What's in a Name? A Learning Difficulty in Connection with Cyclic Groups

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> "Don't stand chattering to yourself like that", Humpty Dumpty said, looking at her for the first time, "but tell me your name and your business "
> "My *name* is Alice, but–"
> "It's a stupid name enough!", Humpty Dumpty interrupted impatiently. "What does it mean?"
> "*Must* a name mean something?", Alice asked doubtfully
> "Of course it must", Humpty Dumpty said with a short laugh: "*my* name means the shape I am – and a good handsome shape it is, too. With a name like yours, you might be any shape, almost "
>
> (Lewis Carroll, *Through the Looking-Glass and What Alice Found There,* Chapter VI)

First sighting of the problem

As part of a research project on students' difficulties with basic concepts in group theory, we interviewed students majoring in mathematics who had passed a required introductory course on algebraic structures [1] In this article, we limit ourselves to reporting data concerning cyclic groups Five students were systematically asked questions on this topic They were interviewed seven to eight months after the end of the course; by that time, all five had finished the second year in their three-year program and two, Geneviève and Hélène, [2] had successfully completed a second, elective, algebra course on algebraic equations and group theory

All students had difficulty recalling what a cyclic group was. Pressed to say something, they all associated a cyclic group with a cycle or a cyclic permutation and all mentioned the idea of returning to a starting point or element The following examples are typical of their remarks. [3]

Eric: It's a group [in which] the elements come back up [...] you perform certain operations [...] then it returns to the same, to the beginning. [...] It must loop.

Geneviève: At some point, it must come back to the starting element

Hélène: The first thing [that comes to mind when I think of a cyclic group] is cycles. [...] [A cyclic group] always comes back to the same starting point

Julien: For a cycle [...] to come back to its [starting] point, it must go through all the elements in the set. It's like a circle. [...] As the name says, a cycle is on a circle.

Karina: A cycle [...] should keep coming back to the same thing. Come back to the first one every time I should find some sort of rule that would make it return to the beginning again and again. It's got to turn round.

Although the five students referred to some sort of cycle, not all of them had the same idea in mind. Geneviève viewed the cycle as being made up of the successive powers of a given element, which at some point yields the identity. Julien thought of multiplying each element in the group by a fixed element. He pictured the elements shifting along a circle The other students were unable to specify the nature of the required cycle

They were also unable to apply the idea to an actual finite group – the symmetric group on three letters, S_3, presented by its Cayley table – in order to decide whether it was cyclic or not. Geneviève and Julien were able to apply the idea of a cycle to S_3, but only Geneviève was able to conclude that it was not cyclic.

Julien started by checking whether each group element appeared exactly once in a given row (this implies that multiplication by the fixed element corresponding to that row is a permutation, but not necessarily a cyclic one). Then he remembered that this property is common to every group. He would have liked to arrange all the elements on a circle, so that each would be transformed into the next one, but he did not know how to accomplish this and gave up.

Geneviève was also the only one who eventually came up with a definition sounding like the standard one: "[In a cyclic group] each element is a power of a fixed element" As it turned out, though, she did not understand it in the standard way, for by 'power' she meant only *positive* power

None of the students recognized that the group $G = \langle a \rangle = \{a^n \mid n \in Z\}$ was cyclic. Three (Geneviève, Hélène and Julien) thought it was not cyclic, and the other two would not venture any opinion. Geneviève argued that repeated application of a^1 would yield a^2, a^3, and so forth, but a^0 and "the others" – that is a^{-1}, a^{-2}, etc. – could not be obtained in this way. She went on to state: "Anyway, it cannot be cyclic, because it is infinite. A cyclic group is finite "

For Hélène, G could not be cyclic because:

> a^n increases up to infinity [...] it keeps increasing [...] I will never get the same thing

In spite of his intuition to the contrary, Julien, too, reached the same conclusion for a similar reason:

> In infinite dimension, I am unable to come back to the starting point [...] I wouldn't be able to make a permutation with it. There are infinitely many elements.

Karina, one of the two who could not conclude anything regarding the group G, attempted to apply the idea of a cycle in order to determine whether or not the group was cyclic. She suggested cycles of one or two elements consisting of an element and its inverse (a^n together with a^{-n}, and the identity by itself), but said that it was not worth pursuing:

> It's not a nice cycle, I feel. [...] A nice cycle has three or four elements.

Taking a second look

We were surprised that these students had so much trouble with such a relatively easy concept and wished to find out whether the same kind of difficulty would be encountered by other mathematics majors. Thus, we designed a questionnaire that we administered to students in another university over a two-term period. Each session, the data were collected during the last class of an introductory algebraic structures course: cyclic groups had been covered two to four weeks earlier. Altogether, twenty-nine students participated in this phase of our study. We shall identify them as S1, S2, ..., S29.

The relevant questionnaire items read as follows [4]

1. For each of the following groups, say whether or not it is cyclic and justify your answer.
   (a) The set Z under addition
   (b) The group $(Z_6, +)$
   (c) $G = \langle a \rangle = \{a^n \mid n \in Z\}$.
2. The following remarks were made by students in an algebra course. For each one, say whether you agree or disagree and explain why.
   (a) In a cyclic group, you start with an element, go round all the elements, then return to the first one.
   (b) Cyclic groups have something to do with circles.
   (c) In a cyclic group, there is an element that yields all the others when it is repeatedly combined with itself.
   (d) All cyclic groups are finite.
3. Lastly, what is a cyclic group for you? Say everything you know about this subject.

Tables 1 and 2 present the frequency of 'yes' and 'no' responses. The view of cyclic groups under investigation – that is, the one we described in the first part of this article – was also present among the students who answered the questionnaire. As was already observed during the interviews, this interfered with recognition of infinite cyclic groups.

When presented with three cyclic groups (items 1a, 1b and 1c), only about one third of the respondents wrote that the two infinite groups (Z, +) and $G = <a>$ were cyclic, whereas twice as many gave the correct answer about the finite group $(Z_6, +)$. Among those who indicated that either Z or G was not cyclic, eleven gave as a reason the fact that these groups are infinite, and five set forth the impossibility of returning to 1 or 0 (in the case of Z) or the fact that there was no sequence of repeating elements. On the same issue, statement 2d ("All cyclic groups are finite.") was agreed to by over half the students.

Table 1 Responses to items 1a, 1b and 1c

Group	Cyclic N	Not Cyclic N	No Response N
1a. (Z, +)	10 (34%)	18 (62%)	1 (3%)
1b. $(Z_6, +)$	20 (69%)	3 (10%)	6 (21%)
1c G = $<a>$	10 (34%)	11 (38%)	8 (28%)

Table 2 Responses to items 2a, 2b, 2c and 2d

Statement	Agree N	Disagree N	No Response N
2a In a cyclic group, you start with an element, go round all the elements, then return to the first one	15 (52%)	12 (41%)	2 (7%)
2b. Cyclic groups have something to do with circles.	13 (45%)	11 (38%)	5 (17%)
2c. In a cyclic group there is an element that yields all the other when it is repeatedly combined with itself.	15 (52%)	10 (35%)	4 (14%)
2d All cyclic groups are finite	17 (59%)	10 (34%)	2 (7%)

About half of the respondents subscribed to statement 2a ("In a cyclic group, you start with an element, go round all the elements, then return to the first one."). Moreover, five of those who disagreed with it still believed in returning to a starting element; they just rejected the part about going round all the elements. One objected to the expression 'going round', two did not think it necessary to go round *all* the elements and two more wished to allow for the possibility of multiple cycles.

The same proportion of students agreed with statement 2c ("In a cyclic group there is an element that yields all the others when it is repeatedly combined with itself.") Only one of those who rejected it did so for the right reason, namely the need to include the non-positive powers of a generator in the case of infinite cyclic groups.

When they justified their answer at all, the remainder gave a variety of reasons. One, for example, wrote:

> It is not combined with itself. It goes through some sort of function linking the [element] and its rank in the group. [S27]

The same student would have replaced the expression "go round all the elements" in statement 2a by:

> the image of each element is another one until you come back to the first.

Thirteen students accepted the suggestion that cyclic groups had something to do with circles (statement 2b). One would have preferred regular polygons, but granted that a circle was also a good illustration of a cyclic group, for:

> you always go over the same path, round the circle, indefinitely. [S12]

Finally, of the twenty students who offered some sort of personal definition or comments in response to the last question, six again expressed the idea that a cyclic group consists of a looping sequence. For instance, they wrote:

> It's a group of elements [...] that after a given element returns to the first one. In short, it goes through a cycle (the name says so!). [S12]

> A group in which the function is repeated after a certain section. [S21]

> [It's] a path through different points that leads back to the starting point. [S26]

Actually, almost every item in the questionnaire gave an opportunity for revealing similar beliefs and, at one point or another, altogether eighteen different students out of the twenty-nine did so in a clear way - not counting, that is, those who assented to statement 2a without making any further corroborative comments.

Of course, there was some individual variation among the views of these students. Some had a geometric image in mind while others did not. Some thought the cycle should be made up of the successive powers of a generator and others did not. One, for example, tried taking successive squares and concluded that Z_6 was not cyclic. [S23] Still, the basic idea of returning to a starting point was present.

Reflecting on this finding

What could be the origin of the difficulty we have observed? We suggest that at least part of the problem lies in the students' tendency to rely on the ordinary meaning of the adjective 'cyclic' and the noun 'cycle' in order to make sense of the term 'cyclic group'.

To experienced (modern) mathematicians, 'cyclic' may be just a code word used to identify, in this case, a type of group: its meaning is completely determined by the appropriate definition, regardless of everyday usage. In many if not most abstract algebra textbooks, for example, no comment is made to the reader concerning the relationship between a cyclic group and a 'cycle', even less about the relationship between a cyclic group and a 'circle'. The nature of the game appears to be the one spelled out by Humpty Dumpty in the following dialogue (Carroll, 1871, Chapter VI):

> "When *I* use a word", Humpty Dumpty said, in rather a scornful tone, "it means just what I choose it to mean - neither more nor less."
> "The question is", said Alice, "whether you *can* make words mean so many different things."
> "The question is", said Humpty Dumpty, "which is to be master - that's all."

As we have shown earlier, though, the apprentices - at least some of them - not sharing yet in the role of the master, tacitly assume that ordinary meanings continue to be relevant and expect a cyclic group somehow to be 'cyclic'. Three of the students who took part in our study (one during the interview and two while answering the questionnaire) explicitly referred to the common meaning of the words 'cyclic' or 'cycle' (our italics):

> Julien: *As the name says*, a cycle is on a circle.

> It goes through a cycle (*the name says so*!) [S12]

> It cannot be infinite, because, *as its name indicates*, it must be cyclic. Sooner or later, it must come back to the identity or to the starting element. [S14]

This interpretation of the terms 'cyclic' or 'cycle' led students to imagine the group elements arranged in a circular sequence or linked by some sort of function mapping each element onto a next one and the last one onto the first. The function could be multiplication by a generator, but could also be squaring or taking the inverse; sometimes, it remained unspecified:

> Karina: I should find some sort of rule that would make it return to the beginning again and again.

> It's a set of [elements] linked together in a cycle. [...] It's like a bijective function; each x has its own y. [S6]

> [It's] a group in which every element is related to the previous one. [S23]

Some students even retained from the everyday meaning the idea of going over the cycle repeatedly:

> It must return periodically. [S8]

> When you reach the last class [modulo 6], you come back to the first one and start the cycle over again. [S9]

It's a cycle that recurs indefinitely [S26]

The word 'cyclic' has an everyday meaning that students are aware of before they are exposed to the formal definition of a 'cyclic group'. It is understandable, then, that in order to make sense of the term 'cyclic group', they refer to the basic, everyday meaning of the word 'cyclic'. However, problems can occur since this everyday meaning conveys only part of the new mathematical definition.

The phenomenon just described is nothing exceptional. It has been observed by several researchers in a variety of teaching and learning contexts. Cornu (1981) and Tall

(1992), for example, have noted similar problems concerning the concept of limit and Zazkis (1998a, 1998b) has done so in the area of elementary number theory, especially in connection with the concepts of even and odd numbers

This same type of phenomenon has been discussed by Pimm (1987, ch. 4), who refers to it as a problem of 'semantic contamination'. As he points out, this kind of problem is inevitable considering that the mathematical class is conducted in a mixture of two different but related registers - the mathematical register and natural language - and that students can easily fail to distinguish the two.

Besides students' natural tendency to reduce new unfamiliar concepts to familiar ones, what can explain the prevalence and persistence of the view we have reported in this article? We see three possible reasons. First, not only does the word 'cyclic' have different meanings in ordinary language and within the mathematical register but it has different meanings also within the algebraic register itself: the word 'cyclic' is used in group theory to designate both a one-generator group and a permutation consisting of one 'cycle'.

Seeing a cyclic group in terms of a cyclic permutation is not wrong However, one has to admit permutations on an infinite set of letters, and then understand a cyclic permutation as a permutation in which each letter is being replaced by the next one. [5] The problem is that this is not how the students thought of it! Rather, they viewed a cyclic permutation literally as something 'circular'. This too, we think, can be explained by semantic contamination of the adjective 'cyclic' and the noun 'cycle' from ordinary language to the mathematical register.

Second, the view previously described can lead to correct answers in a large variety of situations, that is, for all finite groups. For instance, Z_6 features an obvious 'cycle', obtained by either repeated addition of 1 or just counting modulo 6, pointing to the conclusion that the group is cyclic Right answers like this one may mask and reinforce a possible underlying misconception

Third, students viewing cyclic groups as cycles may also hold non-standard definitions of powers and generators, allowing them to produce correct statements even beyond the realm of finite groups. We have mentioned the case of Geneviève, who asserted that:

> [In a cyclic group] each element is a power of a fixed element.

Yet she understood 'powers' as positive powers only.

Similarly, in their answers to question 3, two students [S14 and S16] gave the correct definition of a cyclic group as a group in which an element generates all the others. However, it is clear from their other comments that, like Geneviève, they meant using positive powers only. Unless instructors are privy to extensive data like ours, they may fail to realize that something is wrong with their students' notions of cyclic groups and may therefore let such ideas go unchallenged.

How might the problem be addressed?

Theoretically, the 'cyclic' view we have been discussing might cause mistakes exclusively in the case of infinite groups. Since, to within isomorphism, there exist only one infinite cyclic group and infinitely many finite ones, the significance of the difficulty pointed out in this article could be questioned. All the more so in that the view we have been discussing would not have been judged so faulty in the past.

In fact, infinite groups (as well as permutations on infinite sets of letters) are relative latecomers (Wussing, 1984, pp. 235, 252-253) and famed mathematicians like Ruffini and Cauchy defined cyclic groups in the context of finite groups only (pp. 82, 88) Cauchy and Jordan even used the term 'circular' (*circulaire*) to refer to a cyclic permutation (pp. 82, 88) [6] and Jordan, in describing a 'circular permutation', used expressions similar to those used by the participants in the present research. [7]

Are not the students' difficulties, then, simply a case of a misunderstanding which could be cleared up by informing them that the elements of a group generated by one element comprise *all* the powers of that element, positive, zero and negative, and that the notion of cycle has been extended to include infinite cycles? The hitch is that they have most likely already been told so. Why do they not remember?

According to Fischbein and Baltsan (1998), when a scientific term is borrowed from everyday language, its initial meaning tends to survive as an intuitive, tacit model of the scientific concept They theorize that conflicts between such primitive models and formal mathematical definitions result in the formal definitions being gradually forgotten and replaced by the primitive models, unless a systematic didactical intervention takes place

They illustrate their theory with findings about the mathematical concept of set and its intuitive counterpart, the physical collection Our data, too, are consistent with Fischbein and Baltsan's hypothesis, in that all the students interviewed months after being taught the formal definition of cyclic group had (reverted to) an intuitive view of this concept based on the ordinary meaning of the word 'cyclic'

Would retention be improved if cyclic groups were designated differently, either by a more transparent term or by a less transparent one? The answer is not obvious. Expressions like 'one-generator group' might sound promising, but, as we have seen, students are liable to believe that 'generating' means taking positive powers only We cannot think of any name that would fit the concept exactly and prove a guarantee against misunderstanding

Going the other way, by choosing an opaque term (for instance, naming cyclic groups after a mathematician) would avoid the problem of semantic contamination By the same token, however, by giving no hint about its meaning, such a name would leave stranded those who forget its definition - a situation that we have observed in the case of Abelian groups.

Either way, selecting meaningful names or meaningless ones involves advantages and disadvantages. Anyhow, in the case of cyclic groups, given the widespread and well-established usage of the term, it would not be very realistic to try and supplant it, even if a clearly better alternative were available

Rather than trying to reform the terminology, it would be more productive to look for ways of preventing the erosion of formally-defined mathematical notions and their replacement by various intuitive interpretations To this end, Fischbein and Baltsan (1998) recommend systematic didactical reinforcement. We agree that knowledge of formal definitions is particularly fragile and that repeated intervention is needed in order to improve retention

However, this is not the only means available. As we have noted earlier, the use of everyday words within the mathematics register might cause problems because students are not always aware of the shifts across the two different registers Part of the solution, then, might lie in drawing their attention to the different registers at work in mathematical discourses. As Pimm (1987, p. 109) writes:

> In order to obviate some of these comprehensible failures to communicate, pupils at all levels must become aware that there are different registers and that the *grammar*, the *meanings* and the *uses* of the same terms and expressions all vary within them and across them

Furthermore, we would like to suggest that explicit teaching of the role of definition in mathematics - i e., instruction in Humpty Dumpty's stance that words mean just what we choose them to mean - might prove beneficial in alleviating difficulties of the type we have been discussing.

Finally, we wish to make it clear that we do not deny the value of intuitive images. We do believe that physical experience is the direct or indirect source of mathematical ideas, that intuitive models are essential to the learning process and that they continue to be precious heuristic tools even after formal definitions have been established. However, we also believe that in order to ensure safe use of such tools, one must remain aware of their status and limitations

Epilogue

As an afterthought, we decided to ask a few professional mathematicians, all of them faculty members of mathematics departments, to name three cyclic groups - the first ones that occurred to them. We were curious to see whether the infinite cyclic group, in the guise of $(Z, +)$ or otherwise, would be among the three, and more particularly whether it would be mentioned first.

We thus approached seven mathematicians. Three named $(Z, +)$ as their first example and one named it as his third The other three did not mention any infinite groups at all. When asked why, they said that it had not occurred to them; one added that he only worked with finite groups

Even more interesting are the following comments by two of the mathematicians who did name $(Z, +)$. One remarked:

> I have some doubts, because I am not an algebraist, but I think that it is sufficient to have a generator Even if it doesn't look like it turns I should go and check in an algebra book.

While speaking, he drew a circle in the air with his finger. The other one, an algebraist, asked us whether he should limit himself to finite groups, and added:

> Unless you consider that a cyclic group isn't obtained by the powers of a single generator [..] er [..] that it does not have to loop.

Of course, these last two mathematicians may well have been familiar with the distinction made by certain authors, Bourbaki (1964) in particular, between cyclic groups and one-generator groups (see [4]). However, since they both clearly referred to the image of a circle or a loop, we are rather tempted to think that they were, at least partly, influenced by the ordinary meaning of the word 'cyclic'. This would be consistent with Fischbein and Baltsan's (1998, p. 13) hypothesis that even persons highly trained in mathematics carry traces of intuitive models.

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Notes

[1] All courses mentioned in this article are 45-hour courses extending over fifteen weeks Neither of us taught any of these courses

[2] All the students' names are pseudonyms

[3] All the reported data are translated from French

[4] The questionnaire contained a few further items concerning difficulties for which the interviews had provided some weak evidence, especially the idea that in a cyclic group all elements are equivalent, in the sense that, with the possible exception of the identity, they all have the same order and any one of them can generate the group. The questionnaire results did not yield additional proof of this misconception.

[5] It is implicit here that such a set is countable and that its order type is the same as the order type of Z.

[6] Even in modern times, a few authors reserve the term *cyclic* for finite groups Some of them use a different term for (finite or infinite) groups generated by one element; Bourbaki (1964, p 77), for instance, calls such groups *monogène* All the students who participated in our study, though, had been taught that groups generated by one element are called *cyclic*, and had been exposed to examples of infinite cyclic groups, notably $(Z, +)$

[7] For example, in his work on circular permutations, Jordan (1861, p 122) wrote:

> Soient *a*, *b*, *c*, les *n* lettres de la fonction, *A* une substitution. Si on l'effectue, *a* sera remplacé par une autre lettre *b*, celle-ci par une lettre telle que *c*, etc , jusqu'à une lettre *k*, qui sera remplacée par *a*. Les lettres *abc* *k*, qui se remplacent ainsi en cercle, formeront un *cycle* Si ce cycle comprend toutes les lettres, la substitution sera dite *circulaire*

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