TASK

a. Give the axiom, theorem, or definition justifying each step in the following proof, indicating the mathematical elements that intervened.

Proof	What you say
a. $p(x) = g(x)q(x) + r(x)$	
b. the degree of $r(x)$ is less than that of $g(x)$	
c. if $g(x)= (x-a)$ as the divisor	
d. degree of $r(x)$ as 0	
e. $r(x) = r$	
f. $p(x) = (x-a)q(x) + r$	
g. setting $x = a$ in the above equation	
h. $p(a) = (a-a)q(a) + r$	
i. $p(a) = r$	

b. Identify of premise/statement/proposition, and the different roles they play; state the property that is proved with that proof.

Figure 1. Task corresponding to justifying in algebra.

trying to develop the different parts of our course on specific mathematics for future mathematics teachers. Of course, this is only a first try. A better characterization of intrinsic mathematical knowledge and its transformation into the content of a teacher education programme is an important research agenda, which we believe must be addressed.

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Teacher discourse and the construction of school mathematics

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Mathematics educators and teachers often talk past each other. To outsiders, this might seem strange: people in ostensibly similar professions should see the world more or less the same way. Nonetheless, I have yet to work on a school-based research project where discordant moments do not arise. Having worked as both a teacher and a researcher, I find these moments curious and productive. To me, they do not reveal a fundamental irrationality of either teachers or researchers. Instead, they uncover underlying assumptions about the nature of mathematics and schooling that implicitly frame our discussions.

In this communication, I draw on my work studying secondary school mathematics teachers' conversations. These conversations are useful for understanding teachers' everyday thinking about their work. Specifically, in my work, I examine teachers' collegial dialogue to understand the locally constructed conceptions of teaching, mathematics, and student learning.

Language is a part of the larger meaning systems people use to make sense of the world. With this in mind, I share some teacher perspectives that are seen by researchers as problematic and explain why they make sense when viewed in the context of teachers' work. I selected these excerpts because they bothered people on my research team (myself included). I do not abdicate my judgment that the conceptions are ultimately problematic; I simply seek to understand them better so as to address them more effectively as a researcher, teacher educator and teacher advocate. Taking the anthropological imperative that people need to be understood in their own contexts, I arrive at an understanding of these apparently "irrational" perspectives that sees them as sensible in a certain light, namely, within a stance that equates schooling and learning

Paradoxes of schooling

The problematic teacher discourse that I focus on in this communication makes much more sense within a broader analysis of schooling. Schools are contradictory institutions, riddled with paradoxes. Teachers must manage the contradictions inherent in schooling and the bureaucratization of the often messy work of teaching and learning. Managing contradictions is a tough business in any human endeavor and teaching is no exception. Because common forms of teacher discourse naturalize these contradictions, it is useful to revisit them.

The following list of paradoxes is not intended to be comprehensive. I describe these four because they show up in the teacher discourse examined in the next section of this paper.

1. *The paradox of control.* On the one hand, schools are bureaucratic institutions. Central offices make decisions about curriculum and policy which teachers are meant to implement. At the same time, teachers work in relative isolation and with a high degree of autonomy. Loosely coupled control structures minimize the need to coordinate with colleagues and, as a result, the demands remain low for a technical language for teaching.

2. *The paradox of student participation.* In many parts of the world, students' attendance is compulsory. At the same time, those in authority often act with a presumption of voluntary attendance, positioning students who do not wish to attend as deviant, "unwilling," or "unmotivated" (Powell, Farrar & Cohen, 1985).

3. *The paradox of timescales for learning.* The timescales of schooling may not always align with the timescales of meaningful learning (Lemke, 2000). Whether it is the fifty-minute lesson or the nine-month academic year, teachers, who greatly depend on the emotional rewards of their work, look for success in these units of time when, in some cases, students' development may take place on different timescales. Timescales dictate pace and normalize certain patterns of development, positioning different learners as "fast," "slow," "ahead," or "behind."

4. *The paradox of the individual in the crowd.* Classrooms are among the most crowded settings in which people spend time, with more people per square foot than even jails, yet teachers are expected to respond to individual learners, with "adaptation" and "individualization" being highly valued practices. Thus teachers' time and attention becomes a precious commodity. Expeditious systems of reward and evaluation manage to both conserve this resource and sort students, while potentially devaluing deeper and more meaningful student learning.

Managing the contradictions in teachers' talk

The institutional paradoxes described above provide a backdrop for teachers' talk. This talk, while institutionally "sensible", limits what it is possible to do as a teacher, often to the chagrin of researchers and educators who have designs on changing practice. I selected the excerpts discussed below because they represent a common ethos among well-meaning US mathematics teachers. In this sense, they represent the dominant perspective on mathematics teaching and learning. The examples come from different studies in American high schools. To be clear, I have not selected negative examples. The teachers quoted were leaders in their respective departments and well regarded by their school communities. Both showed a genuine concern for children. In both departments, external efforts sought to improve mathematics teaching but took inconsistent hold. The departmental cultures managed to preserve a traditional approach to teaching mathematics and the conceptualization of mathematics underlying the teachers' discourse played into that inertia.

Example 1: Why students stop learning mathematics

This first example comes from an interview with a veteran mathematics teacher in a suburban school district. I asked her why she believed some students struggle with mathematics:

> I've explained to students […] they'll be going, "How come this is so hard?" I say, "Math is something that you only have to remember everything that you've ever learned before. And you get to a point somewhere along the line where your brain says, "My brain is full." And you can't go on. It's not that you *can't*, it's that you don't *want to*. And some people reach that point before others. And some people may have their Ph.D. in math before they decide, "I don't want to learn any more new math." But everybody's going to reach a point where they just don't want to take in any more new math—because you have to remember so much." (Ms. Muller, Greendale High School mathematics teacher, from Horn, 2008).

Ms. Muller's construction of everybody reaching a point where "they just don't want to *take in* any more new math" captured several important assumptions about the nature of mathematics. In this metaphor, *learning* mathematics was a process of *filling the brain* with a substance—in this case, mathematics. One's progress through mathematics learning (notably delimited by schooling, a Ph.D. marking the upper reaches) involved "taking in" this substance. Capacity to take in mathematics was restricted, mainly because one can only remember so much before one gets "full." The main cognitive activity in doing mathematics was remembering.

This talk managed a number of the paradoxes discussed above. Of possible cognitive activities, memorization is low-stakes for students (Paradox 2) and can be easily evaluated in the timescales for learning that organize schooling (Paradoxes 3 and 4). Memorization can also be recognized by multiple levels of bureaucracy (Paradox 1), leading to a hierarchy of students, from those with more to less absorption capacity, simplifying the problem of individualization in a crowd (Paradox 3 and 4). For these reasons, I see Ms. Muller's formulation of mathematical learning struggles as oriented to schooling more than to mathematics itself.

Example 2: Why not all students can learn all mathematics

This excerpt comes from the talk of another veteran teacher in a middle- and working-class high school. She was responding to a colleague's concern about a district plan to detrack (that is, to eliminate courses grouping students by ability):

> Barbara says, "You brought it up, and you're telling the truth that our kids cannot get through our geometry as we teach it now." The teachers discuss whether the kids can get through their geometry course. They talk about the foundations for geometry in the algebra textbook. Barbara says: "What they won't be able to handle is the logic in the two-column proofs. Our regular kids can't handle that. One of the problems with the kids we're putting in [the 2-year algebra course] is they don't have the logic component [...] What you're doing is figuring out our hierarchy and that's good." (South High School Mathematics Department Meeting, from Horn, 2007).

Barbara initially framed students' learning troubles as a teaching problem, yet teaching methods were not questioned. Concern arose about the topic sequence, as the teachers examined the prerequisite course. Ultimately, though, they focused on students and their ability to handle the mathematics. Barbara described students' mathematical limitations in terms of a missing "logic component" needed for "two-column proofs." This discourse simultaneously constructed students and mathematics, with the latter having logic, a universally accepted feature of the subject, and the former having a limited capacity to handle this logic. Two-column proofs, however, are unique to school mathematics and often *obscure* logic (Herbst, 2002). Barbara's final comment naturalized hierarchies among learners based on their possession of logic.

The construction of students and teaching in these excerpts derives from a school-specific version of mathematics. It is school-specific because, while it bears the mark of a broadly accepted feature of mathematics (logic), it is an impoverished representation of logic via two column proofs. Instead, the version of mathematics constructed in this excerpt conveys greater fidelity to the four paradoxes of schooling presented earlier. While outsiders prescribed course content for the teachers, they exerted their autonomy by editing out content that they did not think a subgroup of their students could "handle" (Paradoxes 1, 2, 3, 4). Although we do not know why some students are having difficulties with the proofs, these students most likely did not actually lack logic. In fact, school mathematics often requires students to abandon logic, as they must defy the goals of sense-making by ignoring salient features of their experience and entering the fictional worlds of mathematics word problems (Ladson-Billings, 1995). Resistance may have been at play (Paradox 2) or inadequate time to understand mathematical conventions for representing logic (Paradox 3). Two-column proofs provided an expedient way for teachers to check logic, yet they necessarily limit the representation of that logic (Paradox 4). At the same time, the explicit structuring of hierarchies of learners based on one dimension of cognitive activity (logic) managed the tension imposed by Paradox 4. Barbara's discourse was deeply embedded in the discourse of schooling, collapsing it with a discourse of mathematics.

Conclusion

Being good at mathematics *should* be somewhat distinct from being good at school, but these examples illustrate how the two skills are often conflated. To contrast the way mathematics is constructed in these excerpts to the mathematics itself, consider briefly the solution of Fermat's Last Theorem. Its solution required over 300 years of multi-person effort and the development of entirely new fields of study. The solution arose out of persistence and insight, culminating in a complex mapping between seemingly disparate branches of mathematics. In the two examples analyzed in this communication, mathematics looks different. While its features are invoked (its logical structure, the interdependence of ideas) mathematics, when confused with schooling, becomes a shadow of itself. In Example 1, memorization becomes a valued cognitive activity, more than problem solving and sense-making. Timescales for engaging in mathematics are normalized, positioning some students as smarter based on their facility at memorization over slower cognitive processes like insight, connection making, and problem solving. In Example 2, the school-based representations of mathematics such as two-column proofs become not simply *one* tool for demonstrating logical thinking; they become the only valid instantiation of it. In this way, the epistemology of school mathematics comes from *school* more than from *mathematics* itself.

Some recommendations come out of this analysis. First, researchers and teacher educators can be alert to the conflation of schooling and mathematics often at play in teachers' talk. A consequence of this worldview is the devaluation of some forms of mathematical talent, particularly ones that require a slower pace of thinking, such as problem solving or connecting ideas. Teachers need to understand the distinction between *doing school* and *doing mathematics* or they may continue to have small yields of students who succeed in learning and persisting in mathematics. A wedge needs to be driven between these two forms of competence, as strength in one does not necessarily indicate strength in another. This leads to a second conclusion: changing mathematics instruction is a cultural problem. Because teachers and parents often do not recognize mathematics unless it looks like school mathematics, transforming instruction may require deeper changes to the structures of schooling. Until then, teachers' resolutions of these paradoxes are, in large part, sensible.

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