Geometry Education in the Midst of Theories*

JAN VAN DEN BRINK

1. Introduction

What does one learn from practice? What does one learn from a theory? And what are the links between the two? Topics taken from the practice of mathematics education are often chosen to illustrate theories. Numerous researches in arithmetic education, for instance, are used to obtain statistics to support particular theoretical speculations regarding learning and teaching behaviors. The present article deals with a new topic in geometry and the relations between practice and theory inherent in it. In practice the topic brought out fundamental differences between plane geometry and spherical geometry. But the class discussions could be interpreted, and the place of the new topic within geometry education could be justified, from the perspective of the theory of non-Euclidean geometry [1]. Other theories, too, provided the designer with the sense of assurance needed to deal with a new, divergent, topic. Such theories included the mathematical-didactics theories of Freudenthal and of Van Hiele, and an epistemological philosophy (radical constructivism). Conversely, it appears that these theories can be better understood from the perspective of the classroom. Once one has experienced a number of successful lessons, these experiences seem to provide a kind of safe starting point from which to better fathom or, if necessary, criticize a theory. In this article I describe how these theories elucidate events in the classroom and, conversely, where the perspective of the classroom produces criticism of these theories.

2. Spherical geometry

Six lessons in "spherical geometry" were designed [Van den Brink & Meeder, 1991] and then taught to three general secondary education classes containing a total of 75 16-year-old students [Van den Brink, 1994]. Spherical geometry is a (synthetic) geometry of the surface of a globe. The starting problem was entitled "Where is Mecca?"; the (geometrical) objects involved in the problem included the equator, parallel circles, great circles, pole and anti-pole, and map projections. The discovery that the equator and the parallel circles are not only lines around the globe, but can also be viewed as intersecting circles on planes cutting through the earth, confronted the students with various types of geometry: plane geometry, spherical geometry, and solid geometry. During the lessons the emphasis was primarily on three activities:

1. The various definitions that the students could think up for "great circle". (The sheer volume of definitions was impressive.)

2. It turned out that the diversity of definitions of a great circle arose from the various ways in which the students viewed the earth: as a flat map, as a large curved surface, or as a complete sphere. This variety of viewpoints resulted in conflicts and disagreements between the students.

3. As a consequence of the conflicts, and encouraged by their teacher, the students began to search for even more characteristics pertaining to great circles. They then found links between great circles and other objects (equator, centers, anti-pole, and suchlike). Various "networks" of relations arose in the forms of various types of geometry!

The three points mentioned above will now be described in more detail.

2.1 Great circles

One? No, many definitions

The great circle is an important object in spherical geometry. It is to the globe's surface what the straight line is to a flat surface: the bearer of the "shortest distance" and of the "direction" between two points. But it is also the greatest cross-section of a sphere. Such an object can be regarded from all sides by the students, and this was deemed necessary in order to (a) encourage abstraction and (b) geometrically decipher various "mysteries". It is often argued that "abstraction" is the "abandonment of particulars" [Van Parreren, 1978]. Although this may be theoretically true, in my opinion it sets the teacher off on the wrong foot. My experience in primary school practice showed that in order to get children to abstract from the so-called "bus arrow-language" and develop a bare "arrow-language" it was not necessary to abandon particulars (such as the bus's wheels, etc.); on the contrary, all sorts of decorations from other contexts and events were actually applied [Van den Brink, 1989, p.30 ff.]. Although, theoretically, "abstraction" is the "abandonment of particulars", in order to encourage children to take this step, they needed to be allowed, in practice, to first of all do just the opposite. This also seemed to be the case with great circles. Not just one but many definitions were possible, and these were necessary in order to arrive at the final abstraction.

In considering the geometrical "mysteries" of daily life, geometry can help provide clarification. Take, for instance, the fact that an airplane departing from Amsterdam on its way to Los Angeles will not take off directly to the west (which would seem logical, given a map of the world) but

*This article originally appeared in *Tijdschrift voor didactiek der bèta-wetenschappen,* 12(2), and has been translated from the Dutch by Ruth Raniero rather to the north-west Or the fact that a great circle indicates both the shortest distance and the starting direction between two points on earth. But how does one explain why—sliding along this shortest distance in the direction of your destination—one must keep changing direction? Are there straight lines on earth? Can one have "lines straight ahead" across the earth that are parallel?

Students' own productions

In order to get an idea of what the concept of great circle meant to the students, I talked with them and collected the numerous "definitions" they had thought up

A "two-dimensional definition"

The following assignment (printed in italics) is from our workbook: *Circles pass through points A and B:*

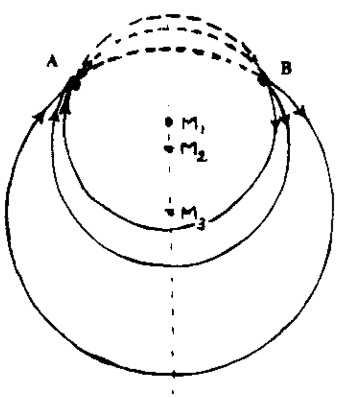


Figure 1

Take your compass and ruler. Copy this figure into your notebook. Can you also find the circle on the earth that has the longest curved distance between A and B?

A teacher told me of one student, whom I'll call Theo, whose idea was that, starting from the *B* on the paper, "You should keep going to the right, going under the earth, and then to the left back up again to *A*. That will then be the biggest circle you can imagine on earth joining points *A* and *B* on your paper."

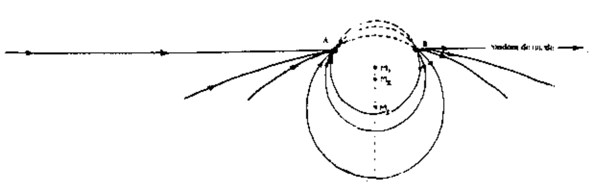


Figure 2

Theo's story clearly demonstrates that the straight line *AB* on the paper is part of the great circle through *A* and *B* around the earth. And because this straight line is the shortest distance between *A* and *B*, the great circle is therefore also the bearer of the shortest distance across the earth's surface. Everything here is, of course, intuitive, but that makes it no less significant. After all, everything remains transparent: the great circle also indicates, for instance, that the longest detour in a line straight ahead from *A* to *B* goes around the "backside" of the earth. I included Theo's story in the student book because of its transparency

A "three-dimensional definition"

Another chapter in the unit deals with intersecting circles on the glove: *Which circle is the largest of the intersecting circles? d or e? Why?*

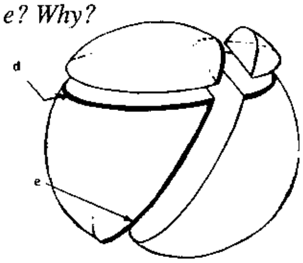


Figure 3

A student chose circle e, "because it's closer to the center". "It's the central cross-section", said another This is a precise formulation of a great circle, regarded, however, as a circle in a section straight through the *three*-dimensional earth On the *two*-dimensional earth's surface, on the other hand, the great circle is a line straight-around-the-earth, as Theo said.

Including students' own productions in the textbook

I collected these and other "definitions" thought up by the students for the concept of great circle Some were correct, others were not. The incorrect descriptions were not exact enough, and only provided one single property of the great circle. I then adapted these productions slightly—both the correct and the incorrect ones—and included them in an assignment in the book. Other students were given the task of evaluating these "definitions"

Here are some examples:

- A great circle is a circle that divides the earth into two halves. Right or wrong? Why? Because.... ..
- A great circle is a circle that divides the earth into two pieces Right or wrong?
- A great circle is a circle that passes through the North Pole and the South Pole. Right or wrong?
- A great circle is a circle whose center is the center of the earth
- If you keep going straight around the earth, you'll make a great circle around the earth.
- A great circle is a circle on the earth that is 40,000 km long
- A great circle is a circle on the earth that is not parallel to the equator. Right or wrong?

In my opinion, every educational designer should know this important "trick" of including student work in a textbook. There are a number of reasons behind this:

a The intention of the assignment was to have the students regard the great circle "from all aspects", each at his or her own level. The students' own productions are indispensable here.

b. Evaluating the work of other students not only has a motivating effect, but aids in clearing up misapprehensions within a student's own theory. A kind of tacit interaction among the students is created here

c. Evaluating the work of students circumvents the authority of the teacher's viewpoint Since the intention of the assignment was to evaluate a variety of geometrical viewpoints, this is of decisive importance

d In a mathematical theory, one is customarily presented with a single definition for an object How the author arrived at this definition, why it was chosen above others—in short, the process of deliberation leading to the ultimate formalization of what was being investigated—all this remains invisible to the reader But if, as was the case in the spherical geometry lessons, the students are allowed to think up and evaluate numerous "definitions" by themselves, then there is a chance the standpoint of the theory's author may become visible to the students as well. The students themselves must be able to deliberate the various definitions: which characteristics are essential, which are not, and which are incorrect? I had my doubts whether this level could, in fact, be reached. But I found myself in good company: Freudenthal, regarding the place of "definitions" in theories, said that they form the "finishing touch" rather than the beginning of an organizing activity [2]. In order to arrive at true formalization, one cannot do without one's own intuitive constructions

e. Including students' own work in a textbook is a recognition of the fact that we consider their own productions to be of importance Did the starting point of viewing the great circle from all sides through the students' own productions pay off? The answer is yes The results of the test I administered later on revealed that all the students were easily able to recognize great circle characteristics in new situations, and they also easily made connections between the great circle and other geometrical objects The classroom quarrels about the various definitions indicate, too, that the great circle was being regarded from all points of view. Different mathematizations were in fact being launched

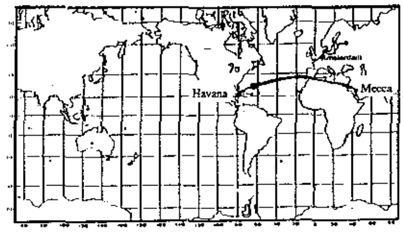
2.2 Various viewpoints

Do you regard the earth as a flat map or as a globe? The geometrical knowledge that you develop and in which your definition of great circle fits (or doesn't fit) depends on your viewpoint Although I had expected this development, I had failed to foresee all the consequences it could have in the classroom. The students' reactions were therefore surprisingly new

Is the direction to Mecca from Havana east or northeast? asks the textbook. "On the map it's east", says Peter Everyone in the group agrees. Then, to my amazement, he adds, "but it's northeast on the earth because the earth curves." Maarten supports this answer by adding, "The shortest distance between two points is more to the north, because that's where the earth is the narrowest." So these children first think in terms of a flat map In their eyes, the map is of higher merit than the globe The fact that the globe is a "truer" model of our earth (after all, the direction is indeed north-east on both of them) is not yet apparent to them. They do, however, construct a globe by making a cylinder of the map and then bringing it together at both poles The distance between the points at the same latitude grows shorter as one goes north. Therefore, in their opinion, the direction (having the shortest distance) should be more northwards.

A bit later, however, at the next question, this constructed link between map and globe has disappeared.

An airplane flying from Havana to Mecca takes off to the northeast and not to the east. Why is this? Peter, Bart and Betrick look at the map and see the curve Havana-Mecca



worldmap with Mecca and Havana

Figure 4

They think up all sorts of possible reasons: so it can refuel in Spain, to have the wind behind them, to avoid the desert. Then Betrick suddenly discovers, "Oh, it's done with the globe; it's the globe-direction, not the map-direction. They've drawn the globe-direction on the map!" The students initially distinguish two separate directions (and shortest distances)—one on the map and one on the globe—and keep these firmly isolated from each other The fact that a *globe* direction can also be depicted on the *map* is new for them, as is the fact that the straight line on the globe will become a curved line on the map. Evidently the students have a natural tendency to keep spherical geometry and plane geometry separate

Intolerance for different viewpoints

Heidy, Anette, Maarten and Jan are busy evaluating the following "definition": *A great circle is a circle whose center is the center of the earth. Right or wrong?*

"Wrong", says Heidy. I can't believe my ears. This is the definition of a great circle. Wrong? "A great circle goes around the earth, not through it, doesn't it?" she says. Anette doesn't agree and tries to explain it: "If you take this ball" (she holds a Styrofoam ball with a seam at the equator and pretends to break it along the seam) "you get two halves That gives you two great circles whose center is in the earth." "Where you draw a line, break it there, that's what you guys mean," says Heidy, repeating Anette's words, "then you get to the center of the earth, yeah, I understand that. But a great circle is still a straight line around the earth and not a section through the earth like you mean. And a straight line doesn't have a center; only a circle has that." Frustrated, she turns to me and asks, "Teacher, who's right?"

Indeed, the great circle as a straight line across the earth's *surface* has no center on the earth's surface, nor is it a circle on that surface. But the great circle as the largest section through the earth as a *globe* does have a center. These are two separate viewpoints, each of which is correct in itself. So I say, "You're both right", and proceed to explain each standpoint. But this bothers them: in their opinion, only one of the two should be right. It is difficult for them to accept the fact that there can be two kinds of geometries, based on different ways of seeing, and that their axioms can fundamentally differ.

Later, thinking back on this event, I realize that attention to various geometries requires that the children demonstrate understanding and tolerance for each other's viewpoints and "rules". Evidently, geometry education can have this pedagogic value as well.

2.3 The origin of different networks and different geometries

An empirical search for links

The teacher continued to fix the class' attention on great circles. Even during the fourth lesson he was still saying, "Name as many characteristics of great circles as you can." "Great circles again?" I thought. Yet it was astonishing how quickly the classes became adept at finding fundamental properties and relations. "The great circles are all the same length", reasoned one student, "because they're all the same size as the equator." The students used the equator as a model for great circles. "Each great circle passes through the center of the earth, because it splits the earth into two equal halves," said one girl. "But parallel circles are different," she continued; "they don't divide the earth into two equal pieces, so their centers are never at the center of the earth."

Another student was struck by the fact that each great circle touches both the northern and the southern hemisphere. Then the class intoned, "Each great circle passes through the equator!" After that it became obvious that "two great circles always intersect one another"; after all, the equator was the model for great circles. Conclusion: great circles are never parallel to one another; so: "lines straight ahead" across the earth will always intersect one another. There simply are no parallel lines on the earth. That is certainly an exceptional discovery.

Non-Euclidean geometry

Through this exhaustive search for characteristics, a network of relations was gradually and deliberately constructed. This network resembled, to a certain extent, a non-Euclidean geometry—that of elliptical geometry. Every pair of straight lines will intersect each other; there is no parallelism on the earth's surface. The children arrived at this axiom—which reveals a fundamental difference between plane and spherical geometry—through their own research and experiences. Another axiom of plane geometry states: "Given two distinct points *A, B*, there is one straight line and only one straight line incident with both *A, B*." This axiom, too, does not hold for spherical geometry (take, for instance, great circles passing through the pole and its antipole). Such a counterexample makes the axiom no longer such an obvious fact and gives significance to "agreements". Would this axiom also be true for elliptical geometry? We will return to this later (see 3.1: Non-Euclidean geometries).

The Van Hieles' educational theories

The development of a relation-network (actually more than one: one for plane geometry and one for spherical geometry) involuntarily brought to mind the research of the Van Hieles. Children began in the first grade with "concrete" quadrangles of paper and had to investigate their characteristics and find relations. The children found themselves on the so-called "zero thought-level" and moved towards a network of relations between the properties. By this means they reached the first thought-level [Van Hiele, 1957, 1986]. In the ninth-grade spherical geometry lessons described here we also saw a "zero-level geometry" à la Van Hiele. And we may say that the "first thought-level" was reached with the research into the properties of the great circle and the construction of a relation network between the great circle and other objects. Moreover, various networks—that is, various geometries—came under discussion and it became clear that true formalization cannot be achieved without intuitive "zero-level geometries".

Didactic activities in class

In the classroom one could see how the "push" took place, from intuitive geometries to a formalized network. First, the students were asked to give their own definition of a great circle (own productions). Then they evaluated each other's definitions. This was followed by discussions where counterexamples were given in order to clear each student's own theory of misapprehensions, and the students were encouraged to search exhaustively together for still more characteristics and relations. Finally, the class set forth on an exciting adventure [3] and the characteristics were weighed against one another [4] (see Figure 5).

Plane geometry	**Spherical geometry**
Point on the plane	Point on the globe
Straight line	Great circle
a. one straight line through two points	a. ....
b. two straight lines are either parallel or intersect one another	b. .....
c. two points on a line divide the line into three segments (one of the three segments can be taken as a measurable distance)	c. ......

Figure 5

The test

The children scored low on topics that were strictly technical and algorithmic and which were based on "petty facts", such as working with coordinate systems on the globe. This was also the case with topics that had been handled only slightly in class, such as the shortest distance and the more northerly path of an airplane in the northern hemisphere. But the children were easily able to work with great circles, poles, and antipoles in all sorts of new situations. They quickly constructed relations between these objects, and some students thought up new links. We may conclude that designing and evaluating "own productions" (definitions of a great circle) is an excellent means of achieving formalization within geometry education. The "intuitive" geometry developed here is a necessary starting point on the path towards more formal geometry.

Worksheets completed by the students during the first lesson were improved upon later; for instance, straight arrows towards Mecca were replaced by curved "great circle" arrows. The students demonstrated concretely here what they had learned. The various kinds of geometry evoked by this unit stimulated the students' powers of reasoning. In a test question cast in the form of a discussion the students consciously took different geometrical standpoints and argued their solutions from these standpoints.

3. Theories

In a recent article on theorizing in development research, Streefland [1994] points out that one is often struck by the strong resemblance between an observed learning process in one research project and that in a different research project, despite the differences in both mathematical topic and context. In another article he describes how children had described a "thaw day" and a "freeze day" in a research into negative numbers. Here Streefland expresses the expectation that making definitions can be part of continued mathematization [Streefland, 1993, p 123]. In the lessons on spherical geometry, such an expectation was certainly fulfilled after the defining of great circles. These theorizing considerations within developmental research provide the designer with a certain confidence in his or her new—and often controversial—design in progress. In addition, as mentioned earlier, existing theories (non-Euclidean geometry, Van Hiele's theory, Freudenthal's analyses) can offer significant support for a new topic. We can estimate the value of events in the classroom from the perspective of theories and explain the importance of the topic for education. Conversely, from the perspective of educational practice, criticism can be expressed regarding the theories. In the following section we will look more closely at non-Euclidean and other types of geometry, as well as at radical constructivism, with respect to their above-mentioned legalizing function. I will also relate them in more detail to the spherical geometry lessons.

3.1 Non-Euclidean geometries

The children discovered that two straight lines on the earth cannot be parallel after all. How does this work, geometrically speaking? The mathematical theory related most closely to this is so-called non-Euclidean geometry. Euclidean geometry itself is a logical system based on a number of axioms, such as:

- Two distinct points are incident with one, and not more than one, straight line.
- Maintaining the order of points, every line is in one-one-correspondence with the number line.
- Through a point *P* outside a line *1* passes one and not more than one straight line *m* that does not intersect *1* (parallel axiom).

These three axioms of Euclidean geometry are true for plane geometry, but not for spherical geometry. For example, an infinite number of great circles can pass through pole and antipole

History

The question of whether the "parallel axiom" is an independent axiom or whether it can be derived from other (apparently simpler) axioms, by the way, occupied mathematicians for more than two thousand years. Gauss was the first to suspect the independence of the parallel axiom and, therefore, draw the conclusion that another geometry founded on a substitute axiom would logically be possible [Struik, 1980, p 211] Through the replacement of the "parallel axiom" by another opposing axiom ("Through a point *P* outside a line *1* pass two or more lines parallel to *P*), a new geometry was formulated. Gauss called this "non-Euclidean geometry". In 1826, Lobatschewsky lectured on topics in this geometry, and Bolyai wrote a treatise on it in 1832. But distrust of non-Euclidean geometry remained. An error in reasoning was suspected. Riemann was the first mathematician of renown who, twenty years later (in 1854), confirmed acceptance of other forms of geometry by suggesting alterations in the axioms [see, for example, Freudenthal, 1971b]. Riemann's repudiation of the Euclidean "parallel axiom" was different and took the form: "There is no straight line parallel to a straight line *1* through a point *P* outside *1*." Two great circles, for example, are not parallel, as our students discovered. Spherical geometry, as far as the parallel axiom is concerned, can serve as a model for Riemann's non-Euclidean geometry

General acceptance of non-Euclidean geometries, however, came only after 1870, when mappings between different geometries were investigated. In 1871, Klein succeeded in demonstrating that non-Euclidean geometry could also be regarded as a projective geometry with an extraordinary metric. This discovery led many mathematicians who had still believed that non-Euclidean geometry must contain a logical error to change their minds. According to Struik [1980, p 225], the idea that an error might, in fact, be located in projective, and therefore in Euclidean, geometry was for many, if not all, too heretical. Of the two non-Euclidean geometries (diverging only in the parallel axiom, the other axioms being maintained), that of Gauss, Lobatschewsky and Bolyai was now generally accepted as a "hyperbolic geometry", while the other form, as demonstrated by Riemann, was termed "elliptical". History shows that the development of non-Euclidean geometry did not progress without a struggle—strikingly similar to the events in the classroom.

Detached?

It is strange, however, that elliptical geometry (Riemann) was not formulated until more than twenty years after hyperbolic geometry (Gauss, Lobatschewsky and Bolyai). Elliptical geometry already had a suitable model: after all, spherical geometry, with its navigational applications, had existed for centuries. These developments perhaps indicate how detached the mathematicians of that time were from "practical life", and how strongly their research focused on formal axiomatics. In this respect, the development of types of geometry. Realistic geometry education is much too connected to the real world for this to be possible.

Spherical geometry

Spherical geometry is a geometry that diverges both from Euclidean geometry and from the two original non-Euclidean geometries. The non-Euclidean geometries we have referred to differ from Euclidean geometry only in the parallel axiom. Spherical geometry also diverges in other respects. Whereas in non-Euclidean geometry exactly one straight line can pass through two different points, in spherical geometry an infinite number of lines can pass through two points. The spherical surface is compact, that is, it contains all condensed points, so that a "straight line" is closed (great circle) and thus cannot be in one-to-one-correspondence with the (open) number line. Nonetheless, Klein found that the sphere could be used as a visual model for elliptical geometry. For this purpose, he defined an "elliptical point" as a pair of spherical points that are one another's antipoles. So exactly one elliptical line then passes through two such elliptical points. The other Euclidean axioms were also satisfied, with the exception of the parallel axiom.

Navigation

In navigation, specific activities involving spherical geometry can suggest classroom activities that are suitable for geometry education in general. For example, two different map projections are cleverly combined in order to chart a course. These are Mercator's projection, where every straight line indicates a fixed course line (a "loxodrome") on the glove, and central projection (gnomonic projection), where every straight line represents a great circle (the shortest distance) on the glove. Navigators prefer to use loxodromes as they eliminate the need to change course continually. In general, however, a loxodrome is not a great circle and, therefore, is not a shortest distance. In order to satisfy both requirements (a fixed course and a shortest distance), a combined course called a "composed track" is charted.[5]

Another geometrical application arises from the fact that parallel circles in a central projection correspond to different types of conic section. These do not arise as slices of one cone by different surfaces, but from the intersections of one surface (the projection surface) with different cones. There is clearly plenty of empirical geometric research to be found within spherical geometry [6]

3.2 Radical constructivism

Ernst von Glasersfeld's radical constructivism (RC) is a philosophical epistemology[7]: it concerns the question of how knowing and knowledge come about. Some of his ideas dovetail with viewpoints regarding "realistic mathematics education", while some are markedly different. The following section will elaborate systematically on several standpoints within this philosophy from the perspective of the lessons in spherical geometry

A world outside

During Heidy and Anette's difference of opinion regarding the definition of a great circle (2.2), Anette used a small Styrofoam ball to prove her point. Heidy, on the other hand, gestured to her surroundings as indicating a part of the entire earth's surface. Nonetheless they were both talking about the same object—a great circle—in the objective world around them. Von Glasersfeld states that we can know nothing about this objective world; what we know is our own knowledge of the world, knowledge created individually by each person. Knowledge of the world around us comes about through personal experience by way of the senses. Knowledge about an object (its characteristics, its relations) depends upon which senses have been used for observing the object: a different sense may result in a different perception of the same object [Von Glasersfeld, 1991]. For this reason, knowledge regarding the same object is a different knowledge when it comes from an "actor" like Anette or from an "observer" like Heidy. One may of course disagree with the radical constructivist viewpoint that "knowing" comes about exclusively by way of the senses, as organs of the human body. The various standpoints from which one views the world are at issue here as well: the world as a flat map, for instance, or as our immediate visible surroundings; the world as a curved surface seen from a satellite; but also the world as a little ball that can be cut through in one's imagination. A globe's surface within a very small area is, perhaps, comparable to a flat surface, but the similarity disappears as soon as the globe's surface is regarded in its entirety. And the knowledge changes here, too [8] Knowledge is, as it were, a close elaboration of the general image first developed.

"Subjective environment"

Constructed knowledge is not limited to separate objects. Knowledge of the environment itself is also constructed subjectively. Each person creates his or her own image of the world. Von Glasersfeld calls this knowledge the "subjective environment" and notes that each of us is of the opinion that his or her world view is the objective world. He rejects this standpoint. One can imagine all kinds of things as being "so real", but the question will remain whether such an image is indeed objective. Nothing can be settled here, as one can only talk of one's own experiences. Anette and Heidy debated one another. Each strongly defended her own standpoint. And yet it disturbed them ("Teacher, who's right?") when it turned out that no objective truth existed. At issue here were two different ways of looking at the world and the ensuing differences in knowledge—in this case, geometries.

"Subjective environment" as a "socket"

A phenomenon must fit into this subjective environment as a lamp fits into a socket. The correctness of a great circle definition depends upon whether or not it fits into the already present geometry. If it does not one will end up with "tacked on" knowledge. Von Glasersfeld speaks here of "matching" instead of "fitting". But this conflict may also lead to a revision of the subjective environment itself and to an adjustment of the restrictions apparently inherent in this environment. There is an obvious similarity here with the development of the various geometries in the classroom. Each student's tinkering in an attempt to get a handle on objects such as great circle, straight line, etc., in order to relate them to one another and to build a network, can be seen as the construction of a part of the subjective environment. Formalized, these may become: spherical geometry, plane geometry, or solid geometry.

Unalterable environment?

Piaget distinguished various forms of adaptation of the organism of the environment (adaptation, accommodation, assimilation), and he must have been aware, particularly in the case of assimilation, of the reciprocity between organism and environment The apparent unalterable imposition of granite's laws on the organisms within it is deceptive Slowly but surely, the organisms also change the granitic environment Much of what has been recorded as unalterable to the end of time appears, upon closer consideration, to be decidedly susceptible to change [Lewin, 1994, p.8]. Von Glasersfeld emphasizes this idea by his concept of subjective environment. In his opinion, "intelligence organizes the world by organizing itself"]Von Glasersfeld, 1982, p.613]. The restrictions of the environment and the means to survive *arise simultaneously*. In other words: adaptation is linked to time, and conversely.

Viability and restrictions

Rather than "adaptation", Von Glasersfeld speaks of "viability". Viability is the power of the organism to develop resources in reciprocity with the restrictions of the environment. He gives the example of understanding language: "We believe we have "understood" a piece of language whenever our understanding of it remains viable in the face of *further* [my emphasis] linguistic or interactional experience" [Von Glasersfeld, 1983, p.213]. Understanding remains "viable" as long as there is a prospect of future activities and of a future increasing deepening of knowledge. The compatibility here with the lessons in spherical geometry is remarkable. Knowledge of independent objects and the construction of a network that increasingly imposed restrictions on these objects progressed concurrently. The viability of knowledge remained guaranteed because the students were stimulated to investigate further by their own discoveries, by their fellow students, and by their teacher It became clear that the type of network was linked to the particular student's assumed view of the world (flat or spherical). The restrictions imposed by this viewpoint and the resources to continue to understand the world (how it is) developed concurrently Knowledge of the environment (the geometries) became an individual design This viewpoint results in the ability to regard mathematics not as a collection of timeless structures imposing their restrictions but, rather, as things designed by human beings Realistic mathematics education shares this viewpoint

The paradox of the happy agreement

One of the criticisms of RC is the so-called "paradox of the happy agreement": "Each knower, independently pursuing his or her best intuitions, somehow comes up with the identical solution as all others We all conclude that alternate interior angles are congruent . ." [Lewin, 1994, p.11]. However different all the individual thought constructions may be, all the answers are eventually the same Praise for this constructivist standpoint in education has been sung in many keys; for example, as "negotiation of meaning" [Cobb et at., 1992] It is my opinion, however, that the "happy agreement" originates in education, rather than in RC.

Unfortunately, in the lessons on spherical geometry, this did not turn out to be the case at all It struck me that 16 year-old students definitely do *not* demonstrate any spontaneous tendency to share another's opinion. The existence of other opinions was only grudgingly put up with while the student's own ideas prevailed, sometimes against better judgment. This attitude, by the way, was in fact laudable in context and benefited the mathematics learning In order to maintain one's own opinion and convince one's fellow students one must persist in researching one's own system with respect to its consequences, consistency, restrictions, and rules This "disagreement" also fits much better into RC, whose starting point, after all, arises from a diversity of subjective world views in the classroom The example (that everyone eventually discovers that alternate interior angles are congruent) is not correct. They are indeed congruent in plane geometry, but not in spherical geometry. So the "happy agreement" does not occur at the level of "alternate interior angles" (within one and the same system), but on the level of which sort of geometry one has in mind, which "rules of the game" one is following In other words, it is much more fundamental.

Paideia and Paidia

Another criticism of RC is that no space is left within the philosophy for upbringing—for "Paideia" [Lewin, 1994]. In a Greek dictionary I found various meanings of "Paideia": "Upbringing", "youth", but also "punishment" according to certain rules I also found the closely related "Paidia", which means: "game", "to play", and also "childish game" and even "jest". Paidia (and Paideia) have to do with all sorts of rules of play Regarded in this way—as games—the Euclidean and non-Euclidean geometries fit wonderfully well Discussions during the lessons based on different geometrical points of view demanded tolerance for and attention to one another's thought constructions, to each other's "rules", and even to each other's "game". This is the pedagogical value of geometry education, based on a fundamentally geometrical topic: investigation into various types of geometry.

Realistic education and RC

From the standpoint of mathematical-didactics realism, RC can be reproached for not being a theory that emanates from education. Indeed, no hints are given in RC for day-to-day education. The philosophy indicates how knowing and knowledge arise, but not how that process can be stimulated and guided This criticism is not entirely justified. The didactic activities which stimulated the students to investigate spherical geometry did come from "education" and not from "philosophy", but these developments can be easily understood from the perspective of RC by calling on concepts such as "subjective environment" and "viability"

Were we to make a comparison between mathematical-didactics realism and RC, we would first be struck by the fact that, according to realism, an objective reality outside the individual does indeed exist that, albeit not in its entirety, can however in part be known by individuals. In radical contrast, according to RC *nothing* objective can be said about the world. RC argues that conflicts between people therefore must arise. Realism states that problems exist which are knowable objectively by everyone—so called "context problems"— and it endeavors to achieve gradualism here within education, a stepwise compromise. RC, on the other hand, even insists that the concept of "context" consists exclusively of subjective ideas and experiences. Realism will have none of this: not enough can be said "objectively" about such "contexts". "Context problems"—that is the point. And yet, there are similarities between realism and RC. Children's ideas can be used in order to create education. They can indicate a path towards a curriculum. The bright ones can lead the class. These similarities are based on great communal respect for the individual with his or her own ideas, both in RC and realism.

4. Notes

[1] This function, by the way, is not only applicable to mathematical theories; a curriculum, too, is capable of justifying the placement of a topic. Only in the unit entitled "Mecca", for instance, do great circles appear in the new mathematics program for Dutch schools [see: Team W12-16: *Background to the new mathematics curriculum* 12-16, 1992]. And yet they do belong in the program as a whole. They constitute a topic within spatial geometry, and they belong with "cross-sections", "sight lines", "place determination" and "calculating in geometry", that is, with the other units that make up the curriculum. For this reason, great circles and other topics in spherical geometry can be expected to receive more attention in the future.

[2] Freudenthal [1971a]: "Definitions are not preconceived to derive something from them, but more often they are just the last element of analysis, the finishing touch of organizing a subject. Children should be granted the same opportunities as the grown-up mathematician claims for himself." (424) But he warns: "Geometrical axiomatics cannot be meaningful as a teaching subject unless the student is allowed to perform these activities himself." (426)

[3] For instance, by going in search of the shortest distance between two points on the earth (following a great circle) without needing to change course (following a "loxodrome"). Some of the students' suggestions were: "Following a meridian?", "Following the equator?", "Following an almucantar?"

[4] We observed the students' tendency to weigh up their "definitions" of great circles (which is the best?, which is the most comprehensive?) as is the case when formalizing a mathematical system. Johan formulated two separate definitions, "The circle that goes straight around the earth" and "Where the earth is the thickest", but he did not choose a "best one". Peter formulated spontaneously, "A great circle is a circle straight through the middle of the earth that comes back at the same spot". He formulated a complex definition of an intersecting circle through the "three-dimensional" sphere and the line straight across the two-dimensional "sphere's surface".

[5] A helmsman once told me: 'First I draw a straight line on the cylindrical projection map. That's a great circle. I then transfer the intersections with straight lines, which are loxodromes on the mercator projection. So I navigate loxodromically towards the points of a great circle. We call this a composed track."

[6] Mathematics is an empirical science—whether one is researching axiomatics or developing a model of the world using great circles and suchlike.

[7] In the United States, radical constructivism is the focus of a great deal of controversy; many criticisms are being raised, and certain aspects are consequently being emphasized and expanded upon (socio-constructivism, social constructivism, didactic constructivism, etc.).

[8] In physics, too, the world (reality) is viewed from various perspectives, resulting in a diversity of theories (Newton's world view, theory of relativity, quantum mechanics)

5. References

Brink, F. J. van den [1989]: *Realistisch rekenonderwijs aan jonge kinderen* (Realistic arithmetic education for young children). Utrecht: OW&OC

Brink, F.J. van den & M. Meeder [1991]: Mecca, *Mathematics Teaching*, 137:20-23

Brink, F. J. van den [1994]: Spherical geometry lessons, *Mathematics Teaching*, 147:26-33

Brink, F. J. van den [1994]: Meetkundeonderwijs te midden van theorieën, *Tijdschrift voor didactiek der bèta-wetenschappen* 12,2 (mei): 130-149

Cobb, P., E. Yackel & T. Wood [1992]: A constructive alternative to the representational view of mind in mathematics education, *Journal for Research in Mathematics Education* 23(1): 1-53

Freudenthal, H. [1971a]: Geometry between the devil and the deep sea, *Educational Studies in Mathematics* 3:413-435

Freudenthal, H. [1971b]: Zur Geschichte der Grundlagen der Geometrie, *Nieuw Archief voor de Wiskunde* 4(5): 105-142 (About the history of the foundations of geometry)

Glasersfeld, E. von [1982]: An interpretation of Piaget's constructivism, *Revue Internationale de Philosophie*, nos 142-143: 612-635

Glasersfeld, E. von [1983]: On the concept of interpretation, *Poetics* 12:207-218

Glasersfeld, E. von [1991]: *A constructivist's view of learning and teaching*. Bremen

Hiele, P. M. van [1957]: *De problematiek van het inzicht, gedemonstreerd aan het inzicht van schoolkinderen in meetkunde-leerstof* (Problems of insight, demonstrated on the insight of schoolchildren in geometry) Amsterdam

Hiele, P. M. van [1986]: *Structure and insight. a theory of mathematics education*, Orlando (etc.), Academic Press

Lewin, P. [1994]: Constructivism and paideia: a paper prepared for a conference in honor of Ernst von Glasersfeld, Atlanta, Georgia, April 2-3

Parreren, C. F. van [1978]: *Niveaus in de ontwikkeling van het abstraheren* (Levels in the development of abstraction) Utrecht

Streefland, L. [1993]: Ontwikkelingsonderzoek negatieve getallen (Developmental research in negative numbers). In: R. de Jong & M. Wijers (eds.) *Ontwikkelingsonderzoek* (Developmental research) Freudenthal Institute, Utrecht: 111-130

Streefland, L. [1994] Theoretiseren in ontwikkelingsonderzoek (Theorizing in developmental research), *Tijdschrift voor Didactiek de bètawetenschappen* (Utrecht) 12:21-34

Struik, D. J. [1980]: *Geschiedenis van de wiskunde* (History of mathematics), Het Spectrum, Utrecht

Team W12-16 [1992]: *Achtergronden van het nieuwe leerplan wiskunde 12-16* (Background to the new curriculum mathematics for 12-16 year olds), 2, Freudenthal Institute/SLO, Utrecht/Enschede

Continued from page 20.

References

Bishop, A. and Nickson, M. *Research on the social context of mathematics education*: NFER-Nelson, London 1983

Conner, C. *Assessment and testing in primary schools*: Falmer Press, Basingstoke, 1991

McCallum, B., *et al. Teacher assessment at Key Stage One*: Research papers in Education, Vol. 8, p. 306, RKP, London, 1993

Macnamara, O., in *Mathematics Teaching* 142: Association of Teachers of Mathematics, Derby, 1993

Wheeler, D. H., in *Examinations and assessment*. Association of Teachers of Mathematics, Derby, 1968