate in this discussion. The present paper is written in response to that call, however, it is not completely tied to it. It aims to present how research conducted with APOS Theory might be of help to instructors by focusing on some relationships between student understanding and teaching.

APOS Theory

APOS Theory is a constructivist framework based on Piaget's genetic epistemology. Research from this perspective concentrates mainly on the construction of mathematical knowledge and has focused particularly on mathematics learning at university level. This kind of investigation can have a research end consisting in identifying the mental constructions needed by students as well as an educational end that takes into account specific learning contexts. In order to reach these ends from a scientific perspective, a key component of the theory is the design of detailed models using the constructs of the theory, called *genetic decompositions,* that can be taken initially as hypotheses about how individuals learn specific mathematical concepts, based on results from literature, historical knowledge or the experience of researchers as teachers. As with other hypothetical descriptions, they need to be tested by experimental research and refined or validated in terms of the results of those studies.

The basic assumptions and objectives of APOS Theory are in line with the limitations that it defines as its domain and object of study. APOS Theory aims at describing, with its detailed "microscopic" approach, the elements of the construction of mathematical knowledge from a cognitive viewpoint. This does not imply an opposition to other theoretical perspectives that address issues in mathematics education from a different and maybe broader perspective. At some point in time it might be possible to contemplate whether these points of view could play a complementary role in providing a deeper understanding of phenomena related to the learning and teaching of mathematics (see, for example, Bosch, Gascón & Trigueros, 2017).

Mental structures and mechanisms

APOS Theory was founded by Dubinsky (1984; 1991) and developed by RUMEC (Research in Undergraduate Mathematics Education Community) (Asiala *et al.,* 1996). According to this framework learning occurs when an individual constructs mental structures through mental mechanisms. The most basic structure is an *Action*, applied to previously constructed mental objects and directed by external stimuli. When an Action is repeated and reflected upon, it becomes a *Process* through the mental mechanism of *interiorization*. Processes in turn are *encapsulated* into *Objects* so that Actions can be applied to them. A *Schema* is a coherent collection of structures connected to each other.

We should emphasize that although there might be a certain relationship between student success and the stages of mathematical knowledge construction according to APOS Theory, there is no strict implication between the two. Sometimes misunderstandings occur regarding the meaning of mental constructions. For example, we have heard people say that if a student cannot perform a task he or she "is at the Action level". However, firstly, Actions, Processes and Objects are not referred to as levels in this theory; they are structures or constructions and represent stages in the learning model which are not necessarily constructed as a linear progression. Secondly, an individual with an Action conception can perform tasks that require this structure; as building blocks, Actions are an important part of the whole construction process. In that sense an Action conception is not associated to the notion of failure, in the same way that a Process or Object conception does not necessarily imply success when dealing with a certain task.

APOS Theory and its methodology

Research performed from an APOS Theory perspective ideally involves three components. *Theoretical analysis,* which consists in the description (genetic decomposition) of mental structures and mechanisms that might be viable in the construction of a mathematical concept; *design and implementation of instructional strategies* that are informed by the theoretical analysis; and *collection and analysis of data* where a comparison is made with the constructions hypothesized in the genetic decomposition which in turn is revised and can be modified if necessary. These three components form a cycle that is repeated until the genetic decomposition and empirical evidence agree; there is also the possibility that a genetic decomposition is refuted. The three components are very closely related. The result of the epistemological work is a cognitive model explaining a possible construction of a certain piece of knowledge. Pedagogical work gives rise to the design of activities, mathematical situations and problems as well as didactical strategies such as using collaborative group work before concepts are formally introduced. Empirical work leads to the identification of the inferred mental structures evidenced by the individuals participating in the research study. In summary, from an APOS Theory perspective, research, teaching and theory development are closely tied to each other. In particular, the application of APOS Theory in design and implementation of instruction, informed by research results, is an explicit interest as well as objective of this theoretical perspective. In this implementation, "The role of the instructor is to identify the mental structures that might be needed in learning the concept and to design activities that help students make the proposed mental constructions." (Arnon *et al.,* 2014, p. 179). In that sense, there is no strict separation between an instructor and a researcher; they share a common goal and participate in activities that would facilitate reaching it.

Each component in the methodological cycle has a method associated with it. The theoretical analysis that leads to a genetic decomposition is based on several factors chosen among aspects such as the researchers' experience, observation of classes, examination of textbooks and review of literature. For the pedagogical component the ACE (Activities—Class Discussion—Exercises) cycle is employed with its variations, where the design of each mathematical situation is informed by the genetic decomposition. For the analysis component, each researcher goes through the data and puts down related observations following a protocol, after which discussion takes place until differences in interpretation are resolved by coming to a mutual agreement.

APOS Theory and teaching

Gascón and Nicolás (2017) question how different theoretical approaches formulate, implicitly or explicitly, their teaching goals (or whether they have goals related to teaching at all). Another way to investigate the relationship between theories and teaching might be starting from the teaching end and asking what goals teaching pursues. The answer depends on the context in which didactical activities are immersed. One objective might be, as Pegg and Tall (2005) state, "to stimulate cognitive development in students" (p. 191). The achievement of this goal involves planning in terms of choosing appropriate activities and strategies, deciding in which order they would be applied as well as establishing evaluation criteria. Approaches such as APOS Theory that focus on cognitive aspects of learning might be useful in serving this goal. In this sense, a theory could respond to an educational need. The characterization of APOS Theory as "a framework for research and curriculum development in mathematics education" (taken from the title of Arnon *et al.,* 2014) can be considered as an evidence for this relationship as well as for the coherence between the research goals and didactic objectives of APOS Theory. These two types of goals are not considered as separate or disjoint. In fact, research and didactic practice feed each other. Apart from having theoretical implications, understanding the construction of a concept has didactical implications; expressed differently, decisions about didactical designs have an epistemological basis in APOS Theory.

The kinds of research questions that can be posed from the viewpoint of APOS Theory might be formulated from a developmental perspective, such as, "How can the function concept be constructed by a generic individual?". They can also be evaluation oriented, such as, "What level of knowledge, ability and understanding regarding the concept of function can these students attain in their first three years of high school?" (Dubinsky & Wilson, 2013, p. 88). These sorts of questions can appear alone or together in the same study where each type determines the role that the theory would play in the research process as well as the nature of results that can be obtained. They also imply the type of didactical component to be included in a specific study.

APOS Theory does not take a prescriptive stance in that it does not state its pedagogical recommendations using the verb 'must'; rather it offers didactical suggestions backed up by theoretical analyses and empirical evidence. Its position with respect to the learning of mathematical concepts is that there might be several ways to construct a piece of knowledge. As such, it does not concern itself with proscriptions impeding certain approaches to the teaching-learning process. According to APOS Theory learning progresses through mental structures; certain pedagogical strategies such as the use of programming activities motivate the development of these conceptions. It is through reflection on their mathematical activity that individuals construct mathematical knowledge that is congruent with those mathematical ideas that have been socially developed. The main purposes of APOS Theory are understanding the construction of mathematical knowledge and using this knowledge to improve learning; however, it is not asserted anywhere in this approach that there is a unique way to achieve this goal. If we consider the improvement of learning as an end (referred to as 'E' in Gascón & Nicolás, 2017), the implicit *value content* (we think it is more appropriate to use this expression instead of *value judgement*) appears in the form of pedagogical suggestions and design of activities based on a genetic decomposition.

Although most studies conducted using APOS Theory concern students' understanding of mathematical topics, this framework can also be used to focus on the teacher's role and how it relates to students' constructions (Gavilán Izquierdo, García Blanco & Llinares Ciscar, 2007) as well as to reflect about teachers' professional knowledge (Badillo, Azcárate & Font, 2011). Even if a course is not designed using the theory, the analysis of reflection opportunities opened by the teacher, the tasks used in class and the way teachers interact with students may be analyzed in terms of their favoring or not the constructions described by a genetic decomposition. Contrasting this analysis with students' actual constructions can also inform both the genetic decomposition and the teacher's practice.

An example

An example may illustrate how research conducted through an APOS Theory lens can identify specificities about teaching and the construction of knowledge. For several years, we have been studying students' construction of linear transformation concepts (see, for example, Romero Félix & Oktaç, 2015). Results from each study gave rise to a refined genetic decomposition, according to which algebraic and graphical Actions are interiorized into the respective algebraic and graphical Processes. Each pair of algebraic and graphical Processes related to the two linearity properties are coordinated to give rise to two Processes: the Process of multiplication by a scalar property and the Process of addition property. These Processes are coordinated to give rise to the linear transformation Process which is encapsulated into an Object. In these studies and another focusing on students' intuitions (Molina & Oktaç, 2007) it was observed that in certain contexts some students associated transformations to individual vectors as opposed to the vectors of the whole domain (a similar observation is reported in Sierpinska, 2000). Molina and Oktaç (2007) suggest that students may associate transformations to single vectors because of the influence of some textbook figures where a single vector is shown together with its image under a linear transformation. Since one of the goals of APOS Theory is the analysis of educational phenomena, and this tendency had not been investigated thoroughly in the literature, we decided to study it taking into account students' constructions and their possible relation to teaching.

In terms of APOS Theory, this phenomenon can be related to the difference between Action and Process conceptions. We used a genetic decomposition to prepare a questionnaire and an interview where one of the questions was designed to specifically address this situation:

Q. What happens to the vector $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ *when the matrix* $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ *is applied to it? Is it rotated or reflected? Justify your response.*

As a result of this study we found that responses to this question clearly pointed out the differences between the types of answers given by students who showed an Action or a Process conception of linear transformations. Students showing an Action conception tended to focus only on the given vector and to find its image by multiplying the given matrix by that vector and/or chose to represent vectors graphically to decide their answer based on their intuition according to visual cues provided by the graphical representation. A Process conception involves a generalization to the vectors of the whole domain; students having constructed this type of conception might choose a generic vector and observe the effect of the matrix on that vector; or, argue in terms of the characteristics of the matrix, observing what kind of transformation is involved.

The questionnaire was applied to undergraduate students from three different institutions in Mexico. From the total of 31 students who answered this question, 23 clearly based their responses only on the given vector and its image. They did not refer to the general effect of the matrix, nor apply it to another vector. They did not take the initiative to choose another vector because their strategy was Action oriented and Actions are externally guided. This does not imply that these students answered all the questions performing Actions; in fact, some of them showed elements of a Process conception. However, for them answering this question does not require more than performing the Action of applying the linear transformation to the given vector.

After the questionnaires were applied, 7 students were interviewed with the purpose to examine their conceptions more closely.

One of the students who considered the transformation as a reflection was asked to choose another vector and find its image under the same matrix. The intention was to confront his first answer with what happens in this new situation. After the student applied the matrix to [5 3] and got the image vector [3 –5] he said it was a rotation. He added that in the case of [2 2] it was a reflection and that the effect of the matrix depended on the vector chosen. This student showed the ability to apply the Action of finding images of vectors under linear transformations given in their matrix representation but had not yet interiorized it into a Process and hence could not think about the transformation as being applied to the vectors of the whole domain.

Another student attempted to generalize and said that the image of a vector [*x y*] would be [*x* –*y*]. This might give the impression of the student showing a Process conception, however this particular generalization did not originate from repeating Actions, reflecting on them and interiorizing them into a Process; even if this student was thinking about the transformation as being applied to the whole domain, the transformation was not actually being applied to all the vectors in it. Hence, the effect is described as a reflection because the student sees a reflection in the given situation.

A Process oriented strategy was to examine the effect of the matrix on a general vector. One student said that the matrix inverts the axes and changes the sign of the y-coordinate; hence it is a rotation by an angle $3\pi/2$. The application of the transformation to the whole domain is clear in this response.

Another student took another vector [1 2] by his own initiative, calculated its image, and from these two cases responded that the transformation is a rotation. This student argued that [2 2] is a particular case where rotation and reflection give a similar result. The Process conception as an interiorization of Actions makes it possible for this student to repeat those Actions, when necessary, as part of a strategy that considers the whole domain of the transformation.

Results obtained indicate that students who participated in this study when faced with this question tend to focus on a single vector to decide about the effect of a linear transformation. The novelty of the situation played a role in their responses. Conceptions do not lie in individuals nor in the mathematical problems; they have to do with a dialectic relationship between the two (Dubinsky, 1997). On the other hand, the fact that students from three very different institutions were involved in the study points out that the results cannot be attributed to an immediate didactical obstacle.

Many students who may be considered to have developed sufficient knowledge about linear transformations after having completed a course in linear algebra, may in fact be functioning under an Action conception. Memorizing procedures, imitating proofs and the related discourse might lead them to succeed in the course, but if the basic structures are not constructed, these strategies fail at some point.

As a next step the instructors of these students were interviewed. T1 is a young teacher with three years of teaching experience. T2 has been teaching for 12 years and has recently obtained a doctorate in Mathematics Education. T3 has been teaching for more than 30 years. All of them have some knowledge of APOS Theory from participation in seminars and T2 used APOS as part of her thesis' theoretical framework. They were asked about their teaching practices concerning linear transformations and the kinds of responses that they would expect from their students regarding the question (Q). They were confronted with their students' responses and their reactions were recorded.

When talking about their teaching none of the instructors referred to using APOS Theory, although all of them mentioned that participating in the seminars made them aware of the need to listen to their students and of discussing their ideas when explaining new concepts. All of them described their teaching as presenting their students with interesting situations, letting them reflect and then using whole class discussion to review students' work as well as to institutionalize the introduced concepts.

All the instructors expected their students to answer Q correctly. They were all surprised to find out that this was not true in most of the cases. For example, when T3 was shown responses from his students such as the one appearing in Figure 1 where it says that the vector is rotated by 90 to the right or it could be a reflection about the x-axis, he made the following comment:

> I was not expecting this, that they wouldn't verify. *[...]* I feel that their knowledge consists of pieces that they do not relate, and perhaps when you do it you think they understood and they connected things, and I know some do, and they are able to solve quite complex problems, but at other times if you present it to them again they don't know it.

Actually, T3 did not expect his students to use a generalized approach in their responses, but he did expect them to use more vectors to verify their answers since he had placed a lot of emphasis on verification in class. He was disappointed that the majority of his students did not verify. When asked about how this difficulty could be overcome, he said that he would emphasize the fact that a transformation is a function and as such, focus more on its domain.

During her interview, T2 said that as a result of the questionnaires and interviews, she now includes different kinds of problems in her teaching, such as asking for the image of whole figures instead of only vectors; however, she cannot use questions that were not solved in class for evaluation purposes in exams.

After seeing the results of this study, T1 also started to employ questions that emphasize the functional character of transformations, including their domains and images. He remarked that through these results he realized that what the teacher knows or does in the classroom has little effect if the teaching is not student centered.

Taking into account the results obtained, it is clear that the teaching component of APOS Theory can help teachers to reconsider their approaches. From an APOS perspective, our suggestion would be to design instruction informed by a genetic decomposition and directed towards the construction of those mental structures called for by it. In particular, based on our observations in this study, problem situations that might be helpful can include asking the students to apply a specific linear transformation to different vectors of the domain and to try to explain its effect on specific vectors as well as on the whole domain; repeating the previous question for different linear transformations as well as different domains and ranges. These situations invite the students to reflect on their Actions, leading to their interiorization, but, as stated before, these suggestions are not normative.

Closing remarks

The hypotheses, context of study and objectives of APOS Theory as well as its limitations are clearly exposed as part of the framework. The theory includes a research methodology which is coherent with its research and teaching ends. It is employed to investigate educational phenomena of interest. It also includes a methodology of curriculum and instrument design consistent with the purpose of helping students to construct mathematical concepts. Genetic decomposition plays a prominent role as a model which predicts possible constructions and guides both the design of research and the development of didactic material.

Confronting this genetic decomposition with observations

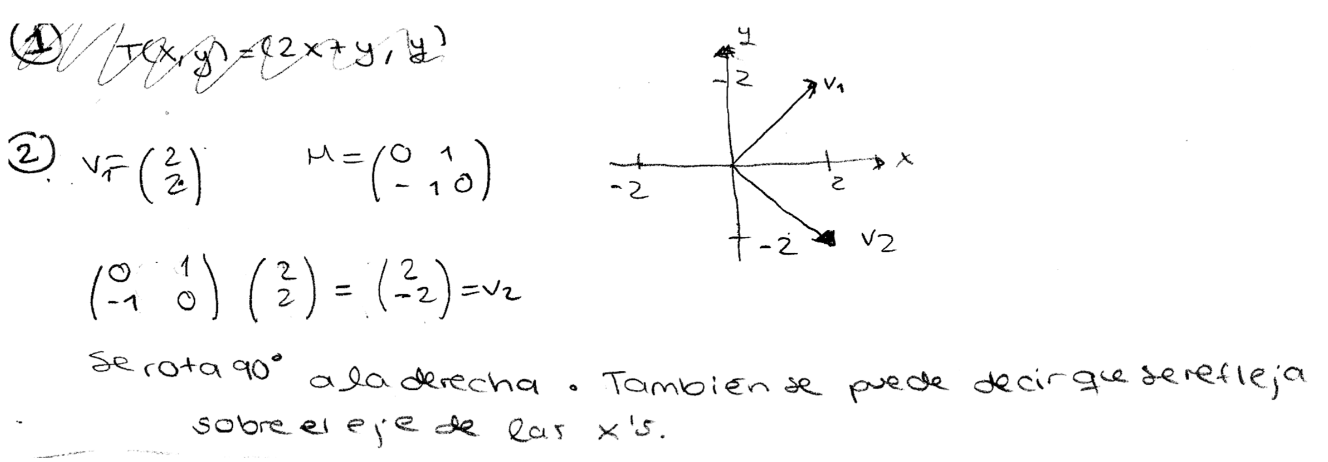

Figure 1. A student's answer.

of individuals while they are constructing the concepts allows recognition and diagnosis of different aspects of learning: characteristics of cognitive structures and relations between them. It also allows the design of theory-based teaching activities as pedagogical proposals that may help students to learn the intended concepts in line with mathematical theory.

APOS Theory focuses on describing how knowledge is constructed. As such it does not emphasize difficulties that occur in the learning of concepts; however, these can be explained naturally from the viewpoint of genetic decomposition and attended to from the same perspective. Knowledge of a genetic decomposition can also help teachers to design teaching activities and didactic strategies as well as to ponder differently the need to open spaces for students to reflect on their Actions and to develop interiorization opportunities. The difference between helping students answer specific types of problems and helping students develop a Process conception implies a strategic plan that can be suggested by a genetic decomposition. The cyclic nature of APOS Theory research makes it possible to reconsider and adjust both the genetic decomposition and the didactic suggestions intended for a generic individual. These cycles reflect the dialectical perspective between the individual and collective dimensions included in the theory.

There might be some tendency to think that cognitive theories are restrictive in explaining educational phenomena and that research should be performed from a broader perspective. We should, however, not forget that in the exact sciences, for example, theories that tackle problems from a microscopic viewpoint have made invaluable contributions to the development of knowledge, although they might have left out important aspects about observable phenomena. Macroscopic and microscopic viewpoints can coexist and play an important role in our understanding of the world. Both perspectives are valid as research fields. Based on these considerations, we posit that the cognitive viewpoint has a lot to offer in our understanding of the teaching-learning process.

On the other hand, during the interviews with instructors we clearly observed the importance of institutional constraints in their decisions. Some of these constraints may be real and some might be related to the teachers' perception. APOS Theory can provide elements for reflection, but this is not enough to overcome difficulties that teachers experience in their classrooms. Instructors need support and accompaniment in the implementation of teaching strategies. It is a long way from reflecting on students' construction of knowledge to designing instructional treatments in line with that reflection. We, as researchers, think that studies with teachers can benefit highly from a combined perspective using APOS Theory and ATD (Anthropological Theory of the Didactic) and plan to undertake this approach in our next study.

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