TALKING ABOUT LOGIC

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David: The article, by Weber and Alcock in FLM **25**(1), raises some interesting questions connected to our past discussions of research on logic, which might be thought-provoking for other FLM readers.

I was struck by the mathematicians' responses to the alleged proof: "7 is prime. If 7 is prime, then 1007 is prime. Therefore, 1007 is prime." They rejected the proof, which is good news, but their reasons, as Weber and Alcock point out, are a bit strange. Stranger still was that what seems to me to be the most straightforward objection was never raised by the mathematicians, or by Weber and Alcock.

If we are to accept that the premise "If 7 is prime, then 1007 is prime" is valid, we need a warrant. If the warrant is

> Direct computation can be used to show that 1007 is prime. Hence, this statement is true using a material conception of implication

then *the proof is circular*. I doubt any mathematician would accept a premise with the conclusion of the proof as the only warrant. To do so would violate the cultural standards of mathematics. So, given such a proof they were forced to assume that the conclusion "1007 is prime" was not already known to be true. Not even thinking of using it as a possible warrant doesn't mean mathematicians are not concerned about material implications – given an alleged proof they, in good faith, assume that it is not blatantly circular.

Matthew: I don't think that Weber and Alcock are suggesting that the *proof* of "1007 is prime" is correct, merely that the *conditional statement* "if 7 is prime, then 1007 is prime" is logically correct: they have given us an example of an incorrect proof that is entirely made up of statements that, when read in isolation, are logically valid. From the point of view of the material conditional, this is absolutely correct. When George Boole wrote down the laws of logic in the middle of the nineteenth century he defined the material conditional "if P then Q" to be true when, and only when, either Q or not-P is true. In this case, since Q is true (1007 is prime), the whole conditional is as well.

Of course you are correct to say that this interpretation of the implication isn't very useful for judging whether the *proof* is correct or not. As Weber and Alcock point out, for this you need to determine whether the conditional is warranted.

David: Perhaps the distinction we need is between "true" and "known to be true". Fermat's Last Theorem (FLT) is true, but if I had submitted a proof a few years ago based on the implication "if 2 + 2 = 4 then FLT" it would have been rejected. That implication is true in some platonic sense, but until it was known to be true it couldn't be used. And in the context of Weber and Alcock's "proof" the implication can not be known to be true based on the known truth of the conclusion, because the fact that we are proving "1007 is prime" implies it is not known to be true.

Seems to me to be a general problem of research that uses such artificial tasks to explore the logical (or illogical) thinking of normal people and mathematicians. Often, complicated explanations are given when quite simple ones would do. I do not think Weber and Alcock have shown that these mathematicians do not use material implications, only that they do so in an intelligent manner, appropriate to the context.

Matthew: This question touches on whether the laws of logic are relevant to how people think at all. Boole believed they were. His famous treatise on logic was entitled *An investigation into the laws of thought* and contained, he believed, an accurate description of how humans think (Boole, 1854/1958). Piaget agreed with this analysis, writing that, in the stage of formal operations, "reasoning is nothing more than the propositional calculus itself" (Inhelder and Piaget, 1958, p. 305).

Since the sixties, however, psychologists led by Wason have fairly conclusively demonstrated that Boole's and Piaget's beliefs are wrong. The so-called deduction paradigm has used a bewildering array of ingenious logical tasks that have demonstrated that people make many logical errors on such problems, and that they are significantly influenced by apparently irrelevant content and context. In short, people tend to interpret 'if ... then' sentences not as material conditionals, but in a fashion more like the warranted conditionals that Weber and Alcock describe (*e.g.*, Evans and Over, 2004).

When discussing the pedagogical implications of their theory, Weber and Alcock worry that students may not understand the role of warrants without explicit teaching. I would argue that research from the deduction paradigm suggests that this is the natural way of judging implications. Weber and Alcock need not worry! Students, and indeed everyone, tend naturally to interpret implications in this way regardless. The difficult part is coming to terms with the sometimes counter-intuitive definition of the material conditional. Arguably, it is an awareness of *this* part of logic that is so crucial if we are to avoid letting contextual irrelevancies draw us into making logical errors.

David: Your answer that we do not need to do anything to get students to judge implications according to sensible criteria is reassuring, though perhaps a bit optimistic. I don't see, however, why learning logic is so crucial. You say "contextual irrelevancies draw us into making logical errors" but I'd be more worried that logical irrelevancies might draw us into making contextual errors.

Given the focus on meaningful mathematics many of us have, I wonder if this whole psychological research agenda belongs in mathematics education. *Why* should we try to do anything about helping students to come to terms with the definition of the material conditional, or anything else about formal logic? Logic is useful only in a few domains of explanation, mathematics (perhaps) being one. Most people spend

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most of their lives in other domains of explanation. Why would a socially responsible teacher teach her students to reason in a way that is not generally useful?

Matthew: I don't accept your claim that logical deduction is "useful only in a few domains of explanation, mathematics being one". It is useful in many domains. An example, almost at random: try understanding the laws of cricket without a basic grasp of conditional logic. Law 34 is about hitting the ball twice and contains many sentences such as this:

> If the conditions in (a) above are met then, if they result from overthrows, and only if they result from overthrows, runs completed by the batsmen or a boundary will be allowed in addition to any penalties that are applicable. (http://www.lords.org)

This seems to be a clear example of where an understanding of abstract logical rules is necessary in a non-mathematical context. Now, you might argue that, like mathematics, understanding cricket is not in itself a desirable goal (although I would, of course, fiercely dispute that!). However, there are many similar situations where we are forced to grapple with complex logical rules like this. Filling in your income tax return; deciding which mortgage is better for you; determining whether your students qualify for special circumstances in examinations … there are lots and lots of further examples.

David: Logic is necessary to play cricket?! That is like claiming a theory of proportions is necessary to drink a gin and tonic. And I have yet to sense any logic in Canada's income tax laws. Perhaps they are different in the UK. Do you have any other arguments?

Matthew: A command of abstract logic may not be necessary for you to *play* cricket, but it is certainly necessary to understand the laws of cricket. The same is true of Canada's income tax laws. To determine successfully whether you are eligible for a tax credit, you must decontextualise. The only relevant facts are those that pertain to the law governing the tax credit we are interested in. These laws are phrased in complex and abstract logical structures.

Stanovich (2004), a firm defender of the deduction paradigm, has a nice line on this. After pointing out that the brain evolved to cope with life in prehistoric times, he writes:

> The issue is that, ironically, the argument that the laboratory tasks and tests are not like 'real life' is becoming less and less true. 'Life,' in fact, is becoming more like the tests! (p. 124)

The meaning is clear. The reason why the brain is so bad at systematic logical reasoning is that it evolved in a time when it wasn't useful. However, times have changed. Nowadays it is very useful, and indeed vital.

There is evidence that success on logical tasks (such as the Wason selection task; Wason, 1968) is correlated with higher Scholastic Aptitude Test scores, and higher 'general intelligence', whatever that means (Stanovich and West, 1998). Whilst I realise that many educationalists dislike phrases like this, I feel they are worth thinking about before being dismissed. We live in a society that rewards people who meet society's definition of 'intelligence'. Meeting society's definition successfully appears to be correlated with success on logical tasks such as the selection task; and success on logical tasks (such as the selection task) appears to be correlated with university level mathematics experience (Inglis and Simpson, 2004). Despite the lack of established causalities in these correlations, these results surely have implications for the value of mathematics education, and merit further investigation and consideration.

David: You raise a vital issue, of the role of logic in today's society, and similar things have been said about mathematics as a whole. Society's concept of intelligence has been horribly skewed by so called 'intelligence' testing, which relies on testing logic because it is easy to test (and because the test designers bought into Boole's and Piaget's theories). The damage this is doing to both society and mathematics has been described by Gould (1981) and Davis and Hersh (1986). Your solution seems to be to change how people think to match what society has become. I wonder if it might be better (and easier) to change the tests and tax forms to match how people think.

Matthew: Yes, the solution has to be to change how people think to match what society has become. Or, to rephrase more palatably: the solution has to be to help people develop cognitive skills that maximise their ability to interact successfully with the society in which they live. Is this not the primary purpose of any education system? It seems quite perverse to design a curriculum to meet the demands of a society we might *like* to exist, rather than the one that actually *does* exist. Furthermore, it seems doubtful, to me at least, that a modern society *could* exist along the lines you describe. Can income tax laws really be rewritten in a way that eliminates the need for formal abstract rules? I am struggling to imagine how such a legal system would operate.

Both: The issues we've discussed in this conversation appear to split opinions within our research community. It would be interesting to hear contributions from others on these issues and other related areas. Specifically:

What role does logic play in an individual's thinking under natural circumstances? To what extent should research on this issue from other academic disciplines be considered part of mathematics education?

What role does the style of logic practised in advanced mathematics play in non-mathematical tasks, such as understanding the laws of cricket and jurisprudence?

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