STRUGGLING TO DISENTANGLE THE ASSOCIATIVE AND COMMUTATIVE PROPERTIES

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Kleiner (1986) states that the mathematician Arthur Cayley gave the first abstract definition of a group in 1854. Cayley's definition included the requirement that the elements be associative under the operation. The way that Cayley articulated this requirement was particularly interesting. Cayley stated that the "symbols are not in general convertible [commutative], but are associative" (Kleiner, 1986, p. 208). It was, of course, unnecessary to explicitly state that the symbols do not commute in general. Even more interesting is the fact that this appears as a sort of caveat to the associativity requirement. Why did Cayley phrase the associativity condition this way?

Interestingly, I have observed undergraduate students including the same caveat in their own definitions of the group concept. In fact, these students experienced a number of difficulties related to the associative and commutative properties. In this paper, I will describe some of these students' struggles to make sense of the associative property and its relationship to the commutative property. Throughout, I draw connections between the students' mathematical activity and the findings from the research literature on children's (and teachers') ways of thinking about binary operations and their properties. Drawing on these connections, I present an explanation for some of the student difficulties related to the associative and commutative properties.

Theoretical perspective

The research presented here is part of an ongoing effort (Larsen, 2009; Larsen & Zandieh, 2007; Weber & Larsen, 2008) to develop an approach to teaching abstract algebra in which the formal mathematical concepts are developed beginning with students' informal knowledge and strategies. This work is grounded in the instructional design theory of Realistic Mathematics Education (RME). Two RME design heuristics were of particular importance as I developed instructional tasks *and* provide crucial theoretical support for the analyses presented here: *guided reinvention* and *emergent models.*

Gravemeijer and Doorman (1999) explain that the idea of guided reinvention "is to allow learners to come to regard the knowledge that they acquire as their own private knowledge, knowledge for which they themselves are responsible" (p. 116). The primary objective of my research was to develop an instructional approach that supported students' (guided) reinvention of the concepts of group and isomorphism. It is crucial that the mathematical activity of the participating students be understood as part of a reinvention process. In particular, the students were not given definitions for concepts and then asked to make sense of them. Instead, they developed these definitions (and the concepts themselves) through their own mathematical activity.

According to Gravemeijer (1999), emergent models are used in the RME approach to promote the evolution of formal knowledge from students' informal knowledge. The idea is that the model initially emerges as a *model-of* students' activity in experientially real problem situations and the model later evolves into a *model-for* more formal activity. Gravemeijer (1999) describes the transition from model-of to model-for in terms of a shift from what he calls *referential activity* to what he calls *general activity.* The students' activity begins in a specific problem context and at the level of referential activity, models-of refer to this context. However, at the level of general activity, models-for "make possible a focus on interpretations and solutions independently of situation-specific imagery" (Gravemeijer, 1999, p. 163).

Background

My design efforts began with a sequence of three teaching experiments conducted with pairs of undergraduate students. Here I will focus primarily on the first two teaching experiments. All of the participating students had recently completed a "transition to proof" course in which they learned to construct basic set theory proofs. None of the students had taken an abstract algebra course prior to their participation.

Jessica and Sandra participated in the first teaching experiment. Jessica was a mathematics major and pre-service teacher while Sandra was not seeking a degree but was taking advanced mathematics to support her interest in science. Erika and Kathy participated in the second teaching experiment and were both mathematics majors. Jessica, Erika, and Kathy all received a grade of "A" in the transition to proof course, while Sandra received a grade of 'C'.

As noted above, the goal was for the students to reinvent the group concept and my efforts to support this were guided by the emergent models construct. My plan was to have the group concept emerge as a model of the students' activity in the context of the symmetries of an equilateral triangle.

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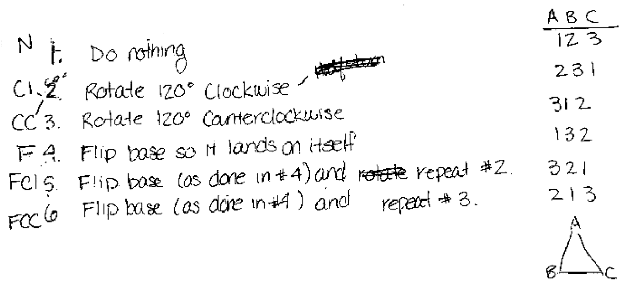

Figure 1. The students' table illustrating the symmetries of an equilateral triangle

Thus, the triangle context was *not* an example of a group for the students. Instead the group concept would only emerge as a result of the students' mathematical activity in this context. The first task given to the students was to describe and symbolize the symmetries of an equilateral triangle. Jessica and Sandra developed the following table (fig. 1) to illustrate the six symmetries of an equilateral triangle.

After the students identified the six symmetries of an equilateral triangle, they were asked to consider each combination of two symmetries and determine to which of the six symmetries it was equivalent. I expected that the students would determine the results by manipulating a cardstock triangle that was provided. The students did use this method to some extent. However, Jessica and Sandra's primary method for determining the results was unexpected. Jessica almost immediately started making observations about the relationships between various symmetries and then used these observations to *calculate* the combinations. For example Jessica quickly noted that, "if we do 'do nothing' and one of these other ones, it's gonna be the same thing."

Jessica then began using her observations to calculate more complex combinations and, after hearing Jessica's explanations, Sandra was able to perform similar calculations.

Jessica: If you combine *F* and *FCL* you're just going to get clockwise. Cause you're going to flip and then you're going to flip again and then so those cancel each other out so you're going to have these left over.

Sandra: You're going to flip …

Jessica: You're going to flip and then you're going to flip it again so both of those flips cancel each other out so this is all your going to have left is this move, clockwise.

This calculation makes implicit use of the associative law (the presence of the composite symmetry, *FCL*, makes it possible to see the combination as involving three symmetries) along with the identity property and some relations specific to the symmetries of an equilateral triangle. As such, *we* can see the group concepts beginning to emerge as a model-of the students' activity. In fact this use of rules to calculate combinations of symmetries was a crucial part of the students' reinvention of the group concept (for more detail see Larsen, 2009). Note however that at this point, the students' activity was still closely tied to the initial problem situation. In particular, an expression like *F FCL* represented for the students a sequence of physical actions and not instance of a binary operation acting on two elements.

Given that a primary goal of the teaching experiments was the formulation of the definition of group, it was important for the students to become explicitly aware of the associative property. So when I observed the students regrouping, I called their attention to this, asked them to describe exactly what they were doing, and asked them to justify that it was a valid technique. The resulting discussions were surprisingly rich and it quickly became apparent that the issue was more complex than I had expected.

Discussions about associativity

Sandra and Jessica

Jessica had used the term "bunching" to describe her regrouping process, so I asked her what she meant by this (in the context of computing the combination *FCL FCC*).

SL: You said something about bunching them up. What did you mean by that?

Jessica: Yeah. Grouping.

SL: Like how do you mean?

Sandra: You can split the compound moves into simple moves and then check your result.

Jessica: We already have what these will do [they had already determined that the combination *FCL F* is equivalent to *CC*] and then counterclockwise and counterclockwise.

SL: So you grouped it like these three together, is that what you did?

Sandra: *FCL* and *F*, these two get counterclockwise, and counterclockwise and counterclockwise give clockwise.

SL: And it doesn't matter how you group them like that?

Sandra: Well *FCL* has to be the first step.

Jessica: No it shouldn't as long as you keep them in order.

In the last two lines above, both Sandra and Jessica emphasize that elements can be grouped as long as the order of the elements is not changed. These comments are reminiscent of the wording of the associativity condition in Cayley's definition of group, and suggest that somehow the associative and commutative properties were linked for these students. This brief exchange foreshadowed a fascinating discussion that took place during the following session.

I resumed the discussion about regrouping by asking about the method Jessica used to compute the combination *FR* followed by R^{-1}. (Earlier in this session, the students had revised their symbol set by replacing *CL* with *R* and *CC* with R^{-1}.) As the students responded to this question, the issue of order and how it might relate to the regrouping procedure became the focus of discussion.

SL: … so she does *FR* dotted with R^{-1}, she'll just put R and R-1 together.

Sandra: Yes.

Jessica: And take them away.

SL: And you know what R and R^{-1} gives right?

Sandra: Right, nothing.

SL: Right but so the question is …

Sandra: Can you do that? Is that legal? … Because you're taking it kind of out of order. I mean you're adding R and R^{-1} together and then doing the flip really.

SL: Right so there's two issues. The first issue is –

Sandra: Does the order matter?

Jessica: As long as it's in order.

Notice that in the previous exchange, Jessica indicated that her regrouping procedure would always work as long as the order of the elements is preserved while Sandra says that, "you're taking it kind of out of order." Keep in mind that the students' written calculations were, for them, closely tied to the physical operation of performing combinations of symmetries. So we are not seeing them confounding the abstract operation with the physical manipulations. Instead they are working through the process of mathematizing this physical operation and figuring out how the mathematized version should work. This would necessarily require the students to refer to the initial physical context to justify their assertions about how the symbols should work.

As our discussion continued, Jessica and Sandra struggled to reconcile their use of grouping (which seemed to say that in some sense order did not matter) with their awareness that the order of the symmetries did matter in some other sense.

Jessica: Because it's associative. Right? Wait, it's associative works.

SL: It's associative?

Jessica: It's not commutative but it's associative. Do you have to prove it?

SL: So I guess what I'm saying is maybe this is an important rule that you're using all the time … and maybe this should be on your list.

Sandra: Then you can combine them however you'd like. But I mean the order isn't, so you're saying the order doesn't really matter then.

Jessica: I think the order that you multiply them isn't important but the order of them is important. It's very important.

In her last statement, Jessica began to tease apart the two kinds of ordering that are in play here. By the "order that you multiply them" she seems to be referring to the order of one's actions while *calculating* a combination of symmetries and by "order of them" she seems to be referring to the order of one's actions as one *transforms* the triangle by performing a combination of symmetries (which is represented by the left-to-right ordering of the symbols). To Sandra this distinction was not yet clear, likely because she was still seeing the symbolic expressions as literal representations of the physical operations which made it difficult for her to see how the "order you multiply" could be different from the "order of them." Using the language of RME, her mathematical activity was still at the referential level, while Jessica was beginning to operate at the general level and could talk about operating with the symbols independently of the physical imagery of the triangle context.

SL: When you say order doesn't matter what do you mean order doesn't matter?

Sandra: It either matters or it doesn't. Doesn't it?

Jessica: You mean the order that you multiply not the order that they're in.

Sandra: What's the difference the order that they're in and the order that you multiply?

SL: So does order matter or does order not matter?

Sandra: Order does matter.

Jessica: That order matters but …

SL: So is this an ordering thing? [Points to the associativity example on the board]

Jessica: Operations. Order of operation … how do you say it? How do you distinguish?

The above excerpt highlights the role of language in the students' reinvention process and in their struggles to make sense of their symbolic regrouping activity. The shift to symbolic calculations introduced a new kind of order to the situation with the result that the word "order" was unhelpful. In order to make herself clear to Sandra, Jessica needed to clarify what she meant by the "order that you multiply them" and in the end she was able to do this by correctly identifying this as the "order of operations."

Erika and Kathy

Erika and Kathy also spontaneously used regrouping to calculate combination of symmetries. I asked Erika about this and, like Jessica, she was not sure what the name of this rule was.

SL: What rule are you using there?

Erika: Oh it's like an associative or commutative or something like that. Like it has to do with uhm …

When I asked whether this was a valid procedure in this context, Kathy said it was, and her explanation was based on the idea that each symmetry symbol represented a single step in a sequence.

Kathy: Well I think that's okay because *FCC* is the same thing as *F* and then *CC*. So *FCC* [pause] *CC* is the same as *F* and then *CC* and then *CC*. So yeah, it works as like associative law because these are all, *F* and *CC* and this *CC*, everything's an independent step all the *F's* and *R's,* each *F* and each *R* 2*R* 3*R* is an individual step.

Kathy's description suggests that she is thinking of a written combination of symmetries as a sequential (left to right) procedure. Her explanation echoes Sandra's observation that, "You can split the compound moves into simple moves and then check your result." Note that the written expressions the students were working with at this time *were* created as symbolic descriptions of the physical process of performing two symmetries in sequence, so it makes sense that they would be thinking of the operation of combining in terms of a sequential procedure. To Kathy, it was clear that in this case one can group (or not group) the symbols at will. In fact she went on to argue that there was actually no need to use parentheses.

Kathy: But I don't think you have to have parentheses. Why do you have to have the parentheses? Can't you just write … like for the *FCC FCC,* can't you write it *FCCFCC* and who cares where the parentheses are…. I mean it could be four steps for all we care.

Freudenthal (1973) noted that automorphisms of a structure under composition are guaranteed to form a group where "rather than by an algorithmic verification, this result is obtained in one conceptual blow, and this is a great advantage" (p. 110). We are seeing this phenomenon in Kathy's explanation. Reasoning at the referential level she argues that when one performs a sequence of three symmetry transformations it clearly does not matter how one groups the symmetries since, regardless, the three transformations will be performed in sequence. The general proof of the associative property for function composition boils down to this very idea: and are both equal to $f(g(h(x)))$.

While Kathy felt that regrouping was trivially valid, Erika experienced difficulty with regrouping. While attempting to compute the combination *RF* followed by 2*R*, she commuted the elements while regrouping to get the combination *F*2*R* followed by *R* (fig. 2). (At this point the students were using a dash to indicate when they were combining two of the six symmetries.)

Kathy and I both tried to explain to Erika that she was doing more than just regrouping. Her response suggested that she was unsure about the *meaning* of regrouping in this context.

Erika: *R F* 2*R* like what does that mean? [Writes (*RF*) 2*R*] Doesn't that mean you do those two first and then that one?

$$(RF)-2R = (F2R)-R$$

Figure 2. Erika commutes elements while regrouping

SL: True. What would be the … what would moving the …

Kathy: But we could make it like that. [Draws parentheses around the *F* 2*R* on the left hand side]

SL: … that's what moving the parentheses would mean you just did these two together, and then that one.

Erika: Right, well that's what I'm saying. Is I can do these two and then that one or you do these …

SL: Right but here your flip actually changed from being after the *R* to being before the *R*.

Erika did not find our arguments persuasive, likely because she was trying to make sense regrouping from the perspective that the parentheses meant that you should "do those two first." (Note that my statement that you "did those two together, and then that one" could easily be interpreted to mean that these two symmetries should be *performed* first rather than *computed* first.) Erika responded to our arguments by asking a question: "But then how … okay can you show me how to do this [*R* (*F* 2*R*)] with the triangle?" This turned out to be productive, because my response seemed to help Erika make sense of the meaning of regrouping in this context.

Erika: … can you show me how to do this [*R* (*F* 2*R*)] with the triangle?

SL: Well I couldn't. … You can't do that with the triangle. Because with the triangle, you're stuck. You have to do things in a certain way. You have to do *R F* 2*R*. Right? … But there's no way to do this one [*F* followed by 2*R*] and then that one [*R*] because you've already done them out of order …

Erika: So then moving the parentheses around is really quite meaningless.

Kathy: Yeah, exactly!

SL: Well unless it's a case like this when you can do a calculation.

Erika: Yeah but it's all a … like it has nothing to do with the actual order you're flipping the triangle in. Like it's all a paper game kinda.

With her last comments, Erika seems to be making the key observation that regrouping is meaningful when doing a calculation (a "paper game"), but not when actually transforming the triangle. This is reminiscent of Jessica's distinguishing between two different kinds of order, and this seems to be a key point in the shift from referential mathematical activity

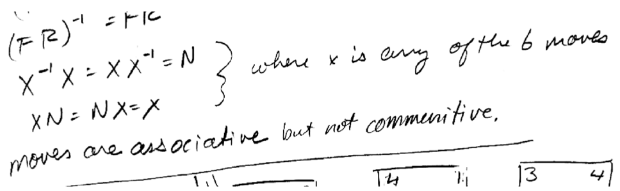

Figure 3. Erika included non-commutativity in her statement that the symmetries are associative

to general mathematical activity. The "paper game" is played at the general level. One manipulates the symbols without referring back to the physical context from which they arose.

From this point on, Erika was comfortable with the associative law. However when listing the rules for the symmetries of an equilateral triangle, like Cayley 150 years earlier, Erika included a non-commutativity caveat in her associativity statement (fig. 3).

It might be tempting to interpret Erika's claim that the "moves are associative" as a sign of confusion on her part since we typically refer to the *operation* as associative. However, since associativity is actually a property of a set-operation pair it is just as reasonable to attribute this property to the set alone as it is to the attribute it to the operation alone (perhaps more reasonable since the operation was often represented implicitly by concatenation). Recall that Cayley's definition was similar in this sense, stating that the "symbols are not in general convertible [commutative], but are associative" (Kleiner, 1986, p. 208).

Exploring connections to the research literature

I was fascinated by the students' struggles to make sense of the relationship between the associative and commutative laws. This was a complexity that I had not anticipated and that led me to wonder if children might have similar difficulties with these important properties in the context of arithmetic. My search of the research literature revealed that such difficulties had been reported both in research with children and in research with teachers.

Research involving children

Kieran (1979) reported that middle school students she observed experienced difficulties with bracketing symbols. For example, she observed that the students did not see the need for brackets in the arithmetic expressions they wrote because they intended them to be calculated in left to right sequence. Kieran (1979) also reported that when asked to bracket the operation at the end of the sequence $4 \times 3 + 1$, a student not only placed parentheses around the $3 + 1$, but also placed this term at the beginning of the sequence and wrote $(3 + 1)\ 4$. As describe above, I observed similar phenomena in my work with the undergraduate students. For example Erika made the error of commuting elements while regrouping them, and Kathy argued that parentheses were not needed in the triangle symmetry context.

Kieran interpreted the children's behavior as the result of a strong tendency to think in terms of a procedure that went from left to right (*e.g.,* $5 + 3 \times 7$ would involve adding 3 to 5 and then multiplying the result by 7). This echoes Kathy's argument that there was no need for parentheses because the various moves were just individual steps that would be performed in order from left to right regardless of how they were grouped. Further, Kathy and Erika explicitly discussed using an addition symbol to indicate the operation of combining symmetries. Their discussion suggested that Kathy was thinking of combining symmetries as a step-by-step process *and* that she saw addition as similar in this sense.

Kathy: So I guess we can call this F plus R and this F plus $2R$ or F plus R plus R.

Erika: Is it really adding though?

Kathy: I mean, well yeah. I mean you do one R and then you do another R.

Jessica and Sandra also described the operation of combining symmetries in this way. They initially used commas to indicate that they were combining symmetries and Jessica explained that, "comma is like a pause and you're moving on to a new clause in the statement. Okay so you do one step and then you pause and then you do another one." It is important to note that they *were* working with a step-by-step procedure that they symbolized by writing the symmetry symbols from left to right in the order they were to be performed. The binary operation concept began to emerge as the students started to develop rules for manipulating the symbols and was only crystallized when they formulated the definition of operation as they wrote their definition of group.

Research involving teachers

Other researchers (Tirosh, Hadass, & Movshovitz-Hadar, 1991; Zaslavsky & Peled, 1996) have reported difficulties with the associative and commutative properties in their work with teachers. For example, many of the participating teachers felt that these properties were logically dependent. This is perhaps not surprising since most operations in school mathematics either possess both or neither of these properties. Interestingly, some teachers argued that the commutative property was more general because it means changing order and the associative law is a special case of this (the order of operations).

Jessica and Sandra also wondered whether the associative law would follow from the commutative law.

Sandra: So if it's commutative, does that mean it's associative also? … I mean I know there are some things that are associative but not commutative, but can they be commutative but not associative?

Jessica: I don't think that the fact that it's commutative implies that it's associative, but it encourages it. I don't think that you could get that it's associative knowing that it's commutative.

Jessica was correct in saying that the commutative property does not imply the associative property. She was also correct in saying that commutative law can "encourage" the associative law. For example, cases of the form $(ab)a = a(ba)$ could be established using the commutative property. Furthermore, if one is thinking in terms of a procedure that goes from left to right, commuting elements seems to be a more fundamental violation of order than is regrouping, and so it could make sense to think that if elements commute they should necessarily be associative.

Zaslavsky and Peled (1996) concluded that, "mathematics teachers and student teachers both struggle with the concept of binary operation as well as with the commutative and associative properties that relate to it" (p. 77). They conjectured that these difficulties could be explained in part by the fact that both properties deal with the issue of order. They note that, in arithmetic, shortcuts using these properties are often justified with the same assertion that "order doesn't matter." Recall that Jessica and Sandra struggled to determine (and then articulate) in which sense order did or did not matter when combining symmetries of an equilateral triangle.

Discussion and concluding remarks

Zaslavsky and Peled's (1996) conjecture that much of the teachers' difficulties could be traced to the issue of order was based on the fact that a number of the teachers' errors were related to order, and the fact that these two properties are related to order in ways that tend to not be carefully distinguished in school mathematics. My observations of the undergraduates' mathematical activity, and their own explanations of their thinking, lend strong support to this theoretical explanation. Jessica and Sandra explicitly wrestled with the meaning of the term *order* as they discussed the associative property while reinventing the group concept.

Kieran (1979) conjectured that the middle school students she observed struggled with regrouping largely because they were thinking of the arithmetic operations in terms of a left to right procedure. The undergraduate students who participated in my study had similar difficulties, and the way they talked about the operation lends weight to Kieran's explanation. As was illustrated above, the undergraduate students' language suggested that they were thinking of the operation of combining symmetries largely in terms of a left to right procedure. This way of thinking makes perfect sense given that they were reinventing the group concept by mathematizing the physical (and sequential) activity of performing combinations of symmetry transformations. Thus, these struggles seemed to be a necessary part of the emergence of more general mathematical activity in which the operation was to be seen in terms of the emerging group axioms, rather than the in terms of the physical actions of the original symmetry context.

In the case of the undergraduate students I was working with, it appears that it was necessary for them to develop a language for distinguishing two kinds of order in the symmetry context. The first kind of order was the order in which transformations were to be performed on the triangle. This kind of order was very tangible in the original task context. When they began to develop rules for calculating combinations using written symbols, they moved to the level of referential activity - the sequences of symbols referred to ordered sequences of transformations. Later, there was a shift and the students' activity was focused on the rules for manipulating their symbols. Thus while they were still often operating at the referential level, they began to do some work with the symbols that was independent of the initial context. In this new mathematical world of rule-based calculations, a new kind of order became tangible – the order in which the students calculated the combinations. Because, in part, their initial language was not precise enough to capture the distinction between these two types of order, the students struggled first to separate them and then to articulate the difference. Ultimately they were able to successfully distinguish them, but like Cayley before them, their final definitions of group still suggested a connection between the associative and commutative properties.

What should we make of the connections between my analyses of the undergraduate students' mathematical activity and the difficulties observed in the research with children and teachers? Above I have argued that the three cases taken together suggest that difficulties with the associative and commutative properties may stem from 1) a tendency to think about expressions involving binary operations in terms of a sequential procedure and 2) a lack of preciseness in the informal language used in association with these properties. I would further argue that my findings and those of Kieran (1979) and Zaslavsky and Peled (1996) suggest that there is important mathematical work to be done by students on the way to developing a strong understanding of binary operations and their properties, and that if students do not have the opportunity to do this work (as the undergraduates did as they reinvented the group concept) difficulties can persist.

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