The Case For and Against "Casting out Nines"

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The chances are that you haven't done much in the way of casting out nines recently. But once upon a time it was very fashionable (or so we are led to believe) and the procedure is as old as the hills We are not going to argue the case for its immediate reinstatement in the curriculum—we suspect that its use as a check for arithmetical calculations is a lost cause, justifiably so. But as a part of a small topic in "number theory" (i.e., criteria for divisibility) it still has a place, at least as an enrichment topic, and can be used, among other things, to help foster number sense [1] With this aspect in mind (i.e , as enrichment for the existing curriculum), we shall discuss here some "moments" of interest from the 1000-year long history of casting out nines In particular, we bring some of the views of those who wanted to banish the topic many, many years ago, long before the calculator relegated it to the sidelines, as well as some discussion of the probability that "casting out" checks reveal an error We suggest that history used in this way provides an interesting context for genuine and relevant mathematical activity

Casting out nines

For those who have never heard of casting out nines—and our experience suggests this may be a lot of people [2]—we begin with a brief description It is only a special case of modular arithmetic applied to the detection of errors in arithmetic calculations If $a = a'$ (mod 9) and $b = b'$ (mod 9), then

$$a + b = a' \pmod 9 \quad \text{and}$$
$$a \times b = a' = a' \times b' \pmod 9$$

Thus we can use the possibly very much simpler addition $a' + b'$ as a check on the correctness of the addition $a + b$. And similarly for multiplication [3], to which we shall restrict ourselves in the following For example, suppose we had performed the calculation

$$3257 \times 48329 = 157407563,$$

then, reducing each numbers modulo 9, we obtain

$$8 \times 8 \pmod 9 = 2$$
$$\text{or} \qquad 1 = 2,$$

which statement is manifestly false. Hence our calculation *must* also be incorrect But it is clear that if the check leads to a true statement, this tells us nothing about the correctness of our calculations, as the ridiculous counterexample

$$3257 \times 48329 = 1$$

shows [4] However, checks were not devised to reveal arithmetic criminals, but rather, since it is human to err, to be a safely net for honest arithmeticians. As incomparably expressed nearly 250 years ago [Wilson, 1752]:

> When any arithmetical operation is performed with attention, that very attention is a presumption that the operation is right; and that presumption is heightened to a very great degree of probability, if the proof [5] by casting out of the 9's succeeds

We shall have cause to return to Wilson later.

So far casting out nines means dividing by 9 and recording the remainder But why 9? Why not 7 or 53? The reason is simply that finding a remainder after division by 9 does not require division—all one needs to do is add the digits of the given number, and this sum, modulo 9, is the same as the original number, modulo 9 For example, the digit sum of 157407563 is 38, which gives 2 modulo 9, as we saw above. This short cut for finding the remainder after division by 9 is obvious to any mathematician who notices that $10^n = 1$ (mod 9) But it is not that obvious in a classroom investigation. And there is a lot more to be discovered, such as that we don't have to add up all the digits—we can cast out 9's wherever and whenever we see them For example

$$157407563 \rightarrow 157407563 \rightarrow 1707563 \quad \ldots$$

It will also quickly be noticed that the digit-sum operation can be iterated In the example we are using the first digit sum is 38, and the digit sum of *that* number is 11, which again gives 2 modulo 9 (Since it is not immediately relevant to our story we postpone to the Appendix the observation that one can also find the quotient on division by 9 without needing to divide.)

Rejectionists

Casting out nines is so easy and elegant that it is difficult to believe that anyone could ever have been against its use as a *check* However, let us consider Cocker's *Arithmetick*

> This remarkable book went through upwards of a hundred editions and had great influence upon British textbooks for more than a century [Smith, 1958]

The first edition was published in 1677. De Morgan [1847] was of the opinion that it was a forgery, and at the end of a long discussion writes:

> The famous book looks like a patchwork collection, .. The reason for its reputation I take to be the intrinsic goodness of the process, in which the book has nothing original; . I am of the opinion that a very great deterioration in elementary works on arithmetic is to be traced from the time at which the book called after Cocker began to prevail

The *famous* Cocker [6] has the following to say about casting out nines

Multiplications proved by *Division* and, to speak truth, all other Ways are false; . . There is a Way (at this Day generally used in Schools) to prove *Multiplication*, which is this; . . . [there follows a verbal description of casting out nines] . . and lastly, cast the Nines out of the total product, and if this Remainder be equal to the Remainder last found, then they conclude the Work to be rightly performed; but there may be given a thousand (nay infinite) false Products in *Multiplication,* which after this Manner may be prov'd to be true; and therefore this Way of Proving doth not deserve any Example; but shall defer the Proof of this Rule [multiplication] till we come to prove *Division*, . .

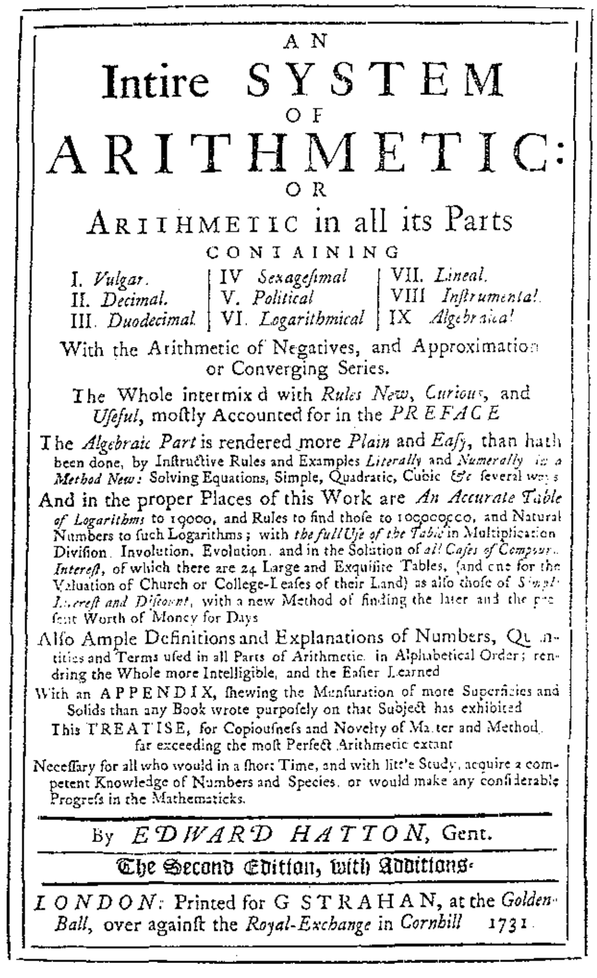
Cocker's rejection can be profitably compared with the quote from Wilson above; there is much to be said on both sides. But another rejectionist, Hatton [1731], gives more explicit reasons for his rejection, not just that the test isn't foolproof.

The Proof of Multiplication
This can only be done by Division: . . . But to pretend to prove Multiplication by casting out the Nines, is a Mistake, as I have elsewhere demonstrated; for why divide by 9 more than 2, or any Digit, which would prove the Work as well? But the easiest way is to divide the Factors by 10, and the Product of the Remainers by 10, which will leave a Remainer equal to that of the Product divided by 10. But the mischief is, that if there be a Mistake in the Product of just your Divisor, or any Power thereof, this Way of proving will not show it. [7]

Indeed, "why divide by 9 more than 2", or 10? And does the last sentence really cover all possible types of "mischief"? There is plenty to discuss here.

Protagonists (and why 9?)

The previous extract leads directly to one of the most interesting points raised by those in favour of casting out nines



AN
Intire SYSTEM
OF
ARITHMETIC:
OR
ARITHMETIC in all its Parts
CONTAINING

I. *Vulgar.*	IV. *Sexagesimal*	VII. *Lineal.*
II. *Decimal.*	V. *Political*	VIII. *Instrumental.*
III. *Duodecimal.*	VI. *Logarithmical*	IX. *Algebraical*

With the Arithmetic of Negatives, and Approximation or Converging Series.

The Whole intermix'd with *Rules New*, *Curious*, and *Useful*, mostly Accounted for in the *PREFACE*.

The *Algebraic Part* is rendered more *Plain* and *Easy*, than hath been done, by Instructive Rules and Examples *Literally* and *Numerally* in a *Method New:* Solving Equations, Simple, Quadratic, Cubic &c several ways.

And in the proper Places of this Work are *An Accurate Table of Logarithms* to 19000, and Rules to find those to 100000000, and Natural Numbers to such Logarithms; with *the full Use of the Table* in Multiplication, Division, Involution, Evolution, and in the Solution of *all Cases of Compound Interest*, of which there are 24 Large and Exquisite Tables, (and one for the Valuation of Church or College-Leases of their Land) as also those of *Simple Interest and Discount*, with a new Method of finding the later and the present Worth of Money for Days.

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This TREATISE, for Copiousness and Novelty of Matter and Method, far exceeding the most Perfect Arithmetic extant.

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By *EDWARD HATTON*, Gent.

The Second Edition, with Additions.

LONDON: Printed for G. STRAHAN, at the *Golden-Ball*, over against the *Royal-Exchange* in *Cornhill*. 1731.

(Frontispiece of Hatton's book)

and similar checks: which is the "best" number to choose? "Best" has at least two aspects: *ease* (e.g., 10 is the "easiest" according to Hatton above) and *reliability* (i e , the "probability" of detecting an error). The two aspects are not unrelated. "Casting out" 2, or 5, or 10, is very easy, but also extremely unreliable; for example, any error in any digit of the product, other than possibly the units digit, will go undetected. Historical sources will, as usual, provide us with cues and clues, and we begin with the remarkable Chuquet [Flegg, Hay & Moss, 1985] from 500 years ago

> There are several kinds of proofs such as the proof by 9, by 8, by 7, and so on by other individual figures down to 2, [8] by which one can prove and examine addition, subtraction, multiplication and division. Of these only the proof by 9, because it is easy to do, and the proof by 7, because it is even more certain than that by 9, are treated here. . . There is a difference [between proof by 9 and proof by 7], for the proof by 9 can easily be carried out by adding up the figures, but not here [in the proof by 7] This proof by 7 is wrong less often than that by 9 because 7 has less in common with numbers than 9

Chuquet is clearly aware of both ease and reliability—9 is easier but 7 is more reliable—although what exactly he means by "7 has less in common with numbers than 9" is not immediately clear Perhaps what he had in mind can be understood from the more extreme cases of the numbers 2, 5, and 10 mentioned above, which because of their very special relation to our base 10 numeration system, are useless for error detection Nine also has special properties because it is one less than the base 10 One of its special properties is, of course, that which makes the test by 9's so simple In the part we omitted from the above quote, Chuquet cites two types of error which casting out nines will not reveal.

- The addition or removal of 9's or 0's as digits in a number, as in, say, writing 9539 or 53 instead of 953, or 34 or 3400 instead of 340.
- The *mutation* of a digit or part of it from one order to another, as in 342 or 1143 for 243

In all these examples, 7 does reveal the error, but if 342 is "mutated" to 2421, neither 7 nor 9 will help. Pathological examples, however, are not really to the point; reliability would seem to depend on at least two things: the likely sources and types of error that occur in multiplication, and which of these are detected by casting out 7, 9, or whatever [9].

Cox [1975] discusses *systematic* errors in the four elementary arithmetic operations, and we checked the 22 multiplication examples in 7 categories which he gives Both 7 and 9 each fail to reveal 2 errors At least in these cases there is no advantage to 7. This does not mean that the same would hold for non-systematic errors, which might be more prevalent than systematic errors. Workman [1908] obviously speaks from experience:

> Note You are strongly advised to apply this method [casting out nines] to *every* multiplication sum you do; but you must be warned that this test will not detect your mistake if the amount of the error happens itself to be a multiple of 9
>
> In particular, if *you happen to put one of the partial products in the wrong place* before you add up, the test will not show your error, ... So also if you have written down the right figures at any stage, but happen to have put two or more in their wrong places .

The second type of error could very reasonably occur when copying a number [10], as in writing 73421 for 74321

It would also seem a reasonable guess that the misplacement of one or more rows in a partial product is a common error in student work, as, for example

	342705	instead of		342705
×	2067		×	2067
	2398935			2398935
	20562300			20562300
	68541000			685410000
	91502235			708371235

These errors are related to Chuquet's types of error, and maybe he had these in mind but did not explicitly connect them to the process of multiplication. Casting out nines will not detect these types of error, whereas casting out sevens will. So why did casting out nines become so popular, and casting out sevens rarely get a mention in arithmetic books? Clearly the simplicity of the rule for casting out nines, which avoids division altogether, was the decisive factor. There is no such simple rule for casting out sevens

But what about casting out other numbers? Single digit numbers do not offer us anything to entice us away from 9, but here is Workman again.

> Proving by Elevens. This method is much less subject to error than that of *Proving by Nines*, but is a little harder to acquire It is well worth the trouble. To find the remainder after dividing by 11, *add together ALTERNATE digits, beginning with the units digit and, when you have reached the left of the number, return, subtracting from the sum you have obtained the digits which still remain, If, when you have again reached the right of the number, your result is less than 11, it is the remainder after dividing by 11 If it is greater than 11, subtract elevens until it becomes less Elevens may be rejected or added at any stage.*

There follow some numerical examples of casting out elevens, and then a note about reliability:

> Note. If one of the partial products of a multiplication be displaced one place only, this test *will* usually detect the mistake. It will not detect it, however, if it be displaced two places for ...

Again there is a lot to discuss here. Why does the "strange" rule for casting out elevens work? Justify the remarks in the note, especially the remark "*will* usually detect the mistake". Why "usually"? And note that it does not detect the mistake in the long multiplication example we gave above, although only one partial product is displaced by only one place. Do we agree that 11 "is much less subject to error" than 9? And, finally does the loss of simplicity (as compared with casting out nines) justify the additional reliability?

In the thirty or more books we have examined, verification by 11s is very rare, whereas 9 is in almost all of them, which would seem to make the historical answer to the last question a firm negative For young students there would seem to be two possible difficulties—the use of alternate rather than consecutive digits, and the need for subtraction The possibility of error in the check may be almost as great as in the calculation itself.

In fact, there is a simpler rule for casting out elevens, but it seems to be even less well known. According to Tropfke [1980] it appears in Arab works as early as 1000 years ago. Instead of adding and subtracting alternate digits, multiply all the digits in even position by 10 and add their sum to the sum of the digits in odd position. This dispenses with subtraction, but not with keeping track of the position of the digits In fact we can make life even simpler, as was noted in a letter to *The Mathematics Teacher* [Chevalier and Irby, 1993] in connection with testing for divisibility by 11. To find the remainder on division by 11, divide the number into pairs of digits from the right:

e.g., 537382231 becomes 5 37 38 22 31.

Now add these two-digit numbers, obtaining 133. The remainder of this number on division by 11 is the same as the remainder for the original number, i.e. 1. In this form the rule for casting out elevens is exactly the same as that for nines, except that the former is performed on pairs of digits instead of single digits. And again short cuts are possible: for example,

5 37 38 22 31 can first be transformed into **5 4 5 0 9**.

The reason why the rule works is not hard to find and is due to the fact that $10^{2n} \equiv 1 \pmod{11}$. (The additional fact that $10^{2n-1} \equiv -1$, explains that "alternate" rule given above.) For some reason the simpler rule for casting out elevens has remained "almost unknown". [11]

Probabilists

We have already touched on the "probability" that the casting out of specific numbers will detect an error in our discussion of reliability We refrained from quantifying this probability because we believe the attempt to be unrealistic. Additional source material for the classroom is provided by the following two quotations, one old and one modern. Wilson [1752] writes:

> It is true, indeed, that if you have miscounted 9, or any of its multiples, this method of proof will not discover the mistake. But there are 8 other digits [12] besides 9; and it is as probable, when an error is committed, that you have mistaken any one of them, as that you have mistaken 9; so the proof, by casting out the 9's besides the probability that arises from attention, has 8 chances against 1, to discover if there be any fault in the operation.

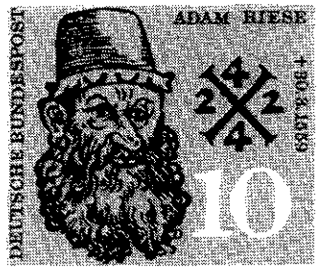
The modern "source" [Lauber, 1990] goes even further

> As explained earlier, the probability of not detecting an error through casting out 9s would be expected to be 1/9. It follows that the probability of detecting an error should be 8/9 For analogous reasons the probability of detecting an error by casting out n's would be $(n - 1)/n$ Thus the smaller the value of n, the less the chance of detecting an error.

A classroom discussion [13] should be able to take issue with these two sources quite successfully, using the "ammunition" provided above

Epilogue

There is so much interesting material in the old textbooks that it is difficult to choose what to leave out. However, we cannot leave the subject without mentioning what we regard as an aside, but nevertheless an attractive one for the classroom. The man depicted on two stamps issued in Germany is the "Rechenmeister" Adam Ries (consistently misspelt Riese) His influence in Germany was (deservedly) even greater than that of Cocker in England.



Although Ries was born 500 years ago, in 1492, a friend who went to school in Vienna only some 60 years ago has told us that, when a calculation was completed successfully, it was acclaimed as "correct according to Adam Riese". [See also Smith, 1958, Vol I, p. 337; or Boyer, 1985] The first stamp was issued in 1959, 400 years after his death, and the second appeared in 1992, 500 years after his birth, a remarkable tribute to a man who was essentially a teacher, as opposed to a creative mathematician

On both these stamps we notice a cross with numbers. Ries used this design as his personal seal, equivalent at that time to a signature. [Deschauer, 1992] But he (and many other authors) also used it as a convenient scheme to record remainders after casting out nines For example, if the exercise was to add two numbers (A + B = C), then the remainders after casting out nines from A and B were recorded in the two regions on the left and right (2 and 2 in the first stamp, and 3 and 3 in the second) The sum of these two remainders (modulo 9, if necessary) was recorded in the upper region, and the result of casting out nines from the sum C in the lower region. If the numbers in the upper and lower regions were the same, the sum was considered correct. We have found a cross used in this context in textbooks well into the nineteenth century.

Notes

[1] Hilton and Pedersen [1981] suggest that it should be reinstated, if for a somewhat different reason: "this first example of a genuine algebraic homomorphism should have been encountered no later than 6th grade." It is also still used to obtain the check digit in some simple numerical identification schemes—for example, US postal money orders and VISA traveller's checks [see Gallian, 1991; or Wheeler, 1994]—and these can be used as further motivation in the classroom The discussion in this paper, restricted as it is to the use of checks in arithmetic calculations, is nevertheless also relevant in some respects to identification schemes—the reliability aspect, in particular.

[2] See also the paper by Hilton and Pedersen

[3] With a little care, subtraction and division can also be checked in this way

[4] Modular arithmetic itself still has a place in the curriculum and we suggest that a "historical" discussion of its use in checking calculations in the pre-calculator era can add interest, enrichment, and contribute to number sense.

[5] On the use of the word 'proof' in this context, we quote a passage from De Morgan [1847]:

> To the commercial school of arithmeticians . we owe the destruction of demonstrative arithmetic in this country, or rather the prevention of its growth It never was much the habit of arithmeticians to prove their rules: and the very word *proof*, in that science, never came to mean more* than a test of the correctness of a particular operation, by reversing the process, casting out the nines, or the like (*At first I had written 'degenerated into nothing more;' but this is incorrect The original meaning of the word *proof*, in our language, is testing by trial)

[6] We quote from the 34th edition published in 1716, but we have examined quite a number of other editions and the wording is identical. It is therefore difficult to understand Smith [1958, Vol II, p. 153], who writes: "In England, however, the influence of Cocker served to make it [casting out nines] very popular "

[7] De Morgan says of Hatton, 'A sound, elaborate, unreadable work . ', but the "sound", presumably, does not refer to the passage quoted.

[8] Interestingly, he doesn't mention proof by 11 (and other two digit numbers), although they do occur in earlier texts [cf Tropfke, 1980] We discuss proof by 11 later

[9] If paper and pencil multiplication were still the only method of performing multiplication, there might be some sense in a systematic study of student errors and their detection. But since much of it is now history, and our interest's sake, we restrict ourselves to examples.

[10] This type of error is of course still relevant in the calculator age, when entering a number or copying it from the display, and this is also a reason why casting out nines to obtain the check digit in identification schemes is unsatisfactory

[11] Even Hilton and Pederson [1981] give the 'alternate' rule.

[12] "A digit is any one of the nine significant figures; for 0 signifies *nothing*" [Wilson, p 5]

[13] We have, in fact, developed a "historical happening" (in Hebrew) for the classroom, similar to that described in Ofir and Arcavi [1992], based on the material in this paper

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Appendix

Finding the quotient on division by 9 without division

It is interesting that, not only the remainder on division by 9 can be found without division, but also the quotient. Of additional interest is the fact that this was pointed out by Charles L Dodgson (Lewis Carroll) in a letter to *Nature* in 1897 (Vol 56, 565-6).

> .. I brought my rule to completion on September 28, 1897 (I record the exact date, as it is pleasant to be the discoverer of a new and, as I hope, a practically useful, truth.)

The first part of the "rule" is the usual one for finding the remainder on division by 9 The second part is the bit of special interest.

> To find the 9-quotient [i.e. the quotient produced by dividing by 9], draw a line under the given number, and put its 9-remainder under its unit digit: then subtract downwards, putting the remainder under the next digit, and so on. If the left-hand end-digit of the given number be less than 9, its subtraction ought to give remainder "0": if it be 9, it ought to give remainder "1", to be put in the lower line, and "1" to be carried, whose subtraction will give remainder "0". Now mark off the 9-remainder at the right-hand end of the lower line, and the rest of it will be the 9-quotient
>
> Examples: 9/<u>75309 6,</u> 9/<u>94613 8,</u> 9/<u>58317 3,</u>
> 83677/3 105126/4 64797/0

Carroll gives no explanation why the rule works, although one is not difficult to supply. Let $n = 9q + r$ be the number to be divided by 9, where q is the quotient and r the remainder Then

$$10q + r - n = q.$$

Given that we know r and n, and want to find q, it might seem strange at first sight that we can in fact determine q from this equation. But we can, step by step, in precisely the way described in the rule above. To find the units digit in q, we notice that $10q$ is irrelevant since it ends in 0. All we have to do is subtract the units digit of n from r, "borrowing" if necessary. Having found the units digit in q, we can insert it as the tens digit in $10q$, subtract from it the tens digit in n, and so obtain the tens digit in q, etc. And this is precisely the rule stated in the letter.

The letter also contains a similar rule for finding the quotient on division by 11, which should be clear from the equation

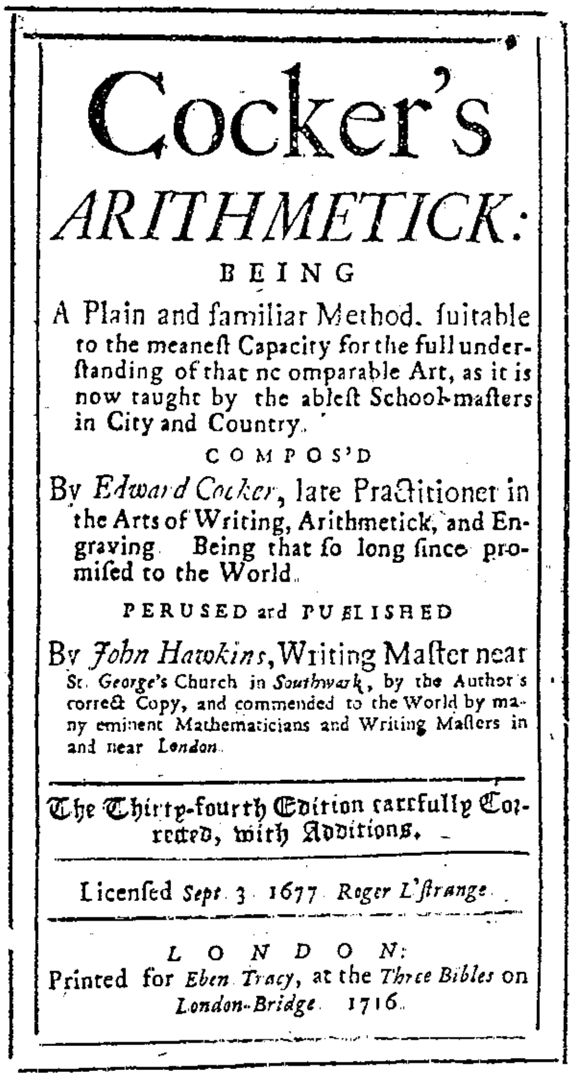
$$n - 10q - r = q.$$

Not only is division avoided, but as the author notes, the rules have another remarkable attribute.

> These new rules have yet another advantage over the rule of actual division, viz. that the final subtraction supplies a test of the correctness of the result: if it does not give remainder "0", the sum has been done wrong: if it does, then either it has been done right, or there have been two mistakes—a rare event.

We may not agree (even had we been teachers of arithmetic in 1897) with the author that these rules "effect such a saving of time and trouble that I think they ought to be regularly taught in schools", but they certainly are interesting, and not only because the writer of the letter was the inventor of the Mock Turtle.

Ingenious COCKER. (Now to Rest thou 'rt Gone
No Art can Show thee fully but thine own.
Thy rare Arithmetick alone can show
Th' vast Sums of Thanks wee for thy Labours owe



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