Axiomatics of Geometry in School and in Science

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The author was invited in November 1986 to conduct a seminar on mathematical didactics at the ETH in Zürich. The organiser, Dr Max Jeger, asked him to address the subject of the axiomatics of geometry and to suggest a background system of axioms suitable for use in schools. Dr Zeitler's paper, which is published in full in German by ETH, comprises two parts. The first and larger part compares in detail a number of axiom systems for geometry; the second part examines the present state of school geometry. The shortened version, published here with the author's kind permission, very substantially reduces the amount of technical mathematical detail in the first part of the paper. The translation into English is by Abe Shenitzer of York University, Canada — *Ed.*

Part 1: Axiomatics of geometry in science
I searched for axiomatic systems for geometry, and tried to compare them and to single out the best background system for teachers. I also examined the significance of axiomatics for modern science

Part 2: School geometry
In order to clarify the problem of a background axiomatic system I decided to analyze the present state of school geometry and to draw conclusions from this analysis.

What came of this and what I said in my Zürich lecture follows.

PART 1: GEOMETRIC AXIOMS

1. Some philosophy

The moment we try to "straighten out" the foundations of geometry we run into philosophical positions such as Platonism, Formalism, Constructivism and other isms. Some mathematicians regard their philosophical positions as matters of faith, as worldviews. The following are stock descriptions of the three explicitly mentioned positions

Platonism (Realism). Mathematical objects — and thus all of mathematics — exist forever, beyond all time frames and independently of human beings. The mathematician's task is to spell out and to investigate these existing truths. The mathematician is a discoverer rather than a creator.

Formalism. Mathematics is a collection of formal systems whose elements are manipulated and combined according to specified rules of the game. These rules of the game, the definitions, and the proving of theorems, are the sole concern of the mathematician. The formalist is a creator and not a discoverer. For him the question of the existence of mathematical objects does not arise. For him it is enough to show that his rules of the game do not lead to contradictions.

Constructivism (Intuitionism). Mathematics is acknowledged to the extent to which its objects are constructed out of certain primitive basic objects in a finite number of steps. The question of constructibility is the dominant and permanent concern of adherents of this position

In the last few decades formalism has been enormously strengthened by the work and activities of N. Bourbaki. His influence reaches into the classroom. This is all the more surprising because most mathematicians are Platonists as well as Formalists. They think of themselves as "secret Platonists with formalist masks", or "Platonists with a small p". The constructivists are a species apart. They form a minority

In view of the tremendous influence of formalism on mathematics and on geometry, and even on school mathematics and school geometry, it is in order to consider it in some detail. Formalism is synonymous with working along axiomatic lines. We might say that formalism is axiomatics in the form of a pure culture. Hence the need to say more about axiomatics.

2. Mathematics as a game

The remarks that follow will give a measure of precision to certain basic concepts. The question to be considered in general terms is "What is axiomatics?" Formalist treatment of an area of mathematics involves the investigation of three groups of problems that we designate as A, B and C. To explain what these groups of problems are about we consider a game, say chess.

(A) *The axiom system*

To play a game we must know all of its rules. What is the role of the knight? How does the queen move? What is a check? What is a draw? The rules of the game form its logical skeleton, its structural background. In mathematics the role of the rules of the game is played by the axioms. The group A of problems concerns the setting up of a system of axioms for a particular discipline such as geometry

(B) *The theory*

Given the rules of the game we can deduce their logical consequences and formulate theorems. In the case of chess, this results in sophisticated opening games and clever end-games. The totality of logical consequences of the axioms is called its theory. Its development is the content of the group B of problems. In this connection we quote Poincaré:

> To see a geometry unfold we must be able to entrust its axioms to a machine.

We emphasize that the use of diagrams in geometry is, in principle, unnecessary and may, at times, be harmful. One plays a blind game

(C) *Models*

We play chess with real pieces. We move them. We see them. We can hold them in our hands. They are a realization, an interpretation, a model. The construction of such models is the content of the group C of problems

3. How do we find axioms?

The Platonists finds the axioms in the world around him. This means that he follows the sequence $C \to A \to B$ (heteronomic system of axioms). In the case of geometry we might use the term "natural" geometry.

Things are very different in the case of the formalist. He is a creator. He plays the role of deity. He follows the sequence $A \to B \to C$ (autonomic system of axioms). He looks for models when the theory is fully developed. In this case we could speak of an "artificial" geometry.

Even the creative freedom of the formalist is circumscribed. Its limits were first formulated by Archimedes.

3.1 NOT TOO FEW AXIOMS!

We require our axiom system to be complete. A statement formulated in terms of the language of the system of axioms must be "decidable"; either the statement or its negation must follow from the system of axioms. Then there are enough axioms. In the case of chess this means that every configuration of the pieces is resolvable; there is always an appropriate rule of the game.

3.2 NOT TOO MANY AXIOMS!

The axioms should be independent. This means that none should be a logical consequence of the others; there should be no superfluous axioms. 3.1 is a must. 3.2 is a desideratum.

3.3 NO CONTRADICTIONS!

No contradictions must arise in the process of developing the theory of a system of axioms (syntactic consistency). A contradictory chess game is a game in which both players could claim victory on the basis of the rules of the game. Most mathematicians view the existence of a model as evidence of consistency (semantic consistency).

5. Some history

5.1 EUCLID

The Greeks set up the first system of axioms for "the usual" geometry. It is contained in the 13-volume *Elements,* an integrated account of the mathematics taught in Plato's school. The work is distinguished by remarkable didactic skill. Euclid, its author, was almost certainly not a brilliant creative mathematician. The most significant and difficult parts of the *Elements* are the work of others. The logical shortcomings of the *Elements* became apparent 2000 years after its creation. (See [11]. — *Trans.*)

5.2 HILBERT'S MODIFICATION OF THE "ELEMENTS"

Hilbert made use of the work of many mathematicians to fill the gaps in Euclid's axiomatization of geometry. His *Foundations of geometry* is the modern version of Euclid's *Elements.*

Hilbert does not regard lines and planes as point sets. He speaks of three different sets of things. That he regarded these as variables ia made abundantly clear by his famous statement that "Instead of point, line, plane we could say table, chair and beer mug. What counts is that the axioms hold." (See [12], Chapters I and II.)

6. Rump geometries

We often axiomatize just parts of geometry. For example, there is a geometry concerned with just incidence properties, or just order properties, or just reflections. We give two examples of such rudimentary geometries.

6.1 R. LINGENBERG'S AFFINE PLANES

If the basic elements, "points" and "lines", of classical geometry satisfy just the incidence axioms and the parallel axiom, then we speak of affine planes. They were investigated by E. Artin and, later, by R. Lingenberg. The aim of these investigations was to demonstrate the close connection between the properties of an affine plane and the algebraic properties of its associated planar ternary ring, the ordered pairs of whose elements serve as coordinates of the points of the plane. For example, if Pappus' theorem holds in the affine plane, then its associated planar ternary ring is a field. The full Euclidean plane is reached by adding order and continuity axioms. (See [6], Chapters IV and V.)

6.2 F. BACHMANN'S REFLECTION GEOMETRY

The core of this geometry is the calculus of reflections. One begins with the group of congruence mappings generated by reflections. Certain reflection axioms yield a geometry called by Bachmann a metric plane. Two additional axioms yield a richer geometry called by him a Euclidean plane.

As in the case of Lingenberg's geometry, one must add further axioms to produce the classical Euclidean plane. (See [3].)

7. We all know the real numbers!

A critical examination of the Hilbert axiom system shows that it includes an axiomatic foundation for the real numbers. Nevertheless, the road to **R** is strenuous and troublesome. Also, order and continuity axioms are indispensable. This being so, it was natural to develop systems of axioms that take the real numbers for granted. One such system of axioms is the basis for L. M. Blumenthal's distance geometry. (See [5], Chapter VII.) Another is the ruler-and-protractor geometry of G. D. Birkhoff, whose basic concern is didactic. He wants to motivate his choice of axioms. He wants the reader to see where the axioms come from and why they are called what they are called. (See [4], [5], and Chapters 3-8 in [14].)

8. Geometry loses its autonomy

Bachmann's system of axioms relies heavily on elements of group theory, and the axiom systems of Blumenthal and Birkhoff assume the real numbers. This reliance on known and well-developed theories has been steadily gaining ground and has resulted in the embedding of classical geometry in a more general context. Thus in Blumenthal's development the larger context is that of the theory of metric spaces. The most popular modern context is that of linear algebra — more specifically, that of a vector space. This development has been propagated by J. Dieudonné with a determination bordering on belligerence. (See [6a].)

The very number of required axioms belies the oft asserted simplicity of this development. On the other hand, this development has various *mathematical* advantages (the possibility of replacing **R** with another field or skew field; the possibility of introducing any dimension $n \in \mathbf{N}$; the possibility of going over to other geometries by modifying a few axioms).

We conclude this section with a listing of the surveyed authors and their systems.

Classical approach	Rump geometries	**R** is assumed	Embedding
Euclid Hilbert	Lingenberg Bachmann	Blumenthal Birkhoff	Dieudonné

9. Systems of axioms for geometry — a playground for geometers and would-be geometers

One can hardly count the possibilities or list the names.

9 1 REFINEMENTS AND MODIFICATIONS

Well known systems of axioms, such as Hilbert's, can be refined in many ways and particular axioms can be replaced by equivalent ones Here R Baldus has done a great deal of work. A more recent example is the work done by E. Sperner and his students, in particular, H Karzel These geometers have handled the concept of order (among other things) in a very effective way It is hardly possible to give a complete listing of relevant names A partial list includes:

> J Diller, A. Kirsch, W Klingenberg, H. Lenz, H Meschknowski, H Karzel, G. Pickert, E Schroeder, K Spallek, E Sperner, . . , A D. Alexandrow, A V Pogorelov, B. Kerekjarto, . . . , G Choquet, A. Delessert, G Papy, . .

Some of the papers of these geometers are of a didactic nature Their authors have tried to adapt various systems of axioms for use in schools.

It seems that every respectable geometer has his own system of axioms and swears by it Without one you are nobody!

9 2 OTHER APPROACHES

A word about researches other than those devoted to improving, varying, and refining, well known systems of axioms.

J V Neumann embedded classical Euclidean geometry (over projective geometry) in lattice theory

A Tarski, W Szmielew, L Dubikaitis, W Schwabhäuser, M Moszynska and others made classical Euclidean geometry into a component of mathematical logic.

P Lorenzen developed geometry along constructivist lines In this approach an instance of a relevant question is what grinding tools are needed to make a plane.

9.3 THE QUESTION OF A BACKGROUND SYSTEM

The poor teacher is bound to be utterly confused by this depressing and chaotic multitude of systems What should he choose? Choice and distress go hand in hand! There are no mathematical reasons for designating one of the many systems as *the* system, the one to be kept within easy reach or under one's pillow. All of these roads lead to geometry but none is the royal road. It is my unalterable view that every prospective teacher should learn all the details of one such system at the university and should regard it as *the* system, as *his* system.

Since no system of axioms is ever likely to be *the* system taught at all universities, we must comfort ourselves with Cantor's famous dictum that "the essence of mathematics is its freedom." One indication of this freedom is the profusion of systems of axioms of our geometry.

This freedom we must all be aware of

10. The triumph and limitations of formalism

10.1 THE TRIUMPH

Guided by Platonist notions, mathematicians have elaborated axiom systems other than those underlying classical Euclidean geometry and have developed the corresponding theories Some of these systems and theories involve projective geometry, spherical geometry, inversive geometry, Minkowskian geometry and the respective rump geometries.

The use of axioms not suggested by experience heralded the triumph of formalism This development resulted in the introduction of hyperbolic geometry, of non-Desarguesian geometries, of non-Archimedean geometries and, finally, of finite geometries We now have geometries in which the number of parallels to a line through a point not on that line is just one, just 225, zero, or infinite It is all a question of individual taste! Axioms can be chosen at will They are all free creations of the human mind. Geometry is reduced to a game and everybody plays his own geometric game!

10.2 "NO SOONER WON THAN LOST"

The first and highest purpose of the formalist was to give complete axiomatic descriptions of various realms Every proposition belonging to such a realm would, presumably, be decidable within it But in 1931 K. Gödel published a revolutionary paper in which he proved the incompleteness of "sufficiently rich" and consistent axiomatic systems "If arithmetic is consistent then it is incomplete" was one of Gödel's shattering conclusions. This means that all such systems include undecidable propositions and that, therefore, the chief aim of the formalists — the complete axiomatization of all areas of mathematics — cannot be achieved *in principle*. Hence all mathematical theories are of limited comprehensiveness

Another key objective of the formalists was to establish the consistency of axiomatic systems not be producing models of such systems but by logical proofs based on the relevant axioms (see 3 3) They felt that existence was a matter of faith, and faith was not an acceptable mathematical proving tool Here again Gödel devastated the extreme formalists by proving that the syntactic consistency of a "sufficiently rich" system of axioms cannot be established within the system Such a proof belongs to a higher (metamathematical) system "If arithmetic is consistent, then its consistency cannot be established by means of a mathematical argument that can be mapped on the formalism of arithmetic " This limits the range of axiomatic methods. All its merits notwithstanding, axiomatics does not do all that was expected of it or that it is still said to be capable of doing; "still" means today, more than fifty years after Gödel's discoveries

10.3 RECENT DEVELOPMENTS

Essential changes have taken place in the philosophical foundations of mathematics in general and of geometry in particular. This is due, in the first place, to the pioneering work of K. Popper in epistemology. The views concerning the sense and methods of proof have also undergone radical changes Thus Imre Lakatos, in his book *Proofs and refutations*, sees mathematics as flawed and very definitely not above doubt. He believes that, like other areas of knowledge, mathematics evolves through criticism and the improvement of theories which are never entirely free of ambiguity or proof against error and oversight. These are certainly utterly surprising views! *They should be borne in mind in discussions about background systems of geometric axioms for use in schools.*

11. What is the Erlangen Program about?

The author's Bavarian origin obliges him to mention the Erlangen Program.

11.1 GEOMETRIES AS THE THEORIES OF INVARIANTS OF GROUPS OF MAPPINGS

The constructivist researches of Lorenzen referred to in section 9 cannot be fitted into the framework of formalism. Even less can the so-called Erlangen Program *(Erlanger Programm)* of 1872. This program has nothing whatever to do with axiomatics. The stated and programmatically formulated objective of its author F. Klein was a group-theoretic classification of geometries. Klein valued intuition far more than axiomatics. In Klein's words, "Instead of using the time set aside for geometry to develop the living visual capacity we use it to master a dead formalism or clever tricks devoid of principles." Is that not true in our own day?

11.2 SOME CONCEPTUAL DETAILS OF THE ERLANGEN PROGRAM

In a sense, the Erlangen program also envisages a stepwise development. Klein singles out the following nested sequence of groups of transformations: the group of congruences, of similitudes (equiform transformations), of affinities and of projectivities (the group of topological transformations also fits into this scheme). As the groups in the sequence increase in "size", the corresponding geometries shrink in the sense that the number of invariants declines. The following table illustrates the relevant changes.

	Congruence geometry	Similarity geometry	Affine geometry	Projective geometry
Length	X			
Area	X			
Angular measure	X	X		
Parallelism	X	X	X	
Simple ratio	X	X	X	
Cross ratio	X	X	X	X

The group of projective mappings (of the plane) yields all (plane) collineations. Hence Cayley's famous "Projective geometry is all geometry."

11.3 VISUALIZATIONS

The captionless Figure 1 illustrates how the above four plane geometries arise. Parallel projection of the spatial configurations to a plane yields in each case two triangles related, respectively, by a translation, a central dilatation, an axial affinity and a perspectivity. These special mappings can be used to build up the four previously-mentioned groups of transformations.

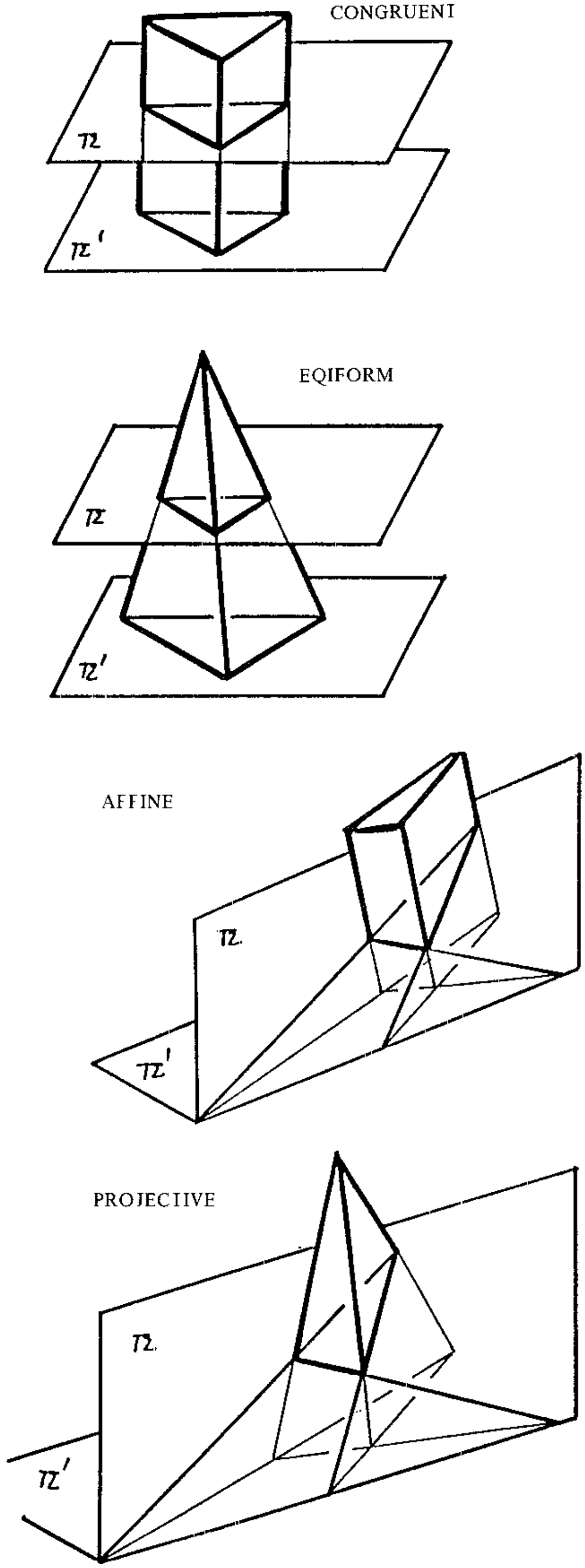
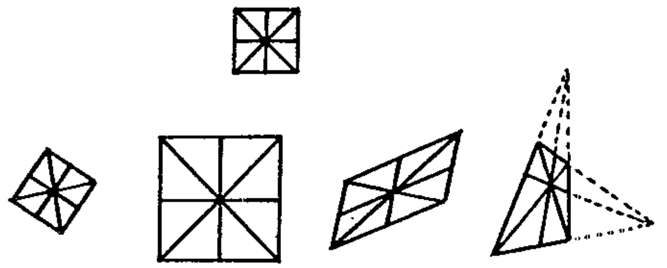


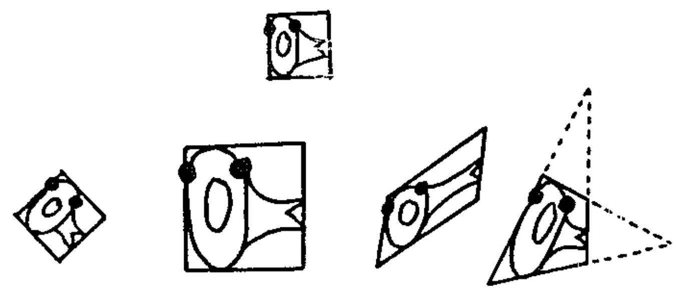
Figure 1

Figure 2 shows the respective congruent, equiform, affine, and projective images of a square (including its diagonals and mid parallels). Figure 3 shows the Erlangen Aquarium



CONGRUENT EQUIFORM AFFINE PROJECTIVE

The Erlangen Program
Figure 2



CONGRUENT EQUIFORM AFFINE PROJECTIVE

The Erlangen Aquarium
Figure 3

12. Foundations of geometry as a topic of instruction at the university

A prefatory remark. The author is familiar with the situation at the German universities and can only guess that it is likely to be symptomatic of the situation at universities elsewhere.

Apart from topology and differential geometry very little time is devoted at German universities to geometry, and still less to its foundations. Sometimes references are made to geometry in connection with the study of linear algebra or in connection with exercises in linear algebra. Recently even this has become taboo. Occasionally, some lectures may deal with Lingenberg's stepwise buildup of geometry, but the aim of such lectures is to demonstrate the close connection between the closure theorems and algebraic structures and not to develop classical Euclidean geometry. This being so, all ends as soon as one gets to Pappus' theorem. In sum, it is fair to say that the status of geometry and of the foundations of geometry at the German universities is deplorable and seems likely to deteriorate further in the future.

PART 2: HIGH SCHOOL GEOMETRY

A prefatory remark. The author is familiar with the situation in German high schools but is convinced that his analysis of the changes in the geometry curriculum in these schools between the late forties and today is of general interest.

13. Geometry in the upper classes of German high schools

13.1 A LOOK OVER THE FENCE

Austria still has a subject of instruction called geometry. In Austria they still teach descriptive geometry. Lucky Austria. The beautiful discipline of descriptive geometry seems to have become a study by Austrians for Austrians.

France and Belgium swear by vector spaces. With champions as intense and articulate as Dieudonné and Papy this ideology was bound to win out. There are many corresponding textbooks on the market.

The amount of geometry in American high schools is, relatively speaking, appreciable. Most of it is of the Birkhoff type. This is probably due to the Birkhoff-Beatley textbook based on Birkhoff's axiomatics.

And what about Germany? The decline of geometry in German high schools can be easily documented by a look at the provincial Bavarian maturity examinations dating from different times.

13.2 THE CONIC-SECTIONS PHASE (up to about 1966)

(In the beginning was the conic section, and the conic section . . .)

Analysis and analytic geometry were given equal weight. They were taught for 5 hours a week over a two-year period.

This was the time of the preeminence of W. Lietzmann. The study centred on conic sections and dealt with various types of "locus problems." In these problems a point traverses a given curve and one is required to determine the curve traversed by the image point determined by a more or less sophisticated prescription. These problems followed a fixed pattern that could be learned by the student.

Students made many constructions and carried out very many calculations. The third dimension was ruled out and the question of applications (available in large numbers) was not raised.

The middle level prepared students — through so-called study of figures and countless triangle constructions — for the crowning study of conic sections. The standard background system was the Hilbert system of axioms (or a system similar to it).

13.3 THE TRANSFORMATION PHASE (up to about 1970)

(In the beginning was the mapping, and the mapping . . .)

Analysis and analytic geometry continued to be given equal weight, and continued to be taught for five hours a week over a two-year period.

This was the era of the transformation geometers. I mention here my fatherly friend K. Fladt as well as O. Botsch and K. Faber. M. Jeger's beautiful *Transformation geometry* fits this phase perfectly.

The main topic was the study of various groups of transformations of the plane, namely congruence mappings, similarities, affinities and projectivities. The study plan was entitled: "Transformation groups and their invariants as an ordering principle in geometry." This was the realization of the Erlangen Program, and thus the pure form of transformation geometry.

There was a great deal of computing (including computing with matrices and determinants). Drawings were few and far between (in contrast to the Jeger book). There was no trace of applications. The third dimension was still prohibited, but so were the conic sections.

The middle level of high school dealt with elementary investigations of special transformations. This served as an excellent preparation for the more advanced study of transformation geometry on the upper level. A single thread ran through all of school geometry. The Hilbert system of axioms — combined with the Erlangen Program — continued its role as a serviceable background system.

13.4 THE FOUNDATIONS PHASE (up to about 1975)

(In the beginning was the axiom, and the axiom . . .)

At this point the so-called college level was introduced. The plan of study was replaced by a curriculum with a great many major and minor learning objectives; in fact, a matrix of learning objectives. The contents had to be modified. Reforms were in. This led to harmful confusion everywhere. The collegians — the new name for students — could now choose the P (for probability) stream or the G (for geometry) stream. If they chose the G stream, then they were taught geometry for three to four hours a week over a two-year period.

The detailed study of the Poincaré half-plane model of the hyperbolic plane was to lead to axiomatics. The existence of "another" geometry was to motivate and stimulate axiomatic thinking. In addition, one used as a master example of a discipline based on axioms a minigeometry, namely the affine plane with emphasis on the finite case. The watchword was "motivation for axiomatizing". This was an attempt to realize A.J. Wittenberg's ideas, as described in his *Bildung und Mathematik*.

This experiment involved a lot of (non-Euclidean) drawing and constructing. The third dimension, conic sections, the Erlangen program — in short, the whole geometric past — were forgotten, were no longer cared for.

Studies on the middle level now lacked plan and objective. One recommendation followed on the heels of another. In view of the later study of the affine plane, the only reasonable background system was Lingenberg's stepwise build-up. But this hardly fitted the hyperbolic plane.

The introduction of the combination of the hyperbolic plane and the affine plane was doomed from the beginning.

13.5 THE VECTOR-SPACE PHASE (up to about 1980)

(In the beginning was the vector space, and the vector space . . . .)

The division of instruction time remained the same but the contents of the geometry were radically changed.

The French-Belgian reform movement spilled over to us. The trinity of set-structure-morphism and the ABC-problem complex (see section 2) formed the ideology dominating all activity. First came vectors. These were variables in the Hilbert sense, that is meaningless elements (much like the points and lines employed in the foundations phase). A system of axioms for these place-holders dropped out of the skies and exploded close to the collegians' heads. The students were now busy with vector spaces over arbitrary fields of every finite dimension. Then came the sequence of affine point space, inner product, Euclidean vector space and, finally, Euclidean point space. In effect, a cheap imitation of the university course — chips off the linear algebra course — was exported to the high school. Teaching was almost completely degeometrized. The sense of familiarity was to be avoided at all costs. The purpose of geometry was to help visualize and interpret structures. The peak of abstraction was reached. A tremendous apparatus was developed and never used. Practical applications (there are such!) were never considered. Conic sections were ruled out. One had no use for drawings and certainly none for constructions.

Middle-level geometry prepared students for work with vectors (translations, central dilatations, group structure). The Dieudonné system sufficed as a background — or rather foreground — system.

This scheme was sure to make every good pedagogue (there were such even then) sick to his stomach.

13.6 THE AMPUTATION STAGE (still on)

(And the earth was desolate and empty . . .)

Now things changed radically. The triumph of stochastics and statistics began. The hours of instruction in the two collegiate years were divided in the ratio of 5:4:3 among *analysis, stochastics* and geometry.* Choices were eliminated.

The content of the geometry section is a thoroughly diluted geometry in $\mathbf{R}^3$. The introduction of vector spaces must now be motivated. This is a clear retreat from abstract linear algebra. There is very little substance but its form is now geometric. Essentially, lines, planes (circles and spheres) are intersected and their mutual disposition is investigated. Transformations have been completely eliminated. Conic sections are not part of the geometry course. Drawings are not part of this truncated construct and applications of this minimal material are hardly possible.

The notions concerning geometric instruction on the middle level are chaotic. At this moment there is a tendency to return to the old study of figures in the conic-section phase. No background system is required for a torso. If one were needed, it would be difficult to make a reasoned choice.

All in all, we are left with a pitiful assortment of geometric fragments — a residue that gives no joy to either student or teacher. I simply cannot imagine that anyone can think of intersecting lines and planes — in a sense, the crowning stage of nine years of the study of geometry — as being much fun. What captivating problems are there for the students? Where has the much-talked-about creativity been relegated to? I think that a student would find the construction of conic sections to be of greater interest than the discussion of linear dependence of vectors. The following two quotations, due to H. Freudenthal and dating back to 1970 (!) are relevant:

> One admits as much geometry as is required for linear algebra, and this puny amount is rolled out and leached to the point of nausea. The geometry that can be done with the aid of linear algebra [brings to mind] turgid waste water.

> Small wonder that mathematics instruction in German schools will soon trail that of the underdeveloped countries.

*This ratio was often changed.

Where do we go from here?

The slice of history presented in sections 13.2 - 13.6 shows that in the last few decades school geometry has shrunk to an absurd torso. Now the new subject of computer science, with — regrettably — political push behind it, is coming to the fore. The newcomer is likely to dethrone stochastics and statistics but may also banish from schools the remnants of geometry.

14. Some of the causes of the disaster

There are very many causes of this depressing state of affairs. I wish to discuss three such, which are closely related

14.1 EXCESSIVE EMPHASIS ON THINKING IN TERMS OF STRUCTURES AND ON FORMALISM

There is no doubt that one cause of the decline of geometry is the spreading plague of abstract structural mathematics, the insistence on axiomatization all the way down the schoolhouse. Formalism is displayed like fashionable clothes. There is hardly anything that has not been tried in an attempt to obtain (even at the middle school level) a global axiomatization of geometry — à la Hilbert, à la Bachmann, à la Blumenthal, ... There is even a complete formalization of school mathematics in the sense of a consistent application of ways of writing and of logical principles. The missing glory of this development is a formal proof of the existence of God and the study of the Löwenheim-Skolem theorem. Where is a place for enjoyment? Bourbakism has triumphed even in school. The investigation of skeletons is the true preoccupation of anatomy. But geometry is akin to life. Grammar and harmony are not the same as poetry and music. In short, I think that the elimination of visual, intuitive geometry from school was a grave error of modern didactics.

14.2 THE FAULT OF THE UNIVERSITY

There is no doubt that the guilt rests largely with the universities. Since visual, intuitive geometry is of less research significance now than at the turn of the century — which translates into fewer publications — it is hardly engaged in by young professors. Today — unlike in the past — the teaching of practicing teachers is for many researchers a horror and a presumption; at the very least it is a demeaning additional burden. Hence, as noted in section 12, there are virtually no lectures on geometry. Teachers have no training in geometry. They know only abstract vector spaces. They are unwilling to teach geometry, they don't enjoy it, and they can inspire no enthusiasm for this subject. A truly vicious circle. Hence my stubborn insistence on a program of sensible lectures on geometry for practicing teachers. The meaning of sensible in the present context would have to be clarified.

14.3 MAKERS OF TEACHING PLANS. TEXTBOOK WRITERS FOR THE SCHOOLS, AND OTHERS

Those mainly responsible for the disastrous misdevelopment of the teaching of geometry are without doubt the many — all too many — developers of teaching plans, superdidactitians, curriculum prophets and textbook writers, myself included. All too often, the starting point of our deliberations has been the university version of the subject rather than child, the student. The aim was to transplant university mathematics with all its rigor and abstraction (necessary at the level of the university) to the school and, above all, to take advantage of this opportunity to show off one's knowledge. Content, spirit and choice of words are appropriate for addressing the mathematician, the scientist. Leafing through the schoolbooks shows that they cannot be read by children. But they radiate remarkable pseudoexactness and bespeak their authors' bombastic pseudolearnedness. One is often tempted to tell these authors: "Say it simply, say it plainly!"

There are many more reasons for the decline and fall of the beautiful subject of visual school geometry, but this is not the place to discuss them.

15. Revitalizing geometry

Some say that geometry should be exhumed. But since it has never died it needs to be infused with new life rather than exhumed. Is this a meritorious undertaking? So much has been said in this connection that I need only sketch a few relevant arguments. Before doing so I wish to quote D. Laugwitz:

> Geometry should be saved because it is beautiful. Linear algebra is useful but hardly beautiful; in fact, once fathomed it is boring.

15.1 THE USEFULNESS OF GEOMETRY

There are many disciplines, such as crystallography, mechanics, kinematics, architecture, cartography, astronomy, and geometric optics, whose very existence depends on visual geometry. It is indispensable for engineers, architects, astronomers and physicists.

15.2 GEOMETRY, A LANGUAGE FOR RESEARCH MATHEMATICIANS

Research mathematicians make constant use of the language of geometry — a language they must really master. They must draw on this language for the visual representations that they need for their spaces, for dimension, for manifolds, bundles, fibres and all the rest.

I am convinced that the very creativity of a research mathematician is closely linked to geometric representations. Everybody paints his or her (possibly secret) little picture.

15.3 GEOMETRY IN SCHOOL

The mental development of a child begins with the visual domain. A child wants to, and must, "seize" and understand his or her world. Children paint and draw. They are little geometers by nature. One cannot begin with sterile, abstract structures. Such structures come at the end of a long development. That is why a school without geometry contradicts the child's nature and mental development.

15.4 GEOMETRY: A CULTURAL TREASURE

Geometry is an essential component of culture, a part of mankind's mental development. This is reason enough to know it and to teach it.

15.5 GEOMETRY IS BEAUTIFUL

"The enjoyment of shape makes a geometer" (C. Clebsch). Geometry — visual geometry — is beautiful. It gives us esthetic pleasure. To mention just one example: can anyone fail to be fascinated by the mosaics of the Alhambra?

Even these few arguments — there are many more — show that it really pays to revive and revitalize geometry in schools and universities.

16. Ways of revitalizing geometry

What to do to make school geometry more attractive and interesting?

16 1 GENERAL REQUIREMENTS

We need less abstraction, less formalism, less or — better still — no axiomatics, less on the foundations of geometry, less linear algebra, less structural geometry, less, less, . .

We need more visual content, more concrete geometry, more interesting (nontrivial) problems, more drawing, more constructions, intuitive, heuristic, unstructured geometry, more joy, more, more, . . .

All of us, geometers and teachers alike, must quickly do something to revitalize school geometry It is five minutes before midnight

16 2 SUGGESTED TOPICS

A number of us have tested many topics of possible use for new geometric instruction in middle and upper high school Some of our findings have appeared in print Here are some cues from our catalogue:

> *Elementary differential geometry*: rolling curves, helices, caustics, maps, . .
> *Coverings, inlays, tessellations*.
> *Convex sets* Linear optimization.
> *Elementary topology*. The Möbius band.
> *Inversive and Minkowskian geometry.* Elements of relativity theory Computing with complex and double numbers. Applications, .
> *Computers in geometry*.

This is far from enough. Many more thematic areas must be opened up, prepared, tested and, finally, put together in book form. Consider yourselves addressed and invited to cooperate I hope to hear about your results. You and your students should get down to work It is time to be creative!

Conclusion

Part 1: *The axiomatics of geometry as a discipline*

There are many very different systems of axioms for classical Euclidean geometry. Each has its advantages and disadvantages. No one has as yet found the royal road to geometry This being so, I cannot, from a purely mathematical viewpoint, recommend any one of the described axiomatizations as *the* background system for the use of teachers I am sorry to have to disappoint expectations of this nature.

I wish to add that the importance of axiomatics for science in general has significantly decreased

Part 2: *School geometry*

The choice of a background system depends entirely on the teaching aims The analysis of school geometry (in Bavarian schools) abounds in clear signs of decline and fall. This shocking picture makes the discussion of a suitable background system virtually irrelevant. The revival of geometry calls for immediate intensive therapy. For this, background systems are useless

This discussion suggests a redistribution of emphases and the choice of a new title for this essay: *The crisis in the teaching of geometry*.

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