Communications

Mathematization matters

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I have various pieces of paper in my drawer that mention mathematization. Some are notes for talks; some are short papers, published and unpublished. I thought that quoting some excerpts might trigger some further ideas and contributions to the discussion.

The first paper was published in *Notes of the Canadian Mathematical Society* (then Congress) in 1974. It begins:

> It is more useful to know how to mathematize than to know a lot of mathematics. Teachers, in particular, would benefit by looking at their task in terms of teaching their students to mathematize rather than teaching them some mathematics.

Subsequent writings cast doubt on the possibility of "teaching .. to mathematize", but I still think this is a useful summary of the two main reasons why mathematization is worth attending to. The paper goes on to lay out some of the grounds that I have since found myself retreading with embarrassing frequency.

> ..Why do the majority of mathematics teachers not encourage their students to "function as mathematicians" rather than try to lead them through a relatively restricted number of skills, mainly practised in a routine imitative fashion, and which bear little obvious relation to mathematics as it exists beyond school? One reason is that they may not know how to do it, and I wouldn't want to minimize the difficulties of teaching to this kind of goal. But I'm sure that a prior reason is that they don't believe it is possible, just as my art teacher didn't believe we could function as artists.

> ..One pointer to an affirmative demonstration that children can function as mathematicians may come from the well-known phenomenon of mathematical precocity which disproves any necessary correspondence between mathematical achievement and chronological maturity. Although the cases are exceptional, they decisively take mathematics out of the class of activities which no one could expect children to be able to do except by imitation. I don't see children, however exceptional, being able to function as historians, or as lawyers, or as psychologists, for instance, since these are extremely complex functionings that involve subtle relationships between several frames of reference. But I would hypothesize that mathematics belongs with art, music, writing, and possibly science, as one of a class of activities that require only a particular kind of response to be made by an individual to his immediate, direct experience. To put it the other way round, these latter activities do not characteristically depend on experience-by-proxy. We remember, in our own field, the story of the child Pascal re-creating for himself the forbidden fruits of Euclid.
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> Now, it would be my contention that school teaching is mostly geared to learning-by-proxy – one learns at school mainly what others have achieved in each particular field. Although most kindergarten and primary grade classes relate the beginnings of mathematics to children's direct experiences of sorting, counting, measuring, etc., it does not seem long before it shifts into "passing on" mathematical skills and concepts, and by the time the students gets into a high school algebra, geometry or trigonometry course, the element of direct experience has almost disappeared.
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> I don't want to oversimplify this. Some learning-by-proxy is inevitable for economical reasons; and desirable because everyone needs the enrichment of his own experience via the experience of others. Also, I don't intend "direct experience" to mean only the experience of physical, tangible activities: personal experience can certainly be intellectual as well as sensory, social or spiritual – a student grappling on his own with a tough geometrical problem is engaged with his own direct experience however many classical theorems he has previously learned by proxy. But, as I see it, the balance of direct and "proxy" learning in mathematics is generally tilted too much one way...

> ..If we see mathematization as a special functioning of the human mind, I think we are liberated intellectually in a number of ways and particularly from the quasi-evolutionary assumption that what is new is necessarily an improvement. We can have a more judicious respect for the past without necessarily putting down our own roots in it. We can catch the excitement of feeling that the future of mathematics is really open, wide open, and its future inexhaustible because there is no limit in principle to the situations and the problems that can be mathematized..

> ..It has been repeatedly stressed by observers that within a very short time of beginning to speak, young children utter grammatically correct phrases and sentences that are not copies of any that they have heard others speak. It is clear that they could not operate

autonomously within the grammar of the language without the capacity to handle classes (nouns, verbs), inclusion and intersection (adjectives, adverbs), relationships (prepositions), transformations (tenses) and substitutions (pronouns). I mention only the simplest and most obvious features: in fact the transformational requirements of a confident use of grammar are very complex and still defy complete analysis But the chief points at issue are that (a) the child's mastery of grammar can only be adequately described in terms of mathematical operations, and (b) this mastery is not derived by imitation.

.. The bonus accruing from the example of speech is that it shows the mental functionings of children are already algebraic long before they attend school and start "learning mathematics", and that teachers can assume from the beginning that the mathematical powers of their students are in operation. Indeed, the teachers' job becomes one of providing children with the raw material that will yield familiar mathematics when it is mathematized

In some notes privately circulated in 1978 I drew attention to the difficulty of identifying acts of mathematization, but also to the significance of trying to do so.

We can't actually observe mathematization – not even, I think, in ourselves when engaged in it. We mathematize without knowing it, only knowing the results, that we have "done the right thing", have acquired a skill or found a path or taken a fresh viewpoint. I am reminded (although mathematization is a more specialized operation) of the way in which we are able to find and utter all the words we want for a normal, easy, conversation without any need for awareness of the mechanism at all. The words come, string themselves in sequence, get uttered, and disappear again, without any awareness on our part of what we had to do at that moment, or what we had to learn some time in the past to allow us to do it. If we are in a situation where we know that what we say may be very important for some reason, the fluency may desert us, and we find ourselves inhibited, "searching" for the right words, "inspecting" what we say just before we say it. We may become aware of the difficulty of what we normally take for granted, but we don't necessarily gain any new insight into the operation itself. In fact we may get frustrated and exasperated because we want to have more control over it and don't know how to exercise it.

..Putting it another way, all of us confronted with the outer manifestations of mathematical activity have met the difficulty of "getting inside it" (or getting it inside us) and have attempted to "read between the lines" or discover "what's *really* going on here". The formal face of mathematics generally hides, rather than reveals, the inner life – at least, until one has enough experience to be able to read its expression. A definition, for example, often covers up the real source of the awareness that "this will be worth pursuing", and a proof can mask the source of the conviction that a result is actually valid. In looking at mathematization we are, it seems to me, trying to get as close as we can to the phenomenology of the awarenesses and convictions that we experience when we are doing mathematics and which power the movements of our mathematical thought. We can try to raise these awarenesses and convictions into consciousness – become aware of our awarenesses if you like – and then we may be able to find a way of talking about them that will make sense of these experiences....

Making the effort to become more concrete I gave the following examples. (They refer to the Gattegno Cubes and Prisms but can be understood without knowing the material, I think.)

(a) It is easy to assemble eight white cubes into a shape equivalent to a red cube ($2 \times 2 \times 2$), and eight yellow cubes into a shape equivalent to an orange cube ($10 \times 10 \times 10$). With this material there are only a few other ways of constructing a red cube, but very many more of constructing an orange cube. In this sense the orange cube contains a much richer variety of dissection possibilities. But if it occurs to one that the *scale* of the configuration may be ignored, the awareness comes that everything that can actually be done to construct an orange cube is potentially available when one considers a red cube, or even a white cube ($1 \times 1 \times 1$). This awareness enables one to "see" transformations of a white cube which are quite unrealizable in physical terms.

(b) A cube equivalent to an orange cube can be constructed from ten flat tiles. Suppose one subjects this arrangement to a physical transformation, pushing each tile at right angles to a vertical face until each tile overlaps the one below by a small amount. The shape is no longer cubical, but since nothing has been added or taken away, the resulting shape is in some sense still equivalent to an orange cube. What attributes remain; which have changed?

One may notice that the surface area has increased. A "dialogue" with the material might take the following direction.

How much has the area increased?
Can it be increased even more?
Can it be increased indefinitely?

No, for it has an upper bound – the total surface area of the ten tiles.

Does the fact that the area increases depend on the number of tiles?
What could be said if the cube was made of twenty tiles, each half the thickness of the original ones?

These tiles do not exist – except in the mind. It may be discovered that the maximum possible area available by "stepping" the tiles has increased.

What would happen if the tiles were halved in thickness again?

> And again?
> Does this increase continue indefinitely?

Perhaps of particular interest are the attempts I made to break down the concept of mathematization further, to try and articulate some of its constituents The first attempt comes from a talk published in *Mathematics Teaching* in 1975.

> In a crude attempt to make explicit the nature of mathematisation, I would include the following ingredients: the ability to perceive relationships, to idealise them into purely mental material, and to operate on them mentally to produce new relationships. It is the capacity to internalise, or to virtualise, actions or perceptions so as to ask oneself the question, "What would happen if. ?"; the ability to make transformations – from actions to perceptions, from perceptions to images, from images to concepts, as well as within each category – to alter frames of reference, to refocus on neglected attributes of a situation, to recast problems; the capacity to coordinate and contrast the real and the ideal and to synthesise the systems of perception, imagery, language and symbolism. When these functionings are applied to pure relationships, detached from specific exemplars, the products will then be mathematics.

A second attempt, worked on at a meeting of the Canadian Mathematics Education Study Group in 1977, came out quite differently

> Although mathematization must be presumed present in all cases of "doing" mathematics or "thinking" mathematically, it can be detected most easily in situations where something not obviously mathematical is being converted into something which obviously is. We may think of a young child playing with blocks and using them to express awareness of symmetry, of an older child experimenting with a geoboard and becoming interested in the relationships between the areas of the triangles he can make, an adult noticing a building under construction and asking himself questions about the design, etc. We notice that mathematization has taken place by the signs of organization, of form, of additional structure, given to a situation.
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> I use these tenuous clues to suggest that mathematization is the act of *putting a structure onto a structure.*
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> Consider the experience of solving a problem, or mastering a new game. In each case there are moments when the whole situation or a part of it is suddenly *seen differently*; the perceptual difference marks a new stage in the mental structuration of the situation.

By the 1980 ICME-IV meeting there seems some lessening of confidence in "structure on a structure" and the addition of two new ideas, neither very convincingly worked out

> In order to help awareness of the activity of mathematization come to the surface, I propose the following *clues* to its presence:
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> (a) *Structuration*
> "Searching for pattern" and "modelling a situation" are phrases which grope towards this aspect. But our perceptions and thoughts are already structured; reality never comes to us "raw". So mathematization is better seen as "putting structure onto structure" Existentially, however, it seems more like discovery or *re*-structuration since what we have brought into being seems new to us. The "eureka" feeling is an extreme case, marking the release of energy brought about by a new structuration.
> (b) *Dependence*
> Mathematization puts ideas into relation and coordinates them; in particular it seeks to establish the dependence of ideas on each other
> (c) *Infinity*
> Poincaré points out that all mathematical notions are implicitly or explicitly concerned with infinity. The search for generalizability, for universality, for what is true "in all cases", is part of this thrust.

The latest triad received its inauguration at Sydney early this year

> (1) *Making distinctions.* This seems to be the fundamental mental action underlying the construction of mathematical sets and mathematical relations
> (2) *Extrapolating and iterating.* These are the main mental actions for producing new things out of old ones
> (3) *Generating equivalence through transformation* This is the most powerful action of all since it generates stability (equivalence) out of flux (transformation).

A couple of quick glosses on these. Dick Tahta suggests "recursion" as an improvement on the second; but I find I have such a hazy idea of recursion that I don't know if he could be right. The "equivalence through transformation" comes from Caleb Gattegno; the earliest reference, I think, is in *Mathematical reflections* (1970). I felt I hadn't understood it until I encountered the following example last year

> A 14 year old boy was asked to solve the problem: "There are chickens and rabbits running around in the yard If there are 94 feet and 35 heads, how many chickens and how many rabbits are in the yard?" The observer, noting that the boy began by writing 90, interrupted and asked him why he had written it. The boy said, "I don't like numbers like 94. I like round numbers So I'm making it 90 feet and 34 heads and I can put a rabbit back at the end."

I have quoted here only from myself. But I don't stake any personal claim to the ideas Maybe recirculating them in the air will freshen them, and it, a little.