Language, Arithmetic, and the Negotiation of Meaning

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Strong support among researchers in mathematics education is given to the "constructivist" paradigm for mathematics learning emphasising the role of language in the construction of knowledge This position and the role of language are summarised by one of the foremost proponents, Ernst von Glasersfeld [1990] as follows:

> Knowledge is the result of an individual subject's constructive activity … language is not a means of transporting conceptual structures from teacher to student, but rather a means of interacting that allows the teacher here and there to constrain and thus to guide the cognitive construction of the student. [von Glasersfeld, 1990, p 37]

It is the nature and role of language, as the teacher "guides" the development of children's knowledge in arithmetic, that forms the basis for the following discussion. the language of mathematics consists of *words* and *symbols* that have *meanings* related to particular *contexts* and to *procedures* for solving problems The nature of children's understanding of the words and symbols will depend upon individual experiences and upon the associations that are used. In the following discussion, I try to show that limitations in children's understanding of the symbols of arithmetic may inhibit choice of appropriate solution procedures for certain problems I argue that the teacher's role in developing understanding will involve "negotiation of new meanings" for words and symbols to match extensions to the procedures that become appropriate for solving problems. New meanings will need to be "reconciled" with children's existing understanding and this reconciliation is part of the negotiation process that takes place between pupils and teachers in the mathematics classroom. In helping children to extend their knowledge, the teacher's actions need to take account of the interpretation brought to a problem by an individual as well as the mathematical significance of any language that is involved.

In recommending a "negotiation of meaning", it is not disputed that mathematical terms and symbols may provide a concise and unambiguous representation of the patterns that exist in mathematics, but I believe that children's understanding needs to accommodate new interpretations as they come to understand the multiple connections that need to be accounted for in the precise formulation of relationships. Teachers and pupils will need to share the same meanings if understanding is to be achieved in classroom discourse

Words and symbols in arithmetic

The vocabulary of mathematics includes a lot of words and symbols with multiple meanings and pupils may not interpret the words as teachers intend them to. A classic example is given by Pimm [1987], who cites two pupils' responses to the questions "What is the difference between 24 and 9?" One says "One's even and the other's odd", and the other says "One has two numbers in it and the other has one", suggesting the pupils' failure to "comprehend the term *difference* as being used in a mathematical sense whose meaning involves the notion of subtraction".

Arithmetic has many words and symbols that may be interpreted in different ways. The symbol "×", for example, may be read as "lots of", "times", or "multipled by", each word or phrase indicating an interpretation that may be matched to a procedure for solving problems Where "3 × 4" is interpreted as "3 lots of 4", for example, a procedure involving sets of 4 objects (4 + 4 + 4) will result in a total of 12. Such a procedure will be of little help with the problem "$^3/_4 \times {}^1/_2$" and the language for interpreting the problem will need to be modified to match this new situation

It is worth noting at this point that the "mathematically correct" interpretation of "3 × 4" is "multiplied by 4" which corresponds to a set of 3 taken 4 times (3 + 3 + 3 + 3) which is not the same as "3 lots of 4". See Anghileri [1991]

Children need to develop flexibility in their interpretation of words and symbols and to associate with them different meanings and procedures for the solution of different problems

$$\begin{tikzcd} \text{words} \arrow[r, leftrightarrow] \arrow[d] \arrow[dr, leftrightarrow] & \text{symbols} \arrow[d] \arrow[dl, leftrightarrow] \\ \text{meanings} \arrow[r, leftrightarrow] & \text{procedures} \end{tikzcd}$$

The facility to "move around" this diagram, adjusting interpretations for each element until there is a "comfortable fit", models the development of mathematical understanding of the operations of arithmetic

Arithmetic understanding

In learning arithmetic, understanding, in the early stages, is characterised by the close association between (even inseparability of) the words describing a real context and the procedure for solving problems associated with that context.

real context ←→ words ⟶ procedure

For example, young children will need to physically "take away" objects from a set to understand the first level of subtraction. At this stage, interpretations of the operations of arithmetic are restricted by a "simple" language with limited meaning.

At a later stage, procedures may be implemented without recourse to any contextual "meaning", as may be illustrated by pupils who are able to find the solution to an abstract problem like "0.2 × 0.3".

words/symbols ⟶ procedure

Pupils' mastery of mathematical language needs to develop from use and understanding of "simple" everyday words and phrases to more formal mathematical terms and symbols capable of expressing complex relationships with precision. Even the "simple" words may have multiple meanings that children come to understand progressively. Development will be evident as pupils are gradually able to articulate the multiple connections between *words, symbols, meanings, contexts* and *procedures* for each of the mathematical operations they encounter.

As children proceed through school, their accumulation of interrelated knowledge must be accommodated when any new operation is introduced. Taking as a focus for study the operation of division, the following discussion attempts to reveal some of the complexities involved in coming to understand one of the operations that children will meet and use in school.

The meaning of division

Most teachers will be familiar with the distinct procedures that may be employed to solve a division problem involving whole numbers. Division and the division symbol "÷" may be identified with *partition* (or *sharing* in which equal "portions" are allocated to a given number of "recipients") and *quotition* (*grouping,* or *repeated subtraction,* with an identified number of elements removed successively from the starting set), and these distinct procedures are both represented by the same abstract symbolisations. When considering the words that are used, it may be claimed that the *correct* mathematical interpretation of the symbol "÷" is expressed using the words "divided by", so that the expression "12 ÷ 3" may be read as "12 divided by 3". But division may also be written in the form $3\overline{)12}$, read as "3 divided into 12". Division may also be interpreted as *complementary multiplication* as when the problem "12 ÷ 3" identified with the questions "How many 3s are there in 12?"

Many alternative phrases are used by both teachers and children, some acknowledging only one procedure for division, e.g., "12 shared by 3", and strongly associated with actions that are familiar to young children, while others, e.g. "12 divided into 3" are adaptations incorporating a more abstract notion of division.

The formal expression "divided by" must be learned in school since it is not found in the normal vocabulary of children and, when learned, it must be interpreted under guidance from the teacher and in the context of children's prior experiences. Even where children use appropriate words to "read" the symbols of written calculation, their success in solving problems will depend on the *meaning* they attach to these words and the procedures they identify with such meaning. The following observations illustrate that interpretations that are appropriate for one problem may inhibit solution of another similar problem.

<u>Case 1:</u> Lisa and Anna

Lisa and Anna, both ten years old, tried to answer the problem 6000 ÷ 6. Having successfully tackled problems like 28 ÷ 7 and 35 ÷ 5 using their table facts, they agreed that the question was asking "How many sixes are there in six thousand?" They collected structured apparatus of rods and blocks to represent six piles of one thousand cubes each. Their next step was to remove groups of six to see how many sixes there were. Clearly this was a very inefficient strategy (even if there was any hope of it being accurate!) and the teacher intervened:

> T: "How many cubes do you have in front of you?"
> L: "Six thousand."
> T: "How many piles of cubes do you see?"
> L: "Six."
> T: "And how many in each pile?"
> L: "A thousand."
> T: "So what is six thousand divided by six?"
> L: "We don't know, so we are just going to find out."
> ... as she and Anna proceed to remove sixes but abandon their effort some minutes later.

This case clearly shows how a successful solution strategy for some problems will inhibit the solution of others. The interpretation "6000 shared by 6" may have been a more useful interpretation enabling the identification of each pile of 1000 as a "portion" for each of 6 "recipients". The teacher's words "6000 divided by 6" were not helpful in guiding the children to a more effective procedure. The question arises what would be the most appropriate teacher interaction at this stage of the problem solving.

<u>Case 2:</u> Lorraine and Jody

Having successfully used the phrase "shared by" to solve the problems

12 ÷ 3   20 ÷ 4   14 ÷ 7

Lorraine and Jody (age 11) spent a giggly five minutes contemplating the problem

6 ÷ 1/2

Their conversation went as follows:

> L: "How can you share 6 with half a person?"

J: "You can't have that because you can't have half a person." (At this stage, Jody drew an illustration of a half person to reinforce the impossibility of the problem.)

L: "I know! Half of 6 is 3."

Here the notion of sharing has been related to having a human recipient for each portion and "half a person" seen to be impossible In contrast to the previous case, the notion of sharing renders this problem conceptually impossible for Lorraine and Jody. In seeking an alternative interpretation, division is related to halving and the phrase "half of 6" is used to formulate an alternative solution procedure.

These cases illustrate that an interpretation that is perfectly acceptable and appropriate for some division problems may cause difficulties in other division problems. There is no single consistent interpretation of the symbol "÷", and the words "divided by" relate to many different procedures that may be implemented to solve problems. While children may come to use the words "divided by" this does not mean that they have a sound understanding of any other than the most naive strategies that accompany the "simple" terminology. There is evidence to suggest that limited interpretations are accepted and encouraged by teachers to enable "entry" into problems that the children cannot themselves interpret. But how do children come to understand the diversity of meanings that can be attached to the written symbols?

Sharing

The idea of "sharing" is important in children's developing experience of division, not least because it incorporates an idea of fairness and equal portions which are built into the mathematical operation. Some children (and some teachers) use the words "shared by" and "divided by" interchangable so that the problems "12 shared by 3" and "12 divided by 3" are read as one and the same Lorraine's question, "How can you share 6 with half a person?" clearly indicates an interpretation that involves human recipients, though no meaning is identified for the abstract 6 in the problem

This notion of sharing may be further reinforced when "remainders" are introduced to account for those elements 'left over' when equal portions have been allocated. The idea of "little bits" may also be used to explain the decimal representation that will arise when a calculator is used to handle such problems as "9 ÷ 4" (= 2.5, or "10 ÷ 3" (= 3 3333333) It is worth noting that certain types of problem, e.g., 34 ÷ 7, will have two solutions: "4 remainder 6" and "4.8571428", whose meanings will have to be reconciled.

But sharing also introduces the common misconception that "division makes smaller" [Bell *et al.*, 1981; Fischbein *et al*, 1985] that persists and presents a conflict in later experiences involving a fractional divisor (for example, $12 \div 1/4 = 48$). The advantages of exploiting the vocabulary identified with everyday experiences must be set against the underlying primitive (and limiting) model that sharing provides for division in later mathematics.

It is evident that some children successfully develop alternative strategies to adapt sharing to different solution procedures. The word "share" may be identified with grouping when the preposition "by" is replaced by the word "into" and the second number seen to designate the size of each group

Case 3: Tina

Tina, an able 9 year old, was having difficulties with the problem 68 ÷ 17, having failed to solve it using her usual pencil and paper method for "long division" After a few minutes she came up with the answer 4, explaining that "64 shared into 17s will be 4 because 17 + 17 is 34 and you double it to get 68".

This explanation does not refer to the usual procedure for sharing items one at a time among 17 recipients but suggests a grouping concept identified with repeated subtraction or complementary multiplication. In this example the precise language used by the child indicates her adaptation, accommodating the familiar language to a procedure identified with the division symbol Such adaptations appear to be commonplace in the classroom, effectively complicating the issue for teachers where not only the actual words but their adaptation and interpretation may vary from child to child.

Listening to children

Recording children and teachers as they work on division tasks has provided evidence of wide variations in the phrases used to read and interpret the division symbol "÷". Consider first the following common phrases that use the expressions associated with "division" for the problem "12 ÷ 3".

12 divided by 3	12 divided into 3
12 divide by 3	12 divide into 3
12 divided into 3s	

There are several points to note about such phrases.

(i) The *passive* [Anghileri, 1991] construction "divided by" may be replaced by the *active* construction "divide by" to indicate a procedure that may be implemented to solve the problem

(ii) Just as a pizza may be "divided into" three portions, some children appear to associate division with partitioning *into* equal subsets.

(iii) There is a subtle difference implicit in the phrases "12 divided into 3" and "12 divided into 3s": the first may be identified with a sharing procedure resulting in 3 "portions" while the second suggests a grouping procedure or repeated subtraction of 3s.

(iv) Depending on the phrase chosen, visual images associated with both the procedure for solution and the outcome will be different.

As well as these phrases using derivatives of the verb "to divide" there appear to be a wealth of other alternatives used by children:

shared into 3, shared into 3s, shared by 3, shared with 3
split into 3, split into 3s
how many 3s in 12?, 3 into 12, 3s into 12
13 grouped in 3s, 12 grouped into 3s, 12 grouped in 3

It appears that teachers may "accept" these phrases from the children while using the formal interpretation "12 divided by 3" themselves. Recorded conversations in the classroom illustrate this point.

Case 4: Salim
Salim was having difficulty with the problem 68 ÷ 17 when this teacher tried to help:

T: "What does it say Salim?"
S: "Sixty-eight shared into seventeen."
T: "That's right, sixty-eight divided by seventeen. How many seventeens in sixty-eight?"
S: "Umm "
T: "What is twice seventeen, Salim?"
S: "Thirty-four "
T: Now can you see how many seventeens make sixty-eight?"

The teacher's assumption here seems to be that Salim could shift from sharing to grouping to doubling, all as possible solution procedures for division The phrase "divided by" was used by the teacher in relation to a particular interpretation of the phrase while the pupil used the specific phrase "shared into". Teachers using the mathematical terminology "divided by" will need to help children *interpret* this phrase and will have to *negotiate* extended meanings as the individual appears ready to progress in understanding the complexity of the terms. Here negotiation of meaning does not involve consensus in the mathematical meaning of division but in the words used to identify an appropriate solution procedure that will match this particular problem and be capable of transfer to different problems

General observations

Clearly the interpretation given to a particular problem will influence greatly the solution strategy selected, but it appears that the words used by children may or may not indicate the procedure they have in mind. Some children adopt a single strategy based on a limited interpretation of the symbols, as the example involving Lisa and Anna illustrates. Some employ a wide range of strategies paying little attention to the words that are used but understanding the diversity of meanings that may be associated with the division symbol It is here that significant differences appear to exist between higher and lower achieving children Able children appear to be able to *select* an interpretation and solution strategy to suit each individual problem while others do not progress from a particular strategy that has given early success.

Where children appear to have a single strategy for solving division problems, teachers will need to encourage alternative strategies by introducing problems and situations that illustrate the effectiveness of alternative approaches Peer group discussion about the methods that may be employed may highlight the alternative interpretations and phrases associated with symbols and procedures

A task I have used successfully at a variety of stages provides groups of children with a list of division problems:

37 ÷ 4 =
6000 ÷ 6 =
68 ÷ 17 =
3 ÷ 12 =
97 ÷ 7 =
636 ÷ 3 =
144 ÷ 36 =

and asks them to decide which are easy and which are difficult. The discussion that ensues within groups and, later, between groups, highlights many considerations and provides much insight into the children's thinking When asked to talk about their work, some children are very reluctant to do so or appear unable to express in words their understanding The above task may include problems selected to make the task challenging but non-threatening because the children are not expected to find solutions (though, inevitably, many do!).

Implications for teaching

It would be nice if each of the operations of arithmetic, for example "division", was represented by a single symbol that was associated with a single meaning and a single calculation procedure and a single set of description words, but the discussion above shows that this is not the case. Children must become accustomed to words and symbols that have multiple meanings and to alternative procedures that may be selected for effective solution of different problems.

In teaching arithmetic, teachers must be mindful of the problems children will face if their understanding of the operations is not extended to include multiple interpretations of the symbols and flexibility in solution procedures. In order to develop the skills needed to solve arithmetic problems, children need to

- realise that some words and symbols have multiple meanings;
- become aware of the limitation of some meanings attached to words and symbols;
- be convinced of the need to progress from naive interpretations to new terminology;
- be aware that teacher's and other children's meanings may differ from their own;
- be able to select appropriate interpretations of words and symbols and adapt procedures to match different types of problems

Teachers need to

- listen to children to access existing meanings associated with words and symbols;
- be aware that there may be discrepancies between teacher meanings and pupil meanings of words and symbols;
- provide experiences that illustrate a diversity of meanings for words and symbols;
- use language that has meaning for individual children as well as mathematical correctness;

Teachers need to devise strategies for classroom interactions, being both responsive to the child's existing understanding and pro-active in negotiating new meanings.

Opportunities to share their thinking with others will encourage children to reflect on the methods and language they themselves use and become aware of alternative interpretations and strategies. Encouraged by teachers, the children can learn effectively by reviewing their own thinking in relation to that of their peers.

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