Maybe a Mathematics Teacher can Profit from the Study of the History of Mathematics

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Introduction

"Never in the history of mathematics has so much been owed by so few to so many" (with apologies to Churchill). *So many* advance the case for the history of mathematics and its value in mathematical education for students and especially for mathematics teachers Through the 20th century, reports [MAA, 1935; Ministry of Education, UK, 1958; among others] and single authors [for example, Barwell, 1913; Jones, 1969; Shevchenko, 1975; Grattan-Guinness, 1978; Rogers, 1980; Struik, 1980] suggest *so much* possible positive influence on attitudes, understanding, etc

In general discussions of the sort mentioned, as in general discussions of almost every educational "case", the influence of whatever one wants to talk about (in our case, history of mathematics), depends not only upon the subject matter itself but upon many other variables rarely explicitly taken into account.*

The case for the history of mathematics has to be carefully stated within a particular context in terms of variables such as: target audience, their mathematical background, the sources, the approach, the topics etc. If, in another context, the variables are not identical, and they never are, the case has to be restated with possible modifications.

Freudenthal [1981] explicitly mentions some of these variables. Our purpose is to refer to some aspects of his argument, in the process of describing an experience of "doing" history of mathematics with in-service teachers. The teachers (as the target audience) are our first variable: they were, in the main, junior high school teachers whose mathematical background barely extends beyond the curriculum which they are expected to teach.

Freudenthal makes the following statements:

1 *History is worth being studied at the source rather than by reading and copying what others have read and copied before.*
2 *The most appropriate way to learn and teach the history of mathematics is through seminars rather than courses.*
3 ... *I stress the history of science as integrated knowledge rather than items stored in well-stocked drawers, each of them labelled and opened when the time-table announces the history of the subject matter.*
4. *The argument closest to hand – and the most often heard – is that knowledge of the history of a subject area helps in understanding the subject matter itself. I doubt it – at last (sic) as far as mathematics is concerned.*

We took the first two of Freudenthal's statements as axiomatic in our particular situation; i e , as two consciously-chosen variables ("Seminar" may have different connotations: in our context the seminar took the form of a workshop in which the participants were active and "worked" most of the time, as will be further described later)

The third statement corresponds in our context to the following. The approach was not chronological, nor biographical (about mathematicians) as such; our purpose was to create a picture of the development of a topic of relevance to the teacher, and by the way, into that picture, enter mathematicians, dates, etc., that are subsidiary to the study of the mathematical story. This was our conclusion on considering a further variable: the objective of the whole exercise. We do not, at this stage, or perhaps ever, expect the teachers to go away and teach the history of mathematics, but what we do expect will be part of the following description.

This article is intended to take issue with the fourth of Freudenthal's statements There is no doubt that this particular statement is seriously dependent upon the variables, and we will describe a situation in which the "doubt" was shown to be unfounded. This cannot be done by arguing in generalities, but only by describing a course, its materials and the outcomes The value of such a description lies in that others who may be interested can then develop and plan their own courses suited to their chosen and enforced variables.

The course and its material

The material for a two-day teacher workshop on *the conceptual development of negative numbers* was created as a series of worksheets whose general form was as follows.

- A *brief* biographical-chronological introduction in order to set the historical scene
- An historical source; as far as possible a primary source.
- Leading questions on the source material and on mathematical consequences thereof.

* We exclude controlled studies such as McBride & Rollins [1977] and Delaney [1980] that confirm some limited aspects of the general claims. These studies, and other unreported courses, which certainly exist, represent "the few". How few, can perhaps be judged from Johnson and Byars [1977], who report that, on average, of a total of more than 30 semester hours devoted to mathematics in pre-service teacher training courses, a mere 0.23 hours was devoted to the history of mathematics

To each worksheet an extensive discussion/solution leaflet was prepared, containing detailed written solutions to the questions and further source material and background, as appropriate to complete the historical-conceptual information

The series of worksheets consists of an introductory sheet, seven worksheets and a summary sheet The relatively short time available for the workshop, and the relatively time-consuming form of activity (two further variables), implied a very careful selection of material

In particular, since the early history of the negative numbers and the original sources are fairly sparse, we chose to introduce the subject using a secondary source [Bell, 1940], which compactly summarizes the history of negative numbers The specific purpose of the *introductory sheet* is twofold:

- to provide the teachers, who knew almost nothing about the topic (see pretest results later) with some orientation; that is, a framework in which to work, and
- to fill in some of the stages of the development of negative numbers not treated in the worksheets

Time was given to read the sheet and then it was discussed. The seven worksheets which then followed are described briefly below The text of one of them (the sixth) and its accompanying discussion/solution leaflet is given, as an example, in the appendix.

1) *F. Viète (1540-1603)*, for an example of
   i) subtraction and the law of signs, but only in those cases in which all the expressions involved are positive, and
   ii) the complete non-recognition of negative quantities.

The primary source on which this worksheet was based, was taken from *In Artem Analyticem Isagoge* (1591) as quoted (in translation) by Struik [1969] Because of the difficulty of reading source material in general, and the unfamiliar language of the older primary sources in particular, the first questions on this and other worksheets were usually devoted to "compelling" the teacher to read In this case he was asked to translate the mathematics of Viète's text into modern symbolism Here, as a byproduct, the teacher is encouraged to practice reading mathematics in a situation which is partly in his favour: the language is unfamiliar but the content, when "translated", is familiar. Thus a possible bonus, which we have not yet seriously evaluated, maybe the enhancing of the teacher's mathematical literacy

2) *Contradictions that arose in the use of negative numbers (17^{th} century).*
In this case we did not bring primary sources but quoted two famous contradictions from Cantor [1901]. One is attributed to Arnauld (1612-1694) among others:

From the relation $-1/1 = 1/-1$ there is an apparent contradiction with the concept of ratio It is "unreasonable" that the ratio of a smaller number to a larger number (−1 to 1) is the same as the ratio of a larger number to a smaller (1 to −1)

The second contradiction, attributed to Wallis (1616-1703) among others, is as follows:
If we extend the sequence of inequalities

$$< 1/3 < 1/2 < 1/1$$

to the left, we have

$$< 1/3 < 1/2 < 1/1 < 1/0 < 1/-1$$

from which we "conclude" that the negatives, which are less than zero are also greater than infinity

The teachers are asked to explain these paradoxes

To answer the Wallis paradox, the discussion/solution leaflet includes a section from Euler's *Vollstandige Anleitung zur Differenzial=Rechnung* (a German translation of the Latin original published in 1790), in which Euler (1707-1783) refutes the argument by considering the sequence

$$\ldots 1/9 , 1/4 , 1/1 , 1/0 , 1/1 , 1/4 , 1/9 \ldots$$

whose general term is $1/n^2$ and from this "surely no one would maintain that they (terms to the right of 1/0) are greater than infinity".*

Is is interesting to note that Euler did no more than repulse the suggestion that the negative numbers were greater than infinity by showing that the same argument led to an absurd conclusion. The teachers in the workshop experienced considerable difficulty in arguing against Arnauld and Wallis, and in discussion we began to sense the main objective source of this difficulty, i.e. the absence of a definition of negative numbers. This turned out to be a continuing theme until, by the time we reached sheet 7, the teachers felt that the definition was begging to be introduced

3. *N. Saunderson (1682-1739)*, for an example of didactics in the 18^{th} century.
Saunderson was essentially a university teacher. We chose a paragraph dealing with multiplication of negative numbers from his posthumously published book *The Elements of Algebra* [1741]. His didactical strategy is what we would regard today as the extension of patterns observed for positive numbers to the negative numbers.

Interesting issues occur, and one of the questions we met was the distinction between definitions and assumptions, which are not distinguished in the text. This led to a discussion of the problem of proof in the absence of the basic definition.

4 *L. Euler*, for another didactical example. We chose a section from the English version of his *Elements of Algebra* (1828, Fourth Edition). Among other things, the worksheet asks the teacher to compare and contrast Euler's approach with Saunderson's, both of them belonging to the period when negative numbers were freely used, but before they had been defined.

Euler's "proof" that the product of two negative numbers is positive is typical of the time:

* Cf. Kline [1972, p. 593] as an example of the unreliability of secondary (tertiary?) sources.

> 33. It remains to resolve the case in which – is multiplied by –; or, for example, – a by – b. It is evident, at first sight, with regard to the letters, that the product will be ab; but it is doubtful whether the sign +, or the sign –, is to be placed before it; all we know is, that is must be one or the other of these signs. Now, I say that it cannot be the sign –; for – a by + b gives – ab, and – a by – b cannot produce the same result as – a by + b; but must produce a contrary result, that is to say, + ab; consequently, we have the following rule: – multiplied by – produces +, that is, the same as + multiplied by +.

The discussion/solution leaflet includes D.E. Smith's [1900] devastating comment (not directed at Euler):

> . These things are easily explained, but the text-book "proofs" of the last generation have now been discarded. The favorite one of these "proofs" was this: Since multiplying – b by a gives – ab, therefore if the sign of the multiplier is changed, of course the sign of the product must also be changed. As a proof, it is like saying that if A, a white man, wears black shoes, therefore it follows that B, a black man, must wear shoes of an opposite color.

5. *W. Frend (1757-1841) and his opposition to negative numbers*:
This worksheet was chosen to represent the diminishing, but still present, opposition to the total use of negative numbers at the beginning of the 19th century. Frend was essentially a teacher, nevertheless, he objects in his book *The Principles of Algebra* (1796), strongly to the use of non-mathematical examples to justify negative numbers:

> ... The first error in teaching the principles of algebra is obvious on perusing a few pages only in the first part of Maclaurin's Algebra. Numbers are there divided into two sorts, positive and negative; and an attempt is made to explain the nature of negative numbers, by allusions to book-debts and other arts. Now, when a person cannot explain the principles of a science without reference to metaphor, the probability is, that he has never thought accurately upon the subject...

The teachers were asked to answer Frend on this point and others concerning his opposition to negatives. We also quoted a brief section of his treatment of the quadratic equation, in which he has to divide the subject into a number of cases because of the rejection of negative roots. We asked the teacher to explain his approach and comment on possible advantages or disadvantages.

The discussion/solution leaflet includes quotations from two considerably more famous mathematicians, which can be used as a basis of a reply to Frend. The first is taken from De Morgan's *A Budget of Paradoxes* (1872). The second is from a much later period: Whitehead's elegant discussion in his *Introduction to Mathematics* (n.d.). Of course, it is not expected of the teacher to argue at this level, but once he has arrived at his own attempt at an answer, the comparison with the answers given by the "masters" has a great deal more meaning.

6. *G. Peacock (1791-1858), an attempt to formalise.*
Peacock begins a period which leads up to the formal mathematical definition, and his work was also in response to the rejectionists. He, apparently, was the first to formulate the *principle of permanence of equivalent forms* (a significant didactical strategy to this day), which formed the basis of his extension from arithmetical algebra to symbolical algebra, the latter being, among other things, the discussion of the extension of arithmetic to negative numbers. The worksheet is based on extracts from his *Arithmetical Algebra* (1842) and *Symbolical Algebra* (1845). The approach is extremely close to some of the modern school treatments of the subject: an interesting discussion is found in Klein (n.d.) and we bring it in the discussion/solution leaflet. This worksheet and its discussion/solution leaflet are given in full in the appendix to this article.

7. *Formal entrance of the negative numbers to mathematics*
This worksheet was not based on historical material but developed, by means of a carefully structured sequence of leading questions, a definition of whole numbers as ordered pairs of natural numbers. The mathematics here is quite formal, except for occasional didactical and historical asides. Without doubt the formalism achieved in this sheet is appropriate to a university pure mathematics course. The teachers' response was extremely interesting: because of the historical preamble and the explicit discussion of the lacunae in previous approaches and attempts to define negative numbers (and the corresponding binary operations), they found this final sheet not only easily within their ability, but much to their liking. There was a clear appreciation of the necessity and role of definition.

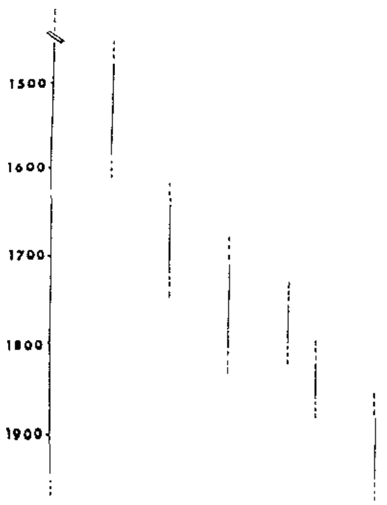
The *final sheet* is intended both as a summary and a warning against the uncritical use of secondary sources. We quote from Kline [1972, pp. 592-3] and ask the teacher to express his opinion and criticism of Kline's summary of the state of the knowledge of negative numbers in the 18th and the beginning of 19th centuries. Finally, in order to summarize all the worksheets and to get a general picture the following question was asked.

In these worksheets we have come across the names of various major and minor mathematicians who have been connected with the history of the negative numbers. Further, we have learnt about the approach and views on this concept. In the following list you will find these mathematicians and a suggested division into stages of the development of the concept.

Viète	Non recognition of negative numbers
Wallis Arnauld	The use of negative with reservations regarding apparent contradictions arising from their use.
Sanderson Euler	The use of negative numbers and their entry into textbooks without mathematical definition.

Frend Masères	Opposition to negative numbers
Peacock de Morgan	An attempt to provide a mathematical basis for negative numbers.
	Formalization: pure mathematics

Mark these different stages on the chronological table provided and attribute to each stage the relevant mathematicians.



The workshop and some results

The workshop was guided by two tutors. Teachers worked in groups or individually, as they pleased. The tutors "interfered" with supplementary questions to individuals or groups according to their progress – or lack of it. They also responded to questions – which were frequent. Generally, after each worksheet, a collective guided discussion of the solutions took place, with appropriate sign-posting of whence we had come and where we intended to go. After the discussion, the prepared solution leaflet was distributed, both for the further information it contained and to provide a record of the activity.

The series of worksheets, the verbal summaries and discussions and the solution leaflets together built the history of the concept, its difficulties, blind alleys, different approaches etc. As we have seen above, attention was also paid to didactical implications and logical concepts such as definitions, assumptions, proofs etc.

About 50 teachers attended the two day workshop. In order not to rely entirely on our subjective evaluation of the workshop, we also had two questionnaires, one at the beginning and one at the end.

The questionnaires were designed to provide information about:

a. previous knowledge of the subject;

b. importance teachers attached to the history of mathematics for themselves and for their students, before and after the workshop;

c. views of the usefulness of the workshop.

An analysis of the responses to the two questionnaires gives the following information:

a. most (about 80%) knew nothing beforehand about the development of the negative number concept. Further, most of them were of the opinion that the mathematical definition preceded the free use of negative numbers. We took this as a possible indicator of the improper picture which many mathematics teachers have of the subject and the nature of mathematical activity.

b. There was significant increase in the importance attached to the history of mathematics after the workshop compared with that before.

c. When asked to chose possible areas in which historical mathematical topics might contribute the following were indicated by a large majority of the participants:
   - to present mathematics as a developing and dynamic subjects;
   - to enlarge and deepen the understanding of certain topics;
   - to increase student interest;
   - to help the teacher better understand errors and misconceptions in certain topics.

   In response to the open question:

What did you learn in the last two days from the point of view of (a) history, (b) didactics and (c) mathematics? the following were among the responses:

History
- I knew about the development from the naturals to the whole numbers, but I never imagined that there had been arguments and battles even when the necessity for extension had been recognized.
- A fascinating illustration of the fact that in many cases the use of concepts far preceded the possibility of human thought to define those concepts in a correct way.

Didactics
- I received support for the view that no student answer should be dismissed, but one should relate to them all, think about them and discover their rationale.
- It became clear to me that one should distinguish between didactics of mathematics and pure mathematics.

Mathematics
- The definition of negative numbers.
- The importance of definitions, proofs and mathematical structure.
- It seems to me that my understanding of the subject of negative numbers has been extended.

Epilogue

We hope that in the above we have shown, in the first place, that Freudenthal's fourth statement in which he doubts that *knowledge of the history of a subject area helps in understanding the subject matter itself* can itself be questioned.

Both the "objective" evaluation, and the subjective evidence of our own observation, show quite clearly that the participants in the workshop approached the seventh worksheet with an appreciation of the logical and mathematical necessity for a proper definition of the negative numbers Moreover, they were in a favourable state of mind to develop and understand the pure definition, and had the ability to develop this for themselves in response to leading questions There is no doubt that the definition of negative numbers can be taught to teachers without the historical preamble, but there is equally no doubt from our observation, that the experience is much more meaningful when preceded by some of the historical attempts, failures, controversies and battles. Further the historical work has more general byproducts. It changed the participants'conception of mathematical activity: it removed the aura of something absolute and divine. It also provided a suitable and interesting context in which to discuss the didactics of the concept, both formally and informally. We are convinced that almost all the participants would sympathise with Whitehead:

> *The idea of positive and negative numbers has been practically the most successful of mathematical subtleties*

True, as we said at the beginning, it all depends on the chosen and enforced variables The history of mathematics in neither useful or useless, relevant or irrelevant to anything or something: it all depends on what you mean by the term, and how you put it into practice.

Appendix

Worksheet 6: GEORGE PEACOCK

The period in which the English mathematician George Peacock (1791-1858) was active was preceded by relatively low scientific activity in general, and in mathematics in particular. After the death of Isaac Newton (1642-1727), the English mathematicians distanced themselves from other European mathematicians, in part because of the dispute between Newton and Leibniz (1646-1716) concerning precedence in the invention of the infinitesimal calculus. This chauvinism on the part of the English mathematicians, caused them to pay insufficient attention to mathematical advances on the continent. In addition, there were at that time men, like Frend and Masères, who wrote books against the use of negative numbers.

Peacock, together with others, began to work in order to "revive" English mathematics, and in particular, to put algebra on a sound scientific basis. To this end he wrote two books, *Arithmetical Algebra* (1842) and *Symbolical Algebra* (1845). In the distinction between these two types of algebra can be found Peacock's response to the arguments of Masères and Frend.

Below we bring sections from these books, in which Peacock explains the difference between arithmetical algebra and symbolical algebra and presents his *Principle of Permanence of Equivalent Forms*, by which he connects these two types of algebra.

About Peacock and his contribution to mathematics one can find further details in* A. Macfarlane, *Lectures on Ten British Mathematicians of the Nineteenth Century*. Wiley, 1916.

> In arithmetical algebra, we consider symbols as representing numbers, and the operations to which they are submitted as included in the same definitions (whether expressed or understood) as in common arithmetic: the signs + and – denote the operations of addition and subtraction in their ordinary meaning only, and those operations are considered as impossible in all cases where the symbols subjected to them possess values which would render them so... thus in expressions .. like $a - b$, we must suppose a greater that b ... all results whatsoever, including negative quantities, which are not strictly deducible as legitimate conclusions from the definitions of the several operations, must be rejected as impossible, or as foreign to the science ...
>
> Symbolical algebra adopts the rules of arithmetical algebra, but removes altogether their restrictions: thus symbolical subtraction differs from the same operation in arithmetical algebra in being possible for all relations of value of the symbols or expressions ..
>
> It is this adoption of the rules of the operations of arithmetical algebra as the rules for performing the operations which *bear the same names* in symbolical algebra, which secures the absolute identity of the results in the two sciences as far as they exist in common; or in other words, all the results of arithmetical algebra which are deduced by the application of its rules, and which are general in form, though particular in value, are results likewise of symbolical algebra, where they are general in value as well as in form: thus the product of a^m and a^n, which is a^{m+n} when m and n are whole numbers, and therefore general in form though particular in value, will be their product likewise when m and n are general in value as well as in form.
>
> This principle, in my former Treatise on Algebra, I denominated the "*principle of the permanence of equivalent forms*," and it may be considered as merely expressing the general law of transition from the results of arithmetical to those of symbolical algebra...
>
> Upon this view of the principles of symbolical algebra, it will follow that its operations are determined by the definitions of arithmetical algebra, as far as they proceed in common, and by the "principle of the permanence of equivalent forms" in all other cases...
>
> The results therefore of symbolical algebra, which are not common to arithmetical algebra, are

* Since we wrote these worksheets, an extensive paper on Peacock has appeared: Pycior H M., George Peacock and the British Origins of Symbolical Algebra *Historia Mathematica*, 8, 1981 pp 23-45

generalizations of form, and not necessary consequences of the definitions, which are totally inapplicable in such cases. It is quite true indeed that writers on algebra have not hitherto remarked the character of the transition from one class of results to the other, and have treated them both as equally consequences of the fundamental definitions of arithmetic or arithmetical algebra: and we are consequently presented with forms of demonstration, which though really applicable to specific values of the symbols only, are tacitly extended to all values whatsoever...

The definition of a *power*, in Arithmetical Algebra, implies that its index is a whole number: and if this condition be not fulfilled, the definition has no meaning, and therefore no conclusions are deducible from it: the principles, however, of Symbolical Algebra, will enable us, not merely to recognise the existence of such powers, but likewise to give, in many instances, a consistent interpretation of their meaning

Questions:

1) What is the difference between the expression $a - b$ in arithmetical algebra and in symbolical algebra?
2) What is the difference between what Peacock calls arithmetic and what we call arithmetic today?
3) In view of Peacock's definition of arithmetical algebra and symbolical algebra and the *Principle of Permanence of Equivalent Forms*, discuss:

 i) We know that $a^m \cdot a^n = a^{m+n}$ (m, n natural numbers)
 Using the *Principle of Permanence of Equivalent Forms*, substitute $n = 0$ to obtain:
 $$a^m \cdot a^0 = a^{m+0} = a^m,$$
 whence $a^m \cdot a^0 = a^m$,
 therefore $a^0 = 1$.

 ii) We know that
 $$(a - b) \cdot (c - d) = a \cdot c - b \cdot c - a \cdot d + b \cdot d$$
 when $c > d > 0$, $a > b > 0$
 Using the *Principle of Permanence*, substitute $a = 0$ and $b = 0$ to obtain $(-b) \cdot (-d) = + b \cdot d$.

 iii) If a and b represent natural numbers, the distance of $a + b$ from 0 on the number line, is equal to the distance of a from 0 plus the distance of b from 0
 Using the *Principle of Permanence*, we conclude that for every a and b, the distance of $a + b$ from 0 is equal to the distance of a from 0 plus the distance of b from 0

 iv) $$a > b > 0,\ c > 0 \rightarrow a \cdot c > b \cdot c$$
 Using the *Principle of Permanence*, we conclude that for every a, b and c
 $$a > b \rightarrow a \cdot c > b \cdot c.$$

 v) For $a > b > 0$
 $$(a^2 - b^2)/(a - b) = a + b.$$
 Using the *Principle of Permanence*, we conclude that for every a and b
 $$(a^2 - b^2)/(a - b) = a + b.$$
 Substitute $a = b = 1/2$ whence $0/0 = 1$

 vi) $$a^m = \underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{m \text{ times}}$$

 Using the *Principle of Permanence*, substitute $a = 0$, whence $0^m = 0$.
 Using the *Principle of Permanence* again, substitute $m = 0$, whence $0^0 = 0$.

4) (This question asked the teacher whether the didactical approach in a local text could be considered as a conceptual inheritance from Peacock.)

Worksheet 6: Solutions and discussion

1) Peacock in his book *Symbolical Algebra* explains this as follows:

> .. in the expression $a - b$, if we are authorized to assume a to be either *greater* or *less* than b, we may replace a by the equivalent expression $b + c$ in one case, and by $b - c$ in the other: in the first case, we get $a - b = b + c - b = b - b + c$ (Art. 22), $= 0 + c$ (Art 16) $= + c = c$: and in the second $a - b = b - c - b = b - b - c = 0 - c = - c$. The first result is recognized in Arithmetical Algebra (Art. 23): *but there is no result in Arithmetical Algebra which corresponds to the second*: inasmuch as it is assumed that no operation can be performed and therefore no result can be obtained, when a is less than b, in the expression $a - b$ (Art 13) *
>
> 545 Symbols, preceded by the signs + or –, without any connection with other symbols, are called *positive* and *negative* (Art. 32) symbols, or *positive* and *negative* quantities: such symbols are also said to be *affected* with the signs + and –. *Positive* symbols and the numbers which they represent, form the subjects of the operations both of Arithmetical and Symbolical Algebra: but *negative* symbols, whatever be the nature of the quantities which the *unaffected* symbols represent, belong exclusively to the province of Symbolical Algebra.

2) Peacock, for example, does not regard the four operations with negative numbers as arithmetic, since the negative quantity does not have any place whatsoever in arithmetic, as we have seen in the previous question. And this is a general approach among many mathematicians in the 19th century. On the other hand, nowadays, operations with negative numbers are usually included in arithmetic

* If we assume symbols to be capable of all values, from *zero* upwards, we may likewise include *zero* in their number: upon this assumption, the expressions $a + b$ and $a - b$ will become $0 + b$ and $0 - b$, or $+ b$ and $- b$, or b and $- b$ respectively, when a becomes equal to *zero*: this is another mode of deriving the conclusion in the text

3) In each of the following sections we have made use of the *Principle of Permanence of Equivalent Forms*. What we have to investigate is whether this use is legitimate or leads us to contradictions.

i) As a comment on this section we quote from S.I. Brown, Signed Numbers: a "Product" of Misconceptions. *The Mathematics Teacher*, 1969, 62, pp. 183-195.

The preservation principle is that if we wish to extend a concept in mathematics beyond its original definition then that candidate ought to be chosen which leaves us many principles of the old system intact as possible. The fact that these principles are left intact does not at all prove that a definition for elements of an extended set is correct. It merely motivates us to make a particular definition.

Suppose we define a^n for n a natural number to be

$$a \cdot a \cdot \ldots \cdot a,$$

where a appears n times as a factor. For m, n, belonging to the set of natural numbers, it is easy to prove that

$$a^m \cdot a^n = a^{m+n}.$$

Now what is a^0? It obviously has no meaning in terms of the definition of a^n, since it is meaningless to select the number a zero times as a factor. "Proofs" are often offered ...

The difficulty with the above "proof" obviously lies with the assertion that

$$a^0 \cdot a^m = a^{0+m}$$

Why should we assume that zero behaves (as an exponent) the same way the natural numbers do? The answer is that we cannot, since the assertion

$$a^n \cdot a^m = a^{n+m}$$

was proven only for natural numbers. There would be nothing inconsistent about our system if we maintained the above equality only for natural numbers and stipulated that it failed in the case of $n = 0$. We could then define a^0 in any way we wished and still have a consistent system. If, however, we wish the principle for multiplication to be preserved in the *new* system (which includes 0), then we are forced to define a^0 to be 1 as the "proof" suggests.

ii) As a comment on this section, we quote from F. Klein, *Elementary Mathematics From an Advanced Standpoint. Arithmetic, Algebra, Analysis*. Dover Publications, n.d., p. 23-28.

...If we now look critically at the way in which negative numbers are presented in the schools, we find frequently the error of trying to prove the logical necessity of the rule of signs, corresponding to the above noted efforts of the older mathematicians. One is to derive $(-b)\ (-d) = +\ bd$ heuristically, from the formula $(a - b)\ (c - d)$ and to think that one has a proof, completely ignoring the fact that the validity of this formula depends on the inequalities $a > b$, $c > d$. Thus the proof is fraudulent, and the psychological consideration which would lead us to the rule by way of the principle of permanence is lost in favor of quasi-logical considerations.

Of course the pupil, to whom it is thus presented for the first time, cannot possibly comprehend it, but in the end he must nevertheless believe it ...

In opposition to this practice, I should like to urge you, in general, never to attempt to make impossible proofs appear valid. One should convince the pupil by simple examples, or, if possible, let him find out for himself that, in view of the actual situation, *precisely these conventions, suggested by the principle of permanence, are appropriate in that they yield a uniformly convenient algorithm, whereas every other convention would always compel the consideration of numerous special cases*. To be sure, one must not be precipitate, but must allow the pupil time for the revolution in his thinking which this knowledge will provoke. And while it is easy to understand that other conventions are not advantageous, one must emphasize to the pupil how really wonderful the fact is that a general useful convention really *exists*; it should become clear to him that this is by no means self-evident.

In sections i) and ii) we dealt with cases in which it was possible to extend, in a "natural way", existing definitions or formulae by use of the *Principle of Permanence of Equivalent Forms*; that is, cases in which the principle "works". Also, we have been warned not to represent these extensions as proofs, which they are not, unless the principle is taken to be an axiom. But, as we shall see in the following sections, we cannot always apply the principle, since in some cases it will lead us to mistaken conclusions and contradictions.

iii) This statement in mathematical form is:

$$|a + b| = |a + b|$$

and this is true if a and b are positive.
If we use the principle to extend the result to negative numbers, we obtain a contradiction, as is clear from the following example.

$$|-3 + 5| = |-3 + 5|$$
$$2 = 3 + 5$$
$$2 = 8.$$

iv) As stated, we assume, by the principle, that for every a, b and c

$$a > b \rightarrow a \cdot c > b \cdot c \qquad (1)$$

One of the useful and fundamental properties of the real numbers is that of order: that is, for any two numbers a and b, we have just one of three possibilities:

$$a = b,\ a < b,\ a > b \qquad (2)$$

If, now we take $a \neq 0$ and assume $a > -a$, then substituting $c = -1$ in (1), we obtain

$$a > -a \rightarrow -a > a$$

which contradicts (2).

In the following, we bring further examples in which the use of the principle leads to contradictions.

v) Exactly in the same way we can show that 0/0 can be anything else Because if $a = b$, then $0/0 = 2a$, for all a.

vi) For example, using the principle in section 1 will lead us to the conclusion that $0^0 = 1$

In conclusion, the *Principle of Permanence of Equivalent Forms* can be used to suggest the extension of concepts, but it is not a universal principle and one has to use it with care. If we look back, at previous worksheets, we can find hints of its implicit use

— Saunderson notices that if we multiply the terms of an arithmetic series (of natural numbers) by a natural number, then the result will be also an arithmetic series. This property he extends to negative numbers and thus makes a justified use of the principle

— Arnauld tries to use the principle on the ratio concept and fails, because in this case it is unjustified

Bell in *The Development of Mathematics* discusses Peacock and his principle and sees in it a further weakness:

> Peacock was not an "important" mathematician in the accepted sense of wide reputation; so possibly the following is a just estimate of his place in mathematics: "He was one of the prime movers in all mathematical reforms in England during the first half of the 19th century, although contributing no original work of particular value "He was merely one of the first to revolutionize the whole conception of algebra and general arithmetic.
>
> Hankel also reformulated the principle of permanence of formal operations, which had been stated in less comprehensive terms by Peacock: "Equal expressions couched in the general terms of universal arithmetic are to remain equal if the letters cease to denote simple "quantities" and hence also if the interpretation of the operations is altered." For example, $ab = ba$ is to remain valid when a, b are complex
>
> It is difficult to see what the principle means, or what possible value it could have even as a heuristic guide*. If taken at what appears to be its face value, it would seem to forbid $ab = -ba$, one of the most suggestive breaches of elementary mathematical etiquette ever imagined, as every student of physics knows from his vector analysis. As a parting tribute to the discredited principle of permanence, we note that since $2 \times 3 = 3 \times 2$, it follows at once from the principle that $\sqrt{2} \times \sqrt{3} = \sqrt{3} \times \sqrt{2}$ But the necessity for proving such simple statements as the last was one of the spurs that induced Dedekind in the 1870's to create his theory of the real number system. According to that peerless extender of the natural numbers, "Whatever is provable, should not be believed in science without proof."

* This comment was critically discussed with the teachers in the workshop.

4) In fact, the approach in the local textbook in question can be seen to be based on a critical use of Peacock's Principle It is assumed that the commutative, distributive and associative laws as well as multiplication by zero can be extended from the rational numbers to the integers.

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