Communications

Response to 'Conceptualiser le périmètre'

CHRISTOPHER DANIELSON

Mathematics thrives on ambiguity. Far from eternal, mathematical truths depend on context. Math lives in the *both/and;* in the *for now* and *it depends.* Mathematics lives in the *what if?*

I was early in my journey of understanding these truths when I wrote 'Perimeter in the Curriculum', in issue **25**(1), more than 15 years ago. I am delighted and honored by Jérôme Proulx's revisiting of the ideas and arguments I presented back then, and I am especially excited by his new classroom example.

The students arguing about whether the perimeter of a 4 cm by 8 cm rectangle, when drawn on a grid, is 28 or 36, and using the idea of *enclosure* to do so is an enlightening example, and it suggests new questions. Would their arguments and/or conclusions have been different if the rectangle had no underlying grid? If so, how? Is there any plane figure for which these two counting strategies agree, either with each other or with the more conventional idea of perimeter? I have come to understand that much more mathematics happens when learners work to resolve such ambiguities than when a neat and tidy definition leads everyone to the same conclusion.

Beyond that single example, I am grateful to Proulx for what his writing highlights about mathematical ideas in general. While he ponders perimeter, many mathematical concepts have the same kind of duality. Is perimeter a 1- or 2-dimensional concept? Few novices are satisfied with the formal answer to this question (which Proulx discusses)—that perimeter is a 1-dimensional measure of an object embedded in 2-dimensional space. So, for many learners the answer shifts depending on how they're looking at it in the moment.

Is surface area a 2- or 3-dimensional concept? Does a circle have 0 sides or 1? Is a cone a kind of pyramid? Is a *dozen* singular or plural? Each of these questions requires setting further conditions—set them one way to get one answer; set them another to get a different one.

A central challenge of teaching and learning mathematics is coming to understand each other's perspectives; to know 'the sense in which' a person's mathematical claims are true, as the mathematician Eugenia Cheng has put it [1]. A common conception of mathematics is that the only sense which matters is the one recorded in textbooks. But math learning does not involve a simple erasure of a learner's previous ideas and replacement with new ones. Instead, learning requires integration and evolution of existing ideas into more complex, robust, and nuanced understandings. I am grateful to Proulx and his students for helping me to understand a sense in which perimeter is about enclosure, and how that enclosure might involve 2-dimensional components. I am grateful for this newly exposed ambiguity.

Note

[1] Eugenia Cheng is a mathematician and author who teaches at the School of the Art Institute of Chicago. In her 2018 book 'The Art of Logic in an Illogical World', she cites her Ph.D. advisor as saying "There is a sense in which" before making claims. Cheng writes, "It reminds us all that mathematics is not just about finding the right answer, but is about finding the sense in which things might or might not be true" (p. 124-5).

Zero probability and dimensional confusions

ALI BARAHMAND

Zero probability and zero- and one-dimensional objects share a counter-intuitive nature that I explore in this communication. I believe that considering the concepts together reveals something about the source of their counter-intuitive characters and leads to observations that could be of importance to mathematics education.

The probability of an event is a ratio comparing the number of ways in which an event can occur to the total number of events in the sample space. An event with zero probability is called an *impossible* event. It is an event that cannot happen. The probability of rolling 7 on a single six-sided dice is zero. But in some contexts, events with zero probability do occur.

When we speak of hitting the centre of a target, we normally refer to an area, the bullseye, and the probability of hitting the bullseye depends on its area compared to the total area of the target, the cross-sectional area of the projectile and the skill of the shooter. If we consider the probability of hitting the mathematical centre with a projectile whose cross-section is a single point, however, the probability is zero. The sample space of points on the target is infinite, and only one of them is the centre. The same applies to any point on the target, yet our projectile must hit somewhere, and an event of zero probability will occur.

The seeming contradiction of an event of zero probability occurring seems to be related to the shift from the realistic situation of hitting a two-dimensional bullseye with a projectile with a two-dimensional cross-section, to the mathematical situation of hitting a zero-dimensional point with a projectile with a zero-dimensional cross-section. But we can also see that if *either* the target or the projectile has area, the problem does not occur. One cannot mark the exact centre of a target, but with a real projectile with a cross-sectional area (say a circle of radius r) it is possible to hit the centre, by hitting any point less than distance r from the centre. Similarly, a projectile whose cross-section is a single point will hit a normal bullseye as long as it hits any point within the bullseye. Perhaps it is because our point-projectile and the centre *both* lack area that the probability becomes zero.

In fact, area is not required for there to be a non-zero probability of a projectile hitting a bullseye. If the bullseye

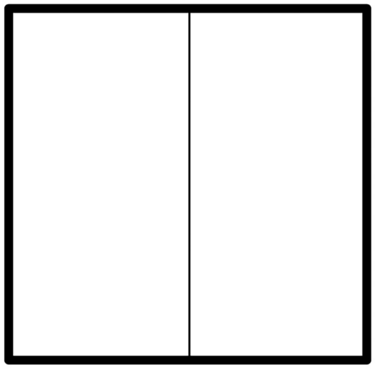

Figure 1. A segment as ‘bullseye’.

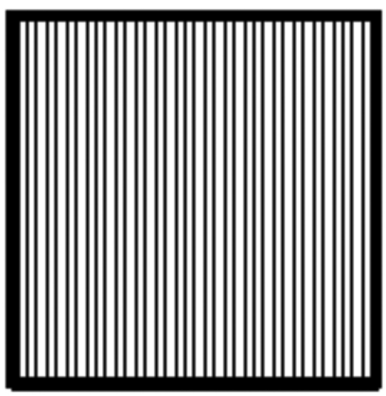

Figure 2. Many segments as ‘bullseye’.

is a segment (see Figure 1), and the projectile’s cross-section is also a segment, the probability of hitting the bullseye is non-zero.

If we aim for a segment with a point-projectile, the probability is again zero, even though the space of successful events now includes an infinite number of points. In a two-dimensional target, the probability of hitting any part with fewer dimensions, such as a zero-dimensional point or a one-dimensional segment, with a point-projectile, will be zero.

We can even cover the target with many parallel segments without increasing the probability above zero (see Figure 2). Intuitively, the greater the number of segments, the greater the chance of hitting them [1]. Individuals might think that the chance of hitting the segments in Figure 2 is higher than hitting the single segment drawn in Figure 1. But in both cases the probability is zero.

The same can be said of the effect of increasing the length of the segment. One may think that the probability of hitting the segment will be higher when the length of the segment is longer. For example, hitting the diagonal (Figure 3) should be easier than hitting the segment in Figure 1. But again, both have zero probability.

This is related to a well-known conflict regarding the area of parallelograms and rectangles. Suppose we put together many equal segments as shown in Figure 1, and diagonals as shown in Figure 3, to obtain a square and parallelogram with identical bases and heights (Figure 4). The shapes appear to be made entirely of segments. “Intuition might suggest that

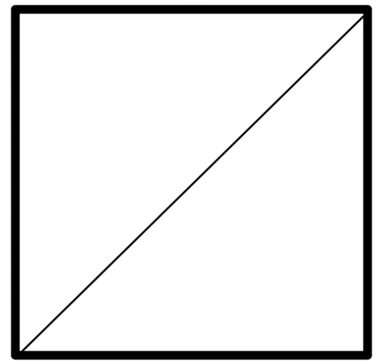

Figure 3. A longer segment as ‘bullseye’.

because the line segments that form the parallelogram are longer than the ones that form the rectangle, the area of the parallelogram is larger than the area of the rectangle.” (Burazin, Kajander & Lovric, 2021, p. 1346)

This is despite the fact that the square and parallelogram with identical bases and heights have the same area from a mathematical viewpoint. The root of this conflict lies in what was discussed about the difference between the dimensions: the two-dimensional figure is not the sum of the one-dimensional line segments. The figure has area, and the segments do not.

The standard solution to this problem is to think of the segments as the limits of sequences of parallelograms of ever-decreasing height. Diagonal parallelograms are longer, but not as wide (see Figure 5).

Here, the smaller side of the parallelogram on the right is as long as the width of the rectangle on the left, a, so its height is $^{a}/_{\sqrt{2}}$. The base-length of the parallelogram equals $\sqrt{2}b$. Therefore, the area of a rectangle and a parallelogram, obtained by partitioning them into equal numbers of ‘segments’ being equal in area, will also be equal.

This suggests a way to resolve the contradictory occurrence of events of zero probability. If we consider the probability of a projectile of cross-section A hitting a bullseye of area A, and

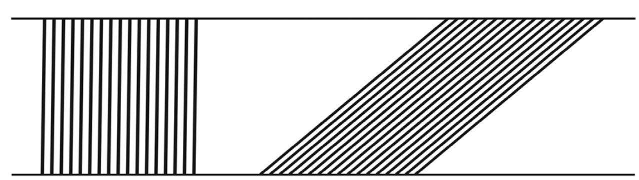

Figure 4. A square and a parallelogram formed of segments (modified from Burazin, Kajander & Lovric, 2021, p. 1346).

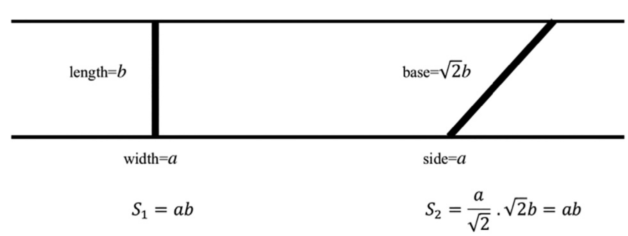

Figure 5. Segments as limits of rectangles.

the limit of these probabilities as A goes to zero, we can state that the limit is zero, without it becoming impossible for a point-projectile to hit a target. Thus it is possible to distinguish between events of zero probability that are impossible, like rolling a 7 on a single six-sided dice, and events that have zero probability *as a limit,* which are not impossible.

Note

[1] We represent points and segments with patches of ink that have width, so that they are visible. We think of segments of zero width, but we draw segments with width. Looking at Figure 2, it is visually clear that the segments cover more of the area, so it is not surprising that one expects the probability of a point-projectile hitting one of the segments is higher. The representation supports the intuition that more segments should be easier to hit.

Reference

Burazin, A., Kajander, A. & Lovric, M. (2021) Reasoning about geometric limits. *International Journal of Mathematical Education in Science and Technology* **52**(9), 1345–1360.

Moser's worm problem: an unsolved problem in plane geometry

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This problem was posed by Leo Moser in 1966 in a mimeographed list of problems [1]:

> What is the region of smallest area which will accommodate every arc of length l?

This problem has come to be known as the worm problem, and can be rephrased:

> Define a *worm* to be any plane curve of length one. Define a *cover* to be a plane region that can accommodate any worm, that is, the worms have a fixed shape that cannot be altered, but they can be rotated and translated to fix into the region. Find, if it exists, the cover of smallest area.

Here we focus on the case in which the cover is additionally required to be convex, which ensures it exists.

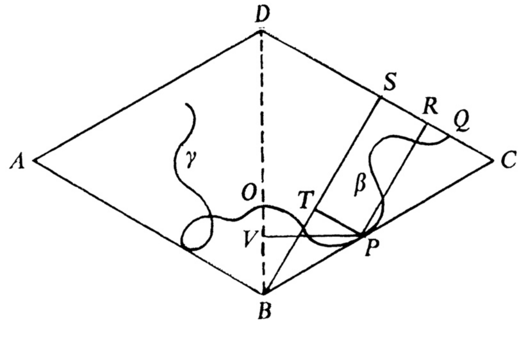

Figure 1. Rhombus cover from Gerriets and Poole (1974, p. 37).

It is easy to find simple regions that can accommodate any worm. For instance, a disk of diameter 1 is a cover. Wetzel (1973) gives a proof due to Meir that a semi-disk of diameter 1 is a cover. Gerriets and Poole (1974) proved that the rhombus with diagonals of lengths 1 and $1/\sqrt{3}$ (obtained by 'gluing together' two equilateral triangles of side length equal to $1/\sqrt{3}$) is also a cover (see Figure 1). I suggest that the reader try to prove this.

Quite recently, Panraksa and Wichiramala (2020) showed that a 30 degree circular sector is also a cover. This is the best convex cover known, at this time.

As with Kobon's Triangle Problem (discussed in **41**(3)), one of the main obstacles is that there is no known general strategy for showing that a given planar region is a cover. Gerriets and Poole tried to find such a strategy. They conjectured that:

> If a convex region contains all "two angle worms" of length L, then the region contains all worms of length L.
>
> By "two angle worms" we mean all arcs formed by joining consecutively, at any angles, three segments whose total length is L. (1974, p. 41)

This was later disproved by Panraksa, Wetzel and Wichiramala (2007). who showed that the conjecture is not true for *any* polygonal worm, that is, any chain of finitely many segments joined end to end.

Notes

[1] The list is reproduced in William Moser's 1991 article 'Problems, problems, problems' in *Discrete Applied Mathematics* **31**, 201–225.
[Editor's Note] This is one of a series of unsolved problems. The first appeared in issue **41**(3).

References

Gerriets, J. & Poole, G. (1974) Convex regions which cover arcs of constant length. *American Mathematical Monthly* **81**(1), 36–41.
Norwood, R. & Poole, G. (2003) An improved upper bound for Leo Moser's worm problem. *Discrete and Computational Geometry* **29**(3), 409–417.
Norwood, R., Poole, G. & Laidacker, M. (1992) The worm problem of Leo Moser. *Discrete and Computational Geometry* **7**(2), 153–162.
Panraksa, C. & Wichiramala, W. (2020) Wetzel's sector covers unit arcs. *Periodica Mathematica Hungarica* **82**(2), 213–222.
Panraksa, C., Wetzel, J.E. & Wichiramala, W. (2007) Covering n-segment unit arcs is not sufficient. *Discrete and Computational Geometry* **37**(2), 297–299.
Wetzel, J.E. (1973) Sectorial covers for curves of constant length. *Canadian Mathematical Bulletin* **16**, 367–375.

Why would a biologist need a logarithm?

IGOR V. ANDRIANOV

Students of our times are very practical persons. "Why do I need mathematics if I plan to study psychology or biology? Or if I will be an artist? Why do I need exponentials, logarithms, *etc.*?". Perhaps future psychologists, biologists and scholars in the humanities will treat mathematics with more attention if they are reminded of the existence of the Weber-Fechner law [1].

The Weber-Fechner law states the logarithmic relationship between perceived intensity and stimulus strength. Weber established this rule as a result of experiments on the subjective perception of the loudness of tones. This observation was formulated mathematically by Fechner:

$$S - S_0 = b \ln(R - R_0)$$

where b is the constant called the difference threshold, $R - R_0$ is the difference in the power of the signal and the nearest subjectively lower signal, and $S - S_0$ is the difference in subjective loudness of these signals.

According to the Weber-Fechner law, a human's response to an external influence is not linear, but proportional to the logarithm of the disturbance. This law, like everything in our life, has both positive and negative sides. It is good that a burn covering 10 cm^2 of skin, does not result in 10 times greater pain than a burn covering 1 cm^2, but only ln(10) times.

The negative side was indicated, for example, by Leo Tolstoy. In a letter to his son Mikhail, warning the latter against being too carried away by various "joys of life", Leo Tolstoy wrote, "There is even a law according to which it is known that pleasure increases in an arithmetic progression, while the means for producing this pleasure must be increased in squares"[2]. The statement of the Weber-Fechner law is not entirely accurate, but the essence of the statement is absolutely correct!

The logarithmic narrowing of the spectrum of external influences is so characteristic of the phenomena of life that it can be used to define it. Molchanov (1992) wrote:

> One of the most characteristic properties of biological objects is the enormous range of irritations within which the system operates normally. The logarithmic scale of responses is the only opportunity to cover all informationally significant irritations, while retaining acceptable size of organs. Systems that could not develop this property in themselves simply could not stand the struggle for existence. (p. 150, my translation)

The Weber-Fechner law manifests itself in rather unexpected places. Scientist and designer Boris Rauschenbach was one of the key figures in the Soviet space program. He studied docking of space vehicles, one of the most challenging aspects of space flight. Soviet spaceships were designed in such a way that the astronaut could only see the docking apparatus on a monitor, that is, on a flat screen. The visual perception of changing distance does not coincide with its actual change. Of course, this is due precisely to the Weber-Fechner law. The image on a flat screen is misleading the astronaut. The solution of this technical problem led Rauschenbach to professional studies of the theory of graphical perspective. Rauschenbach (1983, 1986) found that the systems of perspective in the paintings of artists are associated with their intuitive consideration of the transforming activity of the brain. If you look into the distance, then the apparent change in linear dimensions in the direction perpendicular to the line of sight is proportional to the logarithm of the distance to this subject. A linear perspective takes this into account. For near vision, a logarithmic dependence is also characteristic, which can be approximately replaced by an inverted perspective.

Naturally, all 'laws of nature' are approximations. "As far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality" (Einstein, in Calaprice, 2000, p. 240). If the external influence is too intense, the logarithmic response does not help and the biological system can collapse catastrophically. However, such phenomena are a separate topic; other functions are needed to describe them, in particular, the exponential function.

As for me, I will consider my task complete if students willing to devote themselves to biology or psychology begin to treat the function ln(x) with respect.

Notes

[1] Named after Ernst Heinrich Weber (1795–1878), a German physician and Gustav Theodor Fechner (1801–1887), a German experimental psychologist, philosopher, and physicist.

[2] My translation of "Существует даже закон, по которому известно, что наслаждение увеличивается в арифметической прогрессии, тогда как средства для произведения этого наслаждения должны быть увеличены в квадратах." From a letter from L.N. Tolstoy to his son M.L. Tolstoy dated 16.10.1895, in Tolstoy (1984, Vol. 19, p. 334).

References

Calaprice, A. (2000) *The Expanded Quotable Einstein.* Princeton University Press.

Molchanov, A.M. (1992) *Нелинейности в биологии* [Nonlinearities in Biology]. ОНТИ [Scientific Center of RAN].

Rauschenbach, B.V. (1983) On my concept of perceptual perspective that accounts for parallel and inverted perspective in pictorial art. *Leonardo* **16**(1), 28–30.

Raushenbach, B.V. (1986) *Системы перспективы в изобразительном искусстве: Общая теория перспективы* [Perspective Systems in the Visual Arts: General Perspective Theory]. Наука [Nauka].

Tolstoy, L.N. (1984) *Собрание сочинений в 22 томах* [Collected Works in 22 volumes]. Художественная Литература [Artistic Literature].

Explicit instruction or poor realizations of dialogic instruction: which is better?

TYE G. CAMPBELL, HALEY H. PARKER, ANNA KEEFE

Over the last three decades, there has been much debate surrounding two contrasting modes of mathematics instruction: explicit instruction and dialogic instruction. Proponents of explicit instruction believe that students learn best through explicit guidance from an authority (teacher, textbook, *etc.*) (Clark, Kirschner & Sweller, 2012). In such environments, students are believed to *acquire* knowledge by actively listening to their teacher (or other authoritative resource) and engaging in guided practice. In contrast, proponents of dialogic instruction believe that students learn best when they work on challenging tasks with their peers while the teacher facilitates, rather than explicitly guides, student learning (Munter, Stein & Smith, 2015). In dialogic classrooms, students are believed to *construct* knowledge by sharing their ideas, making sense of others' ideas, drawing on prior knowledge, and participating in meaningful activities.

Proponents of both sides of the debate claim strong theoretical and empirical support for their instructional method (*e.g.* Clark, Kirschner & Sweller, 2012; Kirschner, Sweller & Clark, 2006 versus Boaler & Staples, 2008; Johnson, Johnson & Roseth, 2010). Considering the claims of both sides of the debate, it seems that one can create conditions to support either hypothesis. In other words, the research is inconclusive regarding which approach to instruction is 'better' as a general rule.

In this short communication, we explore a different question than is generally considered in the debate. Whereas most researchers provide empirical or theoretical arguments regarding why *standard* versions of explicit or dialogic instruction are preferable, here, we explore the following question:

> Which is 'better': Standard versions of explicit instruction or *poor realizations* of dialogic instruction?

We clarify the details regarding why this is an important question to address shortly.

Before explaining our arguments, we address a few subjectivities. First, the authors are graduate students, assistant professors, and former educators who draw on their experiences and numerous observations, both formal and informal, of grade 6–12 mathematics teachers and teacher candidates to stimulate conversation in relation to the posed question. This short communication is not empirical, and thus, should be treated for what it is—musings from scholars to add nuance to an important debate in mathematics education. Second, we acknowledge that 'instructional type' is fluid and therefore not a dichotomous variable; however, teachers tend to lean towards particular instructional strategies, and categorical comparisons of these strategies allow researchers and stakeholders to meaningfully deliberate about how teachers *should* teach. Third, we tend to promote and align with dialogic approaches to instruction in our research and practice. We believe, when implemented with fidelity, dialogic instruction is a highly successful instructional strategy. However, we ignore favorable versions of dialogic instruction here, for reasons expounded upon below.

Why the comparison?

Based on our formal and informal observations of novice and experienced teachers, we claim that teachers who use dialogic instruction *usually* implement it in ways that are inconsistent with the literature (*i.e.* 'poor realizations of dialogic instruction'). In contrast, teachers who use explicit instruction often implement it in the way it was intended, as described by proponents of explicit instruction. In dialogic classrooms, we have noticed that teachers face challenges that hinder the goals of dialogic instruction. Teachers often struggle to: (1) keep students on task, (2) ask the right questions to support learners in building conceptual understanding, (3) maintain focus on the lesson objectives, and (4) successfully mitigate group hierarchies that form through discourse. This is not an exhaustive list—indeed, researchers have identified many challenges teachers face that we could expound upon (see, *e.g.,* Franke, Kazemi & Battey, 2007; Heyd-Metzuyanim, 2019), but will refrain for the sake of brevity. In contrast to dialogic teaching, it seems that teachers using explicit instruction are *usually* able to directly teach learners and provide them with guided practice activities, with minimal deviations from the standard expectations of explicit instruction.

It should come as no surprise that teachers have more difficulties implementing dialogic instruction in comparison to direct instruction. Dialogic instruction is much more complex, requiring teachers to possess deep mathematical knowledge for teaching, extraordinary classroom management skills, and knowledge of responsive teaching moves (*e.g.* Franke, Kazemi & Battey, 2007). Teachers who use explicit instruction can, for the most part, follow a predetermined script. Because explicit instruction is teacher- centered, teachers rarely need to modify their plans in the midst of a lesson.

In sum, the reason we compare standard versions of explicit instruction with poor realizations of dialogic instruction is because we *believe* these two types of instruction are most common in their respective categories.

Defining 'better'

The success of classroom instruction is measured in a variety of ways in the literature. Here we focus on three of the most prevalent ways to measure the success of mathematics instruction: (1) mathematics achievement as defined by standardized test scores, (2) opportunities for engaging in mathematical practices (*e.g.* justifying ideas, critiquing the reasoning of others, *etc.*), and (3) student motivation/interest in mathematics. Proponents of explicit instruction generally perceive standardized test scores as the most important tool for measuring the quality of classroom instruction (*e.g.* Clarke, Kirschner & Sweller, 2012). While proponents of dialogic instruction similarly value standardized test scores as a measurement tool, they also believe quality is measured through the opportunities afforded to learners for engaging in meaningful mathematical practices (*e.g.* Boaler, 2002). Research has also documented student motivation/interest in mathematics as an important measurement of classroom instruction, since learners' perceptions towards mathematics can profoundly shape their mathematical trajectories (*e.g.* Heyd-Metzuyanim & Sfard, 2012).

Now let us address the question: Which is better in terms of these three forms of measurement?

Student achievement

In relation to student achievement as defined by standardized test scores, it seems that explicit instruction is preferable to poor realizations of dialogic instruction. During our observations of dialogic instruction, we have noticed that teachers often devote a substantial amount of time to a single mathematical topic, students are often off-task during group work, and learners struggle to make connections between their discoveries and the intended mathematical objectives. Indeed, at the close of many of our observations, we wonder whether students realized the mathematical objectives or could successfully answer questions related to the objectives. In standard versions of explicit instruction, teachers generally cover more mathematical content and the mathematical objectives remain consistent and clear throughout a lesson. Because students receiving explicit instruction have opportunities to experience more mathematical content through an explicit and consistent curriculum with less distractions (*e.g.* off-task talk), we claim they will likely perform better on standardized achievement tests (when confounding variables are similar). Even proponents of dialogic instruction acknowledge that explicit instruction can help students learn mathematical procedures (*e.g.* Boaler, 2002), which is an important component of standardized tests.

Opportunities for engaging in mathematical practices

Proponents of dialogic instruction perceive learning, not only as acquiring information, but also as a form of participation in mathematical practices (*e.g.,* Boaler, 2002). Therefore, from a dialogic perspective, it is important to measure the opportunities students are afforded for engaging in mathematical practices (*e.g.,* conjecturing, justifying, explaining strategies, critiquing the ideas of others) to determine the success of instruction. Within poor realizations of dialogic instruction, learners experience opportunities to share their mathematical ideas and hear others' ideas. In our experience, however, learners often do not engage in *meaningful* practices, such as critically evaluating others' ideas; instead, student dialogue often remains at a surface-level wherein learners share their solutions without engaging in debate or deliberation about mathematical differences. In standard versions of explicit instruction, students generally only have opportunities to imitate strategies that were first performed by the teacher. There are few opportunities to engage in meaningful mathematical practices other than practicing procedural fluency. Taken as a whole, we perceive that learners have more opportunities to engage in mathematical practices during poor realizations of dialogic instruction in comparison to explicit instruction, though learners may not often take up these opportunities meaningfully.

Motivation/interest in mathematics

Finally, neither poor realizations of dialogic instruction nor explicit instruction seem to support learners' motivation/interest in mathematics based on our observations. In poorly constructed dialogic classrooms, we have observed learners talking to one another in ways that could detrimentally influence learners' beliefs about their mathematical abilities (rude remarks, one person assuming authority, *etc.*). Groups often create hierarchies that determine who is 'smart' and worthy of attention, unless teachers remain proactive and create positive group norms (which often does not happen in poor realizations of dialogic instruction). In standard versions of explicit instruction, students can become unengaged and simply 'go through the motions' when instruction becomes monotonous. While students can learn through simply imitating authority figures, such engagement can cause students to lose interest in mathematics because they find it boring or unimportant. In sum, both dialogic and explicit instruction exhibit limitations in relation to supporting learners' motivation/interest in mathematics, albeit for different reasons.

What now?

Now, let us recap the claims we made above based on our observations of teachers and teacher candidates:

1. Explicit instruction seems more beneficial than poor realizations of dialogic instruction in relation to supporting student achievement as defined by standardized test scores.
2. Poor realizations of dialogic instruction provide students with more opportunities to engage in mathematical practices.
3. Neither poor realizations of dialogic instruction nor explicit instruction seem to support students' mathematical interest/motivation.

Naturally, the question is, where does the field go from here? We make provocative claims—claims that are based on anecdotal evidence rather than empirical research. Therefore, the first step is for researchers to empirically investigate our initial claim: dialogic instruction is *usually* realized in ways that are inconsistent with literature. Then, researchers could investigate whether the hypotheses we propose are true regarding which form of instruction is better (*e.g.,* standard versions of explicit instruction lead to higher student achievement than poor realizations of dialogic instruction).

If empirical research upholds our claims, then researchers, policymakers, and practitioners are faced with important decisions. We believe that if teachers understood the hallmarks of dialogic instruction (types of dialogic assessment, questioning techniques, progression of instruction, *etc.*), then they would be more likely to implement dialogic instruction with fidelity. The field still has much work to do to identify these hallmarks, on which research is currently underway (*e.g.,* Campbell, 2021; Gillies, 2019; Langer-Osuna, Munson, Gargroetzi, Williams & Chavez, 2020). In the meantime, teachers and researchers might discuss strategies for using hybrid approaches to instruction depending on the context and level of comfort of the teacher. Teachers who navigate the complexities of dialogic instruction comfortably might rely more heavily on student dialogue and minimal teacher guidance, whereas teachers with less comfort may rely more heavily on explicit guidance. Perhaps such contextual features (*e.g.,* student/teacher characteristics, teacher experience/strengths, *etc.*) should play a more prominent role in decisions about teaching and learning than they have in the past.

We hope this short communication motivates research and discussion regarding two primary modes of instruction. The ideas presented here should not be used in support of either side of the argument. Rather, we shed light on a new aspect of the debate.

References

Campbell, T. (2021) Examining how middle grade mathematics students seize learning opportunities through conflict in small groups. *Mathematical Thinking and Learning.* Advance online publication.

Boaler, J. (2002) The development of disciplinary relationships: knowledge, practice and identity in mathematics classrooms. *For the Learning of Mathematics* **22**(1), 42–47.

Boaler, J. & Staples, M. (2008) Creating mathematical futures through an equitable teaching approach: the case of Railside School. *Teachers College Record* **110**(3), 608–645.

Clark, R.E., Kirschner, P.A. & Sweller, J. (2012) Putting students on the path to learning: the case for fully guided instruction. *American Educator* **36**(1), 6–11.

Franke, M.L., Kazemi, E. & Battey, D. (2007) Mathematics teaching and classroom practice. In Lester, F.K. (Ed.) *Second Handbook of Research on Mathematics Teaching and Learning.* Information Age Publishing.

Heyd-Metzuyanim, E. & Sfard, A. (2012) Identity struggles in the mathematics classroom: on learning mathematics as an interplay of mathematizing and identifying. *International Journal of Educational Research* **51**, 128–145.

Heyd-Metzuyanim, E. (2019) Changing teaching practices towards explorative mathematics instruction: the interweaving of teacher identity and pedagogical discourse. *Teaching and Teacher Education* **86**, article 102862.

Johnson, D.W., Johnson, R.T. & Roseth, C. (2010) Cooperative learning in middle schools: interrelationship of relationships and achievement. *Middle Grades Research Journal* **5**(1), 1–18.

Langer-Osuna, J., Munson, J., Gargroetzi, E., Williams, I. & Chavez, R. (2020) "So what are we working on?": how student authority relations shift during collaborative mathematics activity. *Educational Studies in Mathematics* **104**(3), 333–349.

Kirschner, P.A., Sweller, J. & Clark, R.E. (2006) Why minimal guidance during instruction does not work: an analysis of the failure of constructivist, discovery, problem-based, experiential, and inquiry-based teaching. *Educational Psychologist* **41**(2), 75–86.

Munter, C., Stein, M.K. & Smith, M. (2015) Dialogic and direct instruction: two distinct models of mathematics instruction and the debate(s) surrounding them. *Teachers College Record* **117**(11), 1–32.

Horizon knowledge and the complexities of contingency: a scenario from a senior secondary mathematics classroom

NICOLE MAHER, HELEN CHICK, TRACEY MUIR

A group of senior secondary school students were studying probability distributions. Their teacher, Mr. McLaren, assigned some textbook problems which required them to determine what happens to the variance when random variables undergo a transformation. The students had been given the formula $\mathrm{Var}(aX+b) = a^2\mathrm{Var}(X)$ for determining variance with the original random variable X and the transformation $aX+b$. Just as Mr. McLaren was modelling a specific example, one of the students interrupted him:

Grace — What happened to the b? *[She indicated the absence of b in the right-hand side of the relationship Var(aX+b) =* a^2*Var(X) that was written on the whiteboard.]*

Mr. McLaren — *[After a pause]* It's just gone.

Grace — Really! *[She sounded amazed].*

Mr. McLaren — Yeah. *[He paused for a couple of seconds.]* We could look at why but I'm not too fussed.

As the students continued working, however, Mr. McLaren seemed to ponder Grace's question. He looked down at his open textbook, turned a couple of pages and appeared to think for a while, before turning to the whiteboard again.

Mr. McLaren — If you think in terms of what the variance actually is, I said I wasn't going to show you, but anyway

Mr. McLaren cleared the whiteboard and proceeded to express $\mathrm{Var}(aX+b)$ using the defining relationship $\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$, where $\mathrm{E}(X)$ indicates the expected value of X. He hesitated a little, checked the textbook again, and wrote the following statement on the whiteboard:

$$\mathrm{Var}(aX+b) = \mathrm{E}(aX+b)^2 - [\mathrm{E}(aX+b)]^2 = a^2\mathrm{E}(X^2) + b^2 - [\mathrm{E}(aX+b)]^2$$

At that point Mr. McLaren stopped [1]. He did not attempt to expand and evaluate the final expression, but hastily explained that, "in the end, what will happen is we end up with minus b squared on the end, so the b squared terms end up cancelling out". He added: "I don't think you're going to come across this at all", as if to indicate that the students need not worry about it. He then continued with the rest of the lesson. Later, however, in a post-lesson interview, Mr. McLaren provided a more thorough and detailed justification for why $\mathrm{Var}(aX+b) = a^2\mathrm{Var}(X)$.

> I'm figuring out a way of explaining it better. I think I was fumbling around a bit there, but *[pause]* because *'a'* is having a multiplying effect on all of the *'X'* values and *'b'* is having an additive effect. When it comes to variance, you're not worried about the additive effect because the spread is all added by that *[…]* *'b'* so the variance doesn't get changed by the *'b'*. So *'b'* has no effect on the variance which is a measure of spread. The spread remains the same *[…]* that's why the *'b'* disappears. But the *'a'* does have a multiplying effect and because variance is about a square of the differences the *'a'* has got to be squared.

Commentary

Grace's question illustrates the organic nature of mathematics lessons and how opportunities arise that urge the teacher to make in-the-moment decisions relating to how—or even whether—to embrace a contingent event. Mr. McLaren was initially reluctant to discuss why b disappears, beyond the fact that it is "just gone" because he knew that the course and its external examination did not require students to derive or prove formulae. In the end, however, he did attempt to algebraically verify the relationship between $\mathrm{Var}(aX+b)$ and $a^2\mathrm{Var}(X)$, even though the resulting explanation was vague and incomplete. The lack of rigour in this explanation reflected the fact that Mr. McLaren was not accustomed to explaining why $\mathrm{Var}(aX+b) = a^2\mathrm{Var}(X)$. Nevertheless, his actions were indicative of the kind of *knowing-to-act* that Mason and Spence (1999) discuss. Knowing-to-act is an intricate and dynamic phenomenon that involves more than possessing a bank of knowledge for effective mathematics teaching. As highlighted by Mason and Davis (2013), it is "one thing to notice an absence of something from a learner, but quite another thing to have a sensible pedagogical action come to mind when needed" (p. 183). Knowing-to-act results in "being mathematical with and in front of the learner", a phrase used by Mason (*e.g.*, 2008, p. 307) to describe a teacher's sensitivity to opportunities to initiate action in ways that enable students to become aware of important aspects of the mathematics at hand. In this case, Mr. McLaren was willing to "be mathematical with and in front of" his students but he struggled to assemble and deploy, in the moment, a clear and thorough explanation. Here is where the notion of horizon content knowledge (HCK) is particularly valuable.

HCK is a category of mathematics knowledge for teaching introduced by Ball and her colleagues (*e.g.,* Ball & Bass, 2009; Ball, Thames & Phelps, 2008) that is concerned with "having a sense of the larger mathematical environment of the discipline being taught" (Jakobsen, Thames & Ribeiro, 2014, p. 4). Jakobsen and his colleagues, in collaboration with Ball and Bass, developed the following working definition of horizon knowledge.

> Horizon Content Knowledge (HCK) is an orientation to and familiarity with the discipline (or disciplines) that contribute to the teaching of the school subject at hand, providing teachers with a sense for how the content being taught is situated in and connected to the broader disciplinary territory. HCK includes explicit knowledge of the ways of and tools for knowing in the discipline, the kinds of knowledge and their warrants, and where ideas come from and how "truth" or validity is established. HCK also includes awareness of core disciplinary orientations and values, and of major structures of the discipline. HCK enables teachers to "hear" students, to make judgments about the importance of particular ideas or questions, and to treat the discipline with integrity, all resources for balancing the fundamental task of connecting learners to a vast and highly developed field (Jakobsen, Thames & Ribeiro, 2014, p. 4)

This expansive definition foregrounds the importance of how knowledge is held by teachers, aligning with Schwab's (1978) distinction between substantive knowledge (how skills and concepts are organised and connected within a discipline) and syntactic knowledge (the ways in which knowledge is generated within a specific discipline and how it is deemed valid or otherwise). The definition also alludes to the significance of contingency by highlighting that HCK enables teachers to 'hear' students and to "make judgments about the importance of particular ideas or questions" (Jakobsen, Thames & Ribeiro, p. 4). Contingency, in the work of Rowland and his colleagues, is concerned with the ways in which teachers respond to classroom events as they unfold. It refers to situations that are "almost impossible to plan for" but which instantly demand something of the teacher (Rowland, Huckstep & Thwaites, 2005, p. 263). Rowland and his associates highlight that students' ideas (*e.g.,* unsolicited questions or comments) are indicators of their meaning-making, and that the ways in which teachers engage (or not) with students' unexpected ideas have important implications for the meaning-making process.

Mr. McLaren's HCK was evident, to some extent, in his initial attempt to help the students make meaning of the relationship $\mathrm{Var}(aX+b) = a^2\mathrm{Var}(X)$ because he was aware of how to use a defined relationship to derive the result (*e.g.,* proceeding from definition through manipulation to conclusion), an aspect of his syntactic knowledge. His actions also reflect a willingness to both acknowledge and address an unexpected response from a student. To acknowledge and address is one of the three ways identified by Rowland and his colleagues (*e.g.,* Rowland, Thwaites & Jared, 2015) in which teachers might handle contingent moments in the classroom (the other two are ignoring the response, and acknowledging the response but putting it aside). Most contingent moments in classrooms arise because some unexpected idea or query has been triggered from the original planned activities, placing these moments on the periphery of the original focus. It seems plausible, then, that the nature of a teacher's horizon knowledge, and the ability to draw on that horizon knowledge 'in the moment', will play a significant role in the way that teachers respond to contingency in the classroom.

In the post-lesson interview, Mr. McLaren was able to deploy his HCK with greater substance and clarity by articulating rich connections among the concept of variance and the algebraic processes involved in its calculation. He deconstructed the mathematics, during his reflection-on-action, in ways that were not evident in the moment of teaching. What took place in the interview appeared to be an actual development of Mr. McLaren's horizon knowledge, as he connected the visualisation of a distribution with its algebraic properties. It might be argued that this horizon knowledge was not in Mr. McLaren's view or even accessible to him until he assembled the pieces himself from knowledge that he already held. Mason (2016) advocates that teachers must be consciously aware of the natural complexity of classroom activity and be sensitive to "noticing opportunities in-the-moment to act freshly rather than habitually" (p. 299). We saw that Mr. McLaren initially responded habitually (*e.g.,* "We could look at why but I'm not too fussed"); after all the Mathematics Methods syllabus [2] did not require students to verify or prove such mathematical relationships. In the end, however, he did take up the opportunity to 'act freshly' by attempting to unpack the derivation on-the-fly, and was later able to articulate a more complete and sophisticated explanation.

It is interesting to consider how the broader course context, particularly high stakes external assessment, shapes and influences the enactment and development of senior secondary mathematics teachers' knowledge at the mathematical horizon and the extent to which they are able to be mathematical "with and in front of the learner" in Mason's words. This discussion points to questions relating to how teachers perceive "being mathematical" in the senior secondary mathematics classroom and the extent to which the curriculum, and the broader course context, enhances or constrains the growth of teacher knowledge. If the mathematics course and its assessment underplay the role of mathematical justification, then should the onus be on the teacher to be accomplished and creative enough to do—or know to do—justice to the discipline of mathematics and the meaning of concepts? In this sense, Mr. McLaren acted courageously within a broader context that did not routinely prioritise the kind of mathematical justification in explaining the disappearance of the '*b*'. We might wonder what would happen in next year's iteration of the course, should another student ask, "What happened to the *b*?". Would Mr. McLaren be able to provide a 'better' answer, perhaps by developing his interview response in a pedagogically suitable way or by doing some personal work on his content knowledge in deriving the result algebraically?

Conclusion

We have examined a contingent situation in a senior secondary mathematics classroom that prompted the teacher to confront and reconsider his own thinking about specific mathematical ideas that were not directly addressed in the course but offered rich connections between the mathematical procedures and concepts at hand. The scenario contributes to the body of literature into the ways in which horizon knowledge can be fostered and developed, and the extent to which the broader curriculum and course context supports such growth. While it makes sense that the depth and scope of a teacher's horizon knowledge guides the construction of responses to contingent situations that arise in the classroom, crafting such responses in-the-moment may not be well-supported by the broader course context. Knowledge of what is typically included in, and excluded from, the examination appear to affect the mathematics-related knowledge that a teacher might need or be willing to develop. Such contextual factors may reduce the extent to which teachers are willing (or able) to be mathematical with learners. Nevertheless, teachers who are responsive to students' queries, and alert to opportunities made visible by their own horizon knowledge, can deepen their repertoire of responses in teaching situations that involve ideas from the senior secondary mathematics curriculum.

Notes

[1] There are some errors in what Mr. McLaren has written. The first term should be $E((aX+b)^2)$, and the expansion of this after the second equals sign should involve an additional term. However, these are not relevant to the points we want to make here.

[2] Office of Tasmanian Assessment, Standards, & Certification, Mathematics methods. Online at https://www.tasc.tas.gov.au/students/courses/mathematics/mtm415117-3

References

Ball, D. & Bass, H. (2009) With an eye on the mathematical horizon: knowing mathematics for teaching to learners' mathematical futures. In Neubrand, M. (Ed.) *Beitrage zum Mathematikunterricht 2009,* unpaginated. WTM-Verlag.

Ball, D., Thames, M. & Phelps, G. (2008) Content knowledge for teaching: what makes it special? *Journal of Teacher Education* **59**(5), 389–407.

Jakobsen, A., Thames, M.H., & Ribeiro, C.M. (2013) Delineating issues related to horizon content knowledge for mathematics teaching. In Ubuz, B., Haser, C. & Mariotti, M.A. (Eds.) *Proceedings of the Eighth Congress of the European Society for Research in Mathematics Education,* 3125–3134. Middle East Technical University.

Mason, J. (2008) PCK and beyond. In Sullivan, P. & Wood, T. (Eds.), *International Handbook of Mathematics Teacher Education: Knowledge and Beliefs in Mathematics Teaching and Teacher Development,* Vol. 1, 302–322. Sense Publishers.

Mason, J. (2016) Rising above a cause-and-effect stance in mathematics education research. *Journal of Mathematics Teacher Education* **19**(4), 297–300.

Mason, J. & Davis, B. (2013) The importance of teachers' mathematical awareness for in-the-moment pedagogy. *Canadian Journal of Science, Mathematics and Technology Education* **13**(2), 182–197.

Mason, J. & Spence, M. (1999) Beyond mere knowledge of mathematics: the importance of knowing-to act in the moment. *Educational Studies in Mathematics* **38**(1–3), 135–161.

Rowland, T., Huckstep, P. & Thwaites, A. (2005) Elementary teachers' mathematics subject knowledge: the knowledge quartet and the case of Naomi. *Journal of Mathematics Teacher Education* **8**(3), 255–281.

Rowland, T., Thwaites, A. & Jared, L. (2015) Triggers of contingency in mathematics teaching. *Research in Mathematics Education* **17**(2), 74–91.

It's all Chinese to me—and it makes a lot of sense: a letter to Dave and David

ANNA SFARD

Hi Dave, hello David,

I just read your interesting accounts of what happened in two elementary school classrooms in England in which fifth grade children were being introduced to fractions according to approach known as the 'Shanghai model' (in issue **41**(2)). Situated at a cultural crossroads, your story is a genuine tale of mystery. I am joining this conversation to add to your already quite extensive reflections on possible solutions.

Like you, I puzzled over what happened in the two classrooms. In both cases, I was surprised by what was considered by the teachers as correct responses to the question of whether the shaded rectangles in the drawings matched the given fractions (see Figures 1 & 2). I was taken aback by the fact that even when the shaded part appeared to be describable by the fraction ¼ or ³/₅, the teachers would sometimes express the opposite opinion, pointing out that the big rectangle was not partitioned into identical segments.

I was baffled also by the teachers' initial resistance to your explanations, Dave. And then, after rereading your story several times, I started questioning our shared assessment of the teachers' actions. The first doubt crept in when I read your assertion that the novice and expert teachers in the scene you had watched demonstrated "unquestioning acceptance of [the] received lesson plan", which came "with the apparent authority of being based on a Shanghai approach."

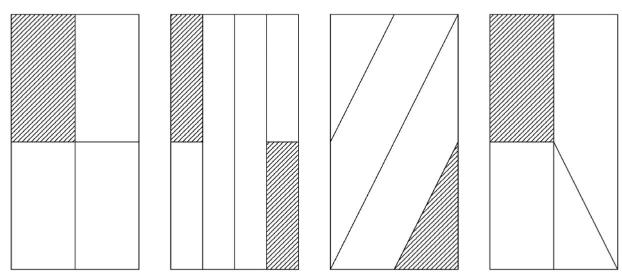

Figure 1. Drawings discussed by Dave, in connection with ¼.

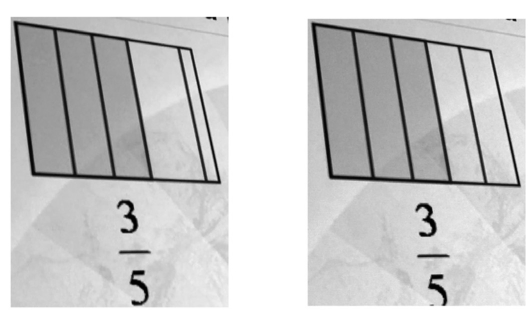

Figure 2. Drawings used in Ms Dai's lesson, in connection with ³/₅.

And, my scepticism intensified when you, David, talked about the Chinese teacher, Ms Dai, who did a similar thing while speaking to a class in England about 3/5. Somehow, I found it hard to believe that these teachers' deference to authority was strong enough to silence their own logic. Ms Dai in particular was an exemplary educator, appreciated enough to be sent to England as a presenter of 'showcase' lessons. If she were a mere rule-follower, what would it imply about the masses of her less outstanding colleagues? And, if ritualistic rule-following is so highly appreciated in China, how do we explain China's high rankings both in TIMSS and PISA?

Listening to Ms Dai and watching her actions [1] I wondered whether it would be possible to work out an alternative, hopefully more charitable, interpretation of this teacher's actions. I assumed that if I only silenced my own spontaneous understanding of mathematical words and symbols, I might, perhaps, find logic where there seemed to be none. In the present case, this meant renouncing my usual ways of describing the familiar situation and trying to retell the story of the Chinese teacher's mathematising within her own mathematical discourse. The odds were that the 'native' lens would glue the apparently ill-fitting pieces together, combining them into a logical whole (I think, David, that this is exactly what you were trying to do while analyzing the structure of Chinese names for fractions, right?). Yet, undertaking this task meant venturing into a language I do not speak and into a culture about which I know close to nothing. It seemed as an unlikely undertaking, but I was determined to try, if only to escape the talk of deficit.

I began by asking myself the question I usually try to tackle when unable to make sense of what a person is doing: Are the participants, in this case the teachers and the students, really trying to accomplish the task I think they are performing? Or, in our case, when asking the question "True or false?", what was the task Ms Dai might have had in mind? Was it possible that she was not an obedient follower of somebody else's instructions but rather a person whose actions were rational and made genuine sense to her? I began my search for an alternative interpretation by scrutinising the verbatim version of what Ms Dai and her students said. Here are the notes I made for myself while watching the first few moments of the relevant segment of the lesson:

> Ms Dai: 'At first, let us review'. She presses on her clicker and three shapes with a fraction beneath each one of them appear on the screen; Figure 2a is considered first. Ms Dai continues: "True or false?… I will give you several seconds.' After a 2-second pause, during which some children start raising their hands, she exclaims, 'Ready? The first one… Go!'"

Later, I found it remarkable that in spite of the brevity of Ms Dai's question, "True or false?", her students were evidently able to understand her intention in a blink. But why was their shared understanding obviously so different from our own?

You made your interpretation of Ms Dai's question explicit when you completed her utterance with the bracketed words: "True or false? *(Can these fractions show the coloured parts?)*". If I wanted to find a reinterpretation that would breathe some sense into the participants' actions, I had to read Ms Dai's question in a different way. Could Ms Dai's query be about the truth of a different statement?

Like you, David, I felt that the key to such alternative reading could be found in some disparities between Chinese and English. And it did not matter that in this case, the classroom interaction was in English only. The important difference is between the English and Chinese speakers' respective mathematical *discourses* and, in particular, that they use the same symbols in not necessarily identical ways. An 'aha' experience came when in my web search I came across a video in which the presenter mentioned in passing that the Chinese character 分, to be read as *fēn* and appearing in the name of every fraction, may signify both a *thing* and an *action* [2]. GoogleTranslate confirmed this claim, saying that, indeed, 分 may serve as a noun corresponding to the English *piece, part,* or *section,* and it can be used as the verb: *divide, separate, partition.* If so, is it possible, I asked myself, that the symbol 3/5 can be interpreted as referring not just to a product of an action but also to the action as such? This would be not unlike the kind of process-product duality of the arithmetic and algebraic notation, where expressions such as $3 + 7$ or x^2 can be read as referring both to an operation and to this operation's outcome. This discovery seemed like a beginning of something, especially considering your remark, David, that Chinese fraction names are reflective of the order of operations one performs to find a part of the whole expressed in a given fraction. I now decided to take this comment even further by hypothesising that ideographs such as 1/4 or 3/5 may be read in Chinese as descriptions of a certain *routine operation.* Yes, I thought, perhaps in Chinese, a fraction may be read not just as a *thing* (a part of something), but also as a story of what was *done* to produce this part? Could, indeed, a fraction be also a coded prescription for a routine?

This sounded strange, but I was sufficiently intrigued to email Jinfa Cai, who is a mathematics education professor and a native Chinese speaker, asking him about "the ways in which the expression 三分之二 (sān fēn zhī èr, which is the Chinese name for 2/3) may be interpreted." And I continued, "Can it be read as a brief narrative, say, 'A thing has been partitioned into three parts of which two have been taken', with the word *partition*, 分, interpreted as the action of dividing into equal parts"? The response arrived promptly, confirming this conjecture. "Based on my understanding, 2/3 is both a noun and a verb", Jinfa was telling me. And he continued, "2/3 is a noun as it is '*a result* of division, signifying a part of the whole or a number', exactly as 'two thirds' is interpreted in English; 2/3 is a verb as it implies '*to* partition or *to* divide." This meant that the story the fraction 3/5 could be telling, rightly or wrongly, about the drawings in Figure 2, might be "A thing has been partitioned into five congruent parts of which three have been shaded". If this latter story was the object of Ms Dai's question, then the students were right when calling this narrative false! With this new interpretation, things were falling into place and the sensibility of the teachers' decisions, as well as their honour, could be saved. Thank you for helping me to see this possibility, Jinfa!

Of course, this new interpretation was highly speculative, but I realized that I had a strong argument in its favor. Introducing fractions as stories of things done rather than of products obtained resonated with my discursive ('commognitive') view of the development of mathematical discourses. According to this approach, every mathematical object, including those to be eventually signified by fractions and called *rational numbers,* emerges in the discursive act of reifying a process. Before a new object is conceived, there must be a certain routine on already existing mathematical objects, in this case integers. Dividing a thing into congruent parts and taking a number of them is, indeed, a procedure that can be described and performed without a reference to fractions. As the learner becomes proficient in the routine and its multifarious applications, she may 'compress' her discourse by reification, that is, by replacing verbal phrases with nominal phrases. For this, she has to start using the symbols that served her so far as a shorthand for the routine—the fraction, in this case—also as a name of this routine's product.

If so, the 'operational' way of seeing fractions, possibly reflected in Ms Dai's question, appears first and may be the only one accessible to beginners. The advantage of the idea of fraction-as-an-action is that unlike that of fraction-as-number, it can be developed within the discourse of integers, or even just natural numbers, in the context of practices with which children are familiar. And it can be done even when something like "this part constitutes $^3/_5$ of the pizza" is still beyond the reach of learners. For instance, an operation such as the one through which the drawing in Figure 2b was produced might constitute a solution to the practical problem of sharing a rectangular pizza between a family of two and a family of three. The idea of fraction-as-an-action is thus an ideal gangway from the child's old discourse of natural numbers to the new one, that of rational numbers. And it is, clearly, advisable if one wishes to ensure that the required transition happens in a meaningful way. In the light of this interpretation, what you witnessed in the two classrooms, Dave and David, does not appear strange any longer. On the contrary, it seems natural. Accordingly, the Chinese teachers, rather than being criticised, should probably be applauded for what they were doing.

This is the end of my story. Well, not really. I may have sounded excited by my take on your puzzle, but I would not like you to think that I am certain of my solution. I have offered it merely as a conjecture that has yet to undergo critical scrutiny of people well acquainted with particularities of both the Chinese language and Chinese pedagogical approaches. But even if my hypotheses prove untenable, I will not regret joining this debate. This collective process forced me to revisit the old question of why it is so difficult to break out from our own ways of thinking. It is our well-developed mathematical discourse that I am inclined to blame in the present context. "The limits of my language means the limits of my world" said Wittgenstein, and this example may serve as evidence. Yes, we are captives of our own ways of talking, with our words functioning like Trojan horses that carry with them armies of hidden assumptions and thwart the very possibility of escape. For instance, while reporting on the lesson like the one you saw, Dave, it was natural for us to say that, "there was a strong emphasis on the need for dividing shapes into *equal* regions" or that, "the students were asked to say whether each of the drawings in Figure 1 represented a quarter of the rectangle *area*." But the words 'equal regions' and 'quarter of the rectangle area', indicate the assumption that the young learners already know what area is. Was it really their familiarity with the idea of area that was supposed to make them able to perform such tasks? The inner logic of the discourse of areas suggests that the reverse is more plausible—that it is only by comparing the numbers of congruent parts into which one can partition different shapes that learners *begin* their travel toward the mathematical idea of area. Eventually, the number of parts obtained by partitioning a shape into elements identical with the standard one called 'unit' will be called the 'area' of this shape. Could this concept be brought into being in any other way?

Let me finish with a summarising reflection. What we collectively did in this conversation may serve as a constant reminder that most of what can be seen from where we have been taken in our decades-long mathematical journey could not be seen from where this travel began. The interpretation I proposed here is the product of an attempt to return to the point of departure. Thank you for inviting me to this exciting time-travel.

Anna

Notes

[1] Online at https://www.ncetm.org.uk/classroom-resources/lv-year-5-shanghai-showcase-lesson/
[2] Online at https://www.youtube.com/watch?v=NSal_TlRpxQ

In the telephone keypad the numbers 1–9 are arranged so that the numbers in each row and column are in increasing order (read from top to bottom and left to right). How many such arrangements are there?

1	2	3
4	5	6
7	8	9