THE ORIGINALITY OF LEV TOLSTOY'S ARITHMETIC

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In the 19th century, educators began to discuss how to introduce arithmetic notions in elementary education and how they should be presented in textbooks in order to foster student learning, and new proposals emerged in several countries. In Russia, Lev Nikoláevitch Tolstoy (1828-1910), unhappy with the way arithmetic was introduced in schools, proposed a new approach and wrote a textbook dealing with arithmetic. Here I will describe some of its notable and visionary features.

The 19th century was a time of educational reform in Russia. Mathematics began to occupy a prominent position in the general education system. Karp (2014) notes the growing role that mathematics began to occupy in Russian classical schools (gymnasia), in which it was considered a discipline that facilitated mental development in addition to being a necessity in schools for the training of specialized technicians and natural scientists.

Tolstoy, best known as an author, was also an educational reformer eager to instruct the Russian population in keeping with the times in which he lived. He wrote textbooks for elementary education, created a school at his home Yasnaya Polyana [1], and experimented there with his own methods of pedagogy.

Tolstoy traveled to France and Germany in order to have contact with educators and to visit schools. For example, Birukov (1906) notes that Tolstoy had the opportunity in Germany to meet the educator and sociologist Karl Friedrich Fröbel who introduced Tolstoy to the ideas of his uncle, the educator Friedrich Fröbel, who had in turn been a student of Pestalozzi.

Tolstoy visited many schools in Marseille and acquired an insight into education in France; in all popular education establishments in this city, programs include catechism, sacred history, in general the four rules of arithmetic, spelling, and French accounting. Tolstoy was not satisfied with the way children learned in the schools in Marseille; he noted that no student in these schools was able to solve the simplest problems of addition and subtraction (Tolstoy, cited in Vessiot, 1895, pp. 210–211).

According to Vessiot, Tolstoy is quite radical in his ideas in his essay 'La liberté dans l'école' [2]. He does not accept schooling by force, but only a free school to which the students go if they want, when they want, as much as they want, and where they learn whatever they want (Robertson, 2016). This shift of control from the teacher to the student is, according to Vessiot, Tolstoy's most daring and original conception in the field of education. It is also the suppression of all pedagogy, as there is no more direction in teaching. Tolstoy

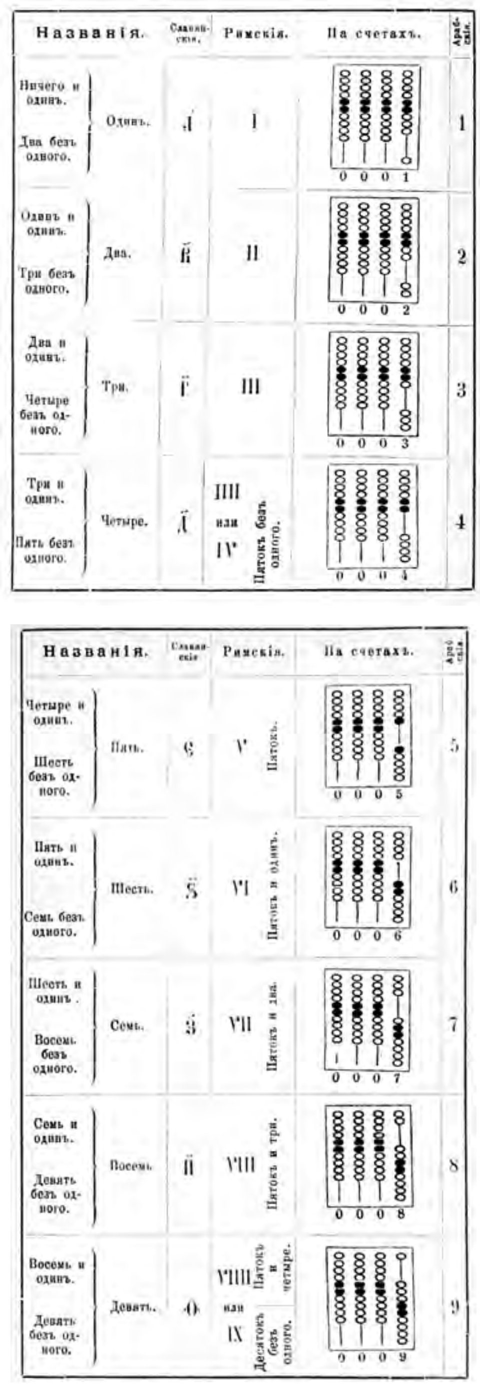

Figure 1. Representations of the numbers 1 to 9.

defends the idea that if the student has difficulties, the teacher should not blame the student, but the teaching method itself; the teacher must invent new methods if necessary, “Teaching is an art, finality and perfection are achievable—and development and improvement are infinite” (Tolstoy, 1936, p. 145–146, my translation).

Tolstoy applied these ideas at this school at Yásnaia Poliana. In this school, children arrive happy because they come freely. They sit wherever they want, on benches, tables and windowsills, and do not need to do homework (Vessiot, 1895, p. 241). It was for this school that Tolstoy wrote his Arithmetic, as part of his 758 page Primer [Азбука], published in several volumes in 1872. In 1874 the Arithmetic was published as a separate book and a revised ‘New Primer’ came out in 1874–75, without the Arithmetic. In 1913, after Tolstoy’s death, an edited version of the Arithmetic was republished, reuniting it with some instructions for the teacher from the original Primer that were omitted from the 1874 version; here I refer only to that edition.

The use of the abacus in Tolstoy’s Arithmetic

Tolstoy begins his book with representations of numbers (see Figure 1). He presents a table, containing five columns:

1) the name of the number;
2) its representation in Slavic numbering (which was used until the 17th century);
3) in Roman numerals;
4) in the abacus, and
5) in Arabic numerals.

The bead abacus, that appears in the fourth column, contains 10 beads in each column, 8 clear and 2 dark. Tolstoy pays great attention to explaining how to use the abacus, using many images. His use of the abacus is notable. The abacus was widely employed in everyday life in Russia, but there are no clear examples of its use in elementary education prior to Tolstoy’s Arithmetic. Pestalozzi advocated the use of visualization in teaching, and this may have inspired Tolstoy’s use of the abacus.

The student sees numbers represented on the abacus first before seeing the representation in Arabic numerals. In his recommendations to the teacher, Tolstoy emphasizes the order of presentation of the representations of the numbers: learn the names of the numbers, read in Slavic numbers, write in Roman numerals, place on the abacus and write in Arabic numerals.

Tolstoy’s Arithmetic continues with tables like that in Figure 1 for several pages, including multi-digit numbers, always maintaining the same five columns. This use of different representations of the same number anticipates the emphasis on multiple representations in mathematics education, from the work of Claude Janvier and Gérard Vergnaud in the 1980 and 1990s, to the present.

The tables include representations of numbers with up to four digits, as shown in Figure 2. This is a departure from the guidance of August Grube (1816–1884), a German author of mathematics textbooks who had a strong influence on Russian pedagogues. He divided the teaching of arithmetic into parts. The part for the first school year should include only numbers up to 100 (Silva, 2015). It also stands in contrast to contemporary curricula which sometimes restrict the numbers learned in the first year to 10, 20 or 31 (so the calendar

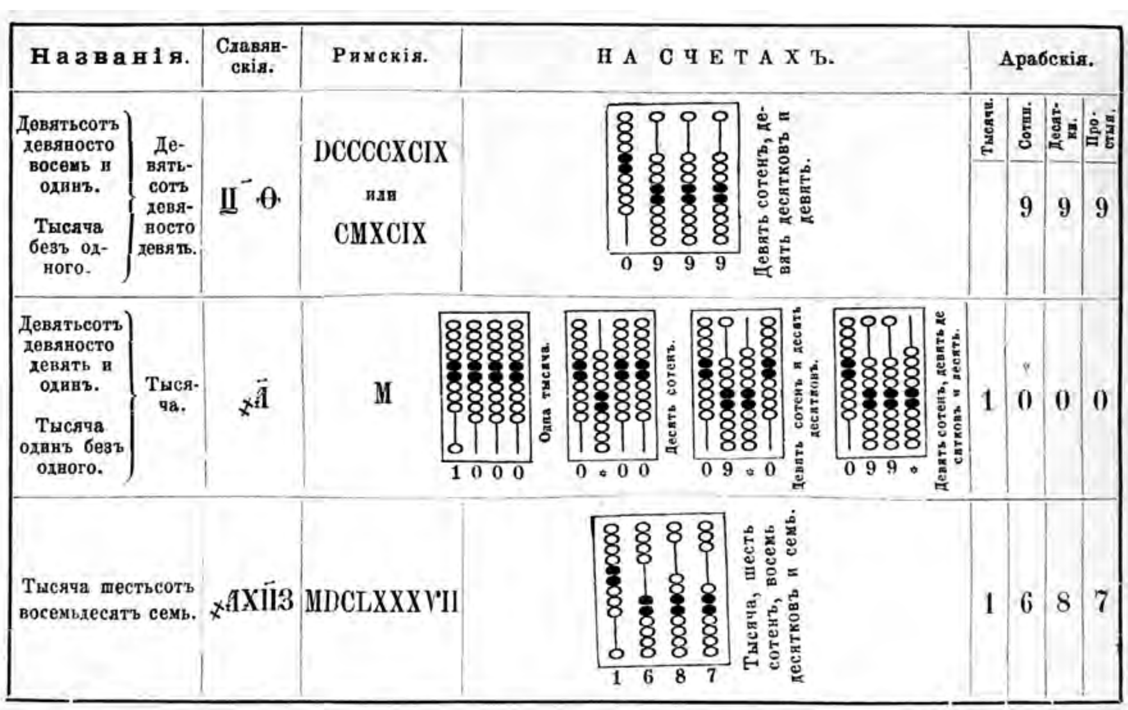

Figure 2. Representations of four digit numbers.

Figure 3. Adding 7 and 2 using the abacus. [7 and 2. Simply put 7 and 2 more. Count how many you are left with and write 9].

can be taught). Four different representations of the number 1000 on the abacus are shown. These different representations of the same number reinforce the understanding of the decimal system.

After the tables showing representations of the natural numbers, material on operations with natural numbers follows. Tolstoy begins by showing the sum using Cyrillic numbers, then with Roman numerals, only after that introducing Arabic numerals and the abacus. He advocates that calculations with Arabic numbers should be based on the use of the abacus. The teacher should not rush to show addition only in Arabic numbers before using the abacus.

An example for addition is shown in Figure 3. The procedure consists of sliding the beads of the abacus downwards, first to represent seven units, and then adding two more. This activity with the abacus, sliding the beads, associates a tactile activity with an action of the mind.

The addition calculations increase in difficulty, starting with those involving numbers with one digit, then with two or more digits such as: 12 + 6; 23 + 45; 1284 + 5413. When the numbers involve 4 digits, the calculations on the abacus are shown step by step (see Figure 4).

Subtraction is also shown on the abacus, in a similar way. An example is shown in Figure 5.

Operating with other number bases using the abacus

Tolstoy explores the possibility of performing arithmetic operations in other bases, such as 5, 6, 7, 9 and 11, on the abacus. He does this to show the existence of other numerical bases and to challenge the imagination of the student. This expansion to operations on other bases is undoubtedly interesting, but even more original is his suggestion to perform them on the abacus. He explains that it is very simple to create an abacus for another base by removing or adding beads to each column.

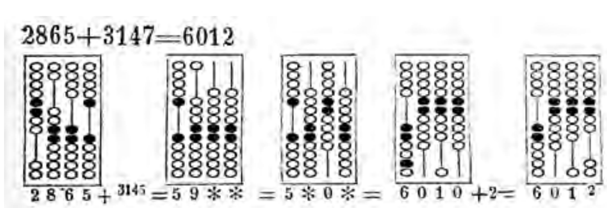

Figure 4. Addition of 2865 + 3147 using the abacus.

Figure 5. Subtraction of 200 – 80.[First, put 200 with two beads of hundreds on the abacus. Next, to take 8 tens, you need to turn one of the hundreds into 10 tens. From these ten tens-beads, take 8 tens. What is left is 120.]

The first suggested activity is to use base 6 to perform an addition and subtraction. Figure 6 shows the addition of 3 and 4 in base 6, resulting in 11_6, along with a representation of the base 6 abacus showing the result. The subtraction is more complicated. The explanation states that one first shifts up one bead from the units column. As one has no more beads in that column, one turns to the second column, shifting up the single bead there and shifting down all the beads in the unit column. Then one can shift up the remaining three beads in the unit column, completing the subtraction and leaving 3_6.

In addition to examples in base 6, Tolstoy suggests activities in bases 9 and 2. In his recommendations to the teacher, Tolstoy explains that it is very useful to perform various addition and subtraction exercises in other number bases in order to better understand the operations and to see that the same numerical laws apply in all bases. This advice anticipates the work of Dienes (1967) in the mid-20th century.

Figure 6. Addition and subtraction in base 6.

1404 − 839.

1404 (=4̇04)—800=604
604 (=5̇04)— 30=574
574 (=56̇4)— 9=565.

1482—789.

3̇7̇
1482̇
— 789
—693

Figure 7. Two subtraction procedures.

687×325=

687
×325
3435
13740
206100
223275

Figure 9. Reduced multiplication algorithm.

Arithmetic operations without the use of the abacus

After giving many exercises on the abacus with numerical examples, Tolstoy presents an algorithm without using the abacus. Figure 7 shows two subtraction algorithms.

In his recommendations to teachers, Tolstoy says that the teacher must go through all the exercises in detail according to the student's capacity, including a mix of addition and subtraction exercises to make the connections between them clear. This same guideline is found in Grube's work (Silva, 2015).

Tolstoy does not use the abacus to explain multiplication and division operations. However, these operations are presented simultaneously to show that they are inverse operations. Multiplication is presented as successive sums, and division as successive decreases or subtractions. At first, multiplication of two digit numbers is presented in a way similar to that of the Egyptians, using successive duplications. For example, in Figure 8 the multiplication 35 × 12 is shown. Another way to perform multiplication is by decomposition. Tolstoy breaks down the multiplication 37 × 45 into five operations:

Multiplying the tens 30 × 40 results in 1200,

30 × 5 gives 150

7 × 40 gives 280,

multiplying the units gives 35,

adding up the products gives 1665.

He shows that this procedure can be done starting with the units or tens, and that the result will be the same.

Tolstoy also includes a reduced multiplication algorithm similar to one currently in use (see Figure 9).

'Word problems' applying arithmetic to more or less 'realistic' contexts, the ages of parents and children, inheritances, buying and selling, land measurements, *etc.* have been a feature of mathematics texts since the time of the Egyptians, and Tolstoy's book does not escape this tradition. The contexts chosen by Tolstoy are oriented towards rural children, that is, the mass of children needing an education with whom Tolstoy worked.

35×12.

1 разъ 35= 35.
1 разъ 35= 35.
2 раза 35= 70.
2 раза 35= 70.
4 раза 35=140.
4 раза 35=140.
8 разъ 35=280.
4 раза 35=140.
12 разъ 35=420.

35×12=420.

35 сложить 12 разъ, будетъ то же самое.

Figure 8. Multiplication by successive duplications.

Decimals and fractions

The second part of the book is dedicated to the theme fractions, and starts with decimal numbers. Here again the abacus is used, as well as tables indicating place values. Tables are used especially when it comes to performing multiplications. Algorithms to perform multiplication with integers and decimal numbers are compared. For example, Figure 10 compares 6784 × 356 and 6,784 × 3,56 [3]. The digits in 6784 × 356 are written in columns according to their place value: 6 in the thousands, 7 in the hundreds, 8 in the tens and 4 in the units. The algorithm is the one known and presented in many current arithmetic textbooks. The digits of 6,784 × 3,56 are written in the same table. In this case, the numbers must be shifted to the right, because when multiplying hundredths by thousandths, the result is in millionths.

Примѣръ умноженія.

	Милліоны.	Сотни тысячъ.	Десятки тысячъ.	Тысячи.	Сотни.	Десятки.	Единицы.	Десятыя.	Сотыя.	Тысячныя.	Десятитысячн.	Стотысячныя.	Милліонныя.
Цѣлыя				6	7	8	4						
				×	3	5	6						
			4	0	7	0	4						
		3	3	9	2	0							
	2	0	3	5	2								
	2	4	1	5	1	0	4						
Дроби							6,	7	8	4			
						×	3,	5	6				
								4	0	7	0	4	
							3,	3	9	2	0		
						2	0,	3	5	2			
						2	4,	1	5	1	0	4	

Figure 10. Examples of multiplication of integers and decimals.[In the columns appear, from left to right: million, hundred thousand, ten thousand, thousand, hundred, ten, unit, tenth, hundredth, thousandth, ten-thousandth, hundred-thousandth, millionth.]

Tolstoy also presents arithmetic operations and word problems with ordinary fractions. For example:

> A man gave the poor $1/2$ of his money, $1/4$ for the church, and had three rubles left for himself. How much money did he have? How much did he give to the poor? What about the church?

Such problems of donations to the poor and the church are frequent in Tolstoy's Arithmetic. They are associated with a voluntary intention to educate children to the practices of charity and religiosity.

Conclusions

Tolstoy's Arithmetic shows his knowledge of contemporary pedagogical thinking, for example, the work of Pestalozzi and Fröbel, as well as his agreement with it. Thus, Tolstoy is part of the pedagogical movement of the 19th century that advocated making use of tactile teaching materials, such as the abacus, in elementary mathematics in order to make the counting numbers and operations on them concrete.

Tolstoy's work was innovative, but seems to have had little impact on the teaching of arithmetic in Russia in the nineteenth century. A possible reason for this is that it departed so much from other textbooks of that century. However, its existence shows that in imperial Russia there was at least one author interested in the mathematical education of the general population and that he was aware of pedagogical proposals that were being made in other countries.

Many innovative features of Tolstoy's Arithmetic anticipate contemporary teaching methods and reforms. He uses the abacus for both learning to represent numbers and to add and subtract. He presents multiple representations of numbers, including Slavic and Roman symbols used in everyday life and suggests that the teacher moves from one representation to another so that the students are not bound to a single numerical system. He uses place value tables to distinguish units, tens, hundreds, thousands, *etc.* and to perform arithmetic operations. He uses the abacus not only in the decimal system but also in other bases. He introduces decimal numbers before ordinary fractions, which was not common at the time. He includes didactic guidance for teachers to know how to use the book.

Many of the approaches Tolstoy introduced in his Arithmetic are being used in contemporary schools or have been advocated in recent reforms. An historical perspective reminds us that these are approaches are not 'new-fangled' but represent educational thinking that goes back many years. They are as applicable to the education of poor children in the 19th century Russian countryside as they are to students in today's schools. Informing teachers of historical methods and their application to current teaching represents an important contribution made by the history of mathematics education, showing a link between the past and the present.

Notes

[1] Yasnaya Polyana (Ясная Поля́на, 'Bright Glade') is about 200 km south of Moscow.

[2] Originally published in Russian in 1862.

[3] Tolstoy followed the European convention of marking decimals with commas.

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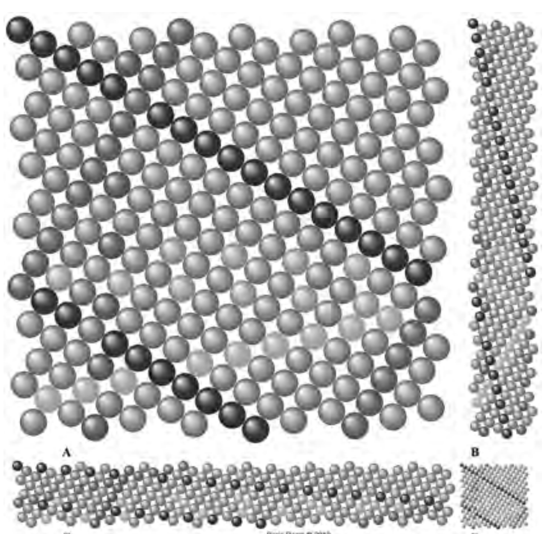
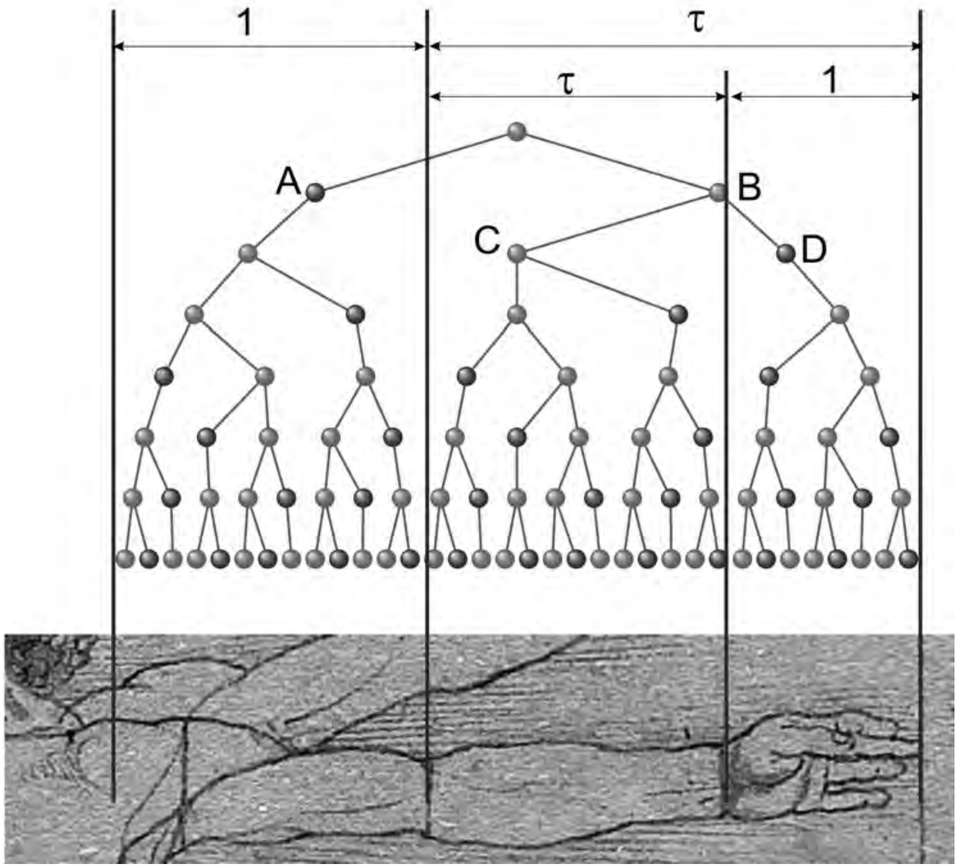


The image on this page, and on pages 7, 29 and 55, are taken from the book "Double Helix of Phyllotaxis: Analysis of the Geometric Model of Plant Morphogenesis" by Boris Rozin, published by BrownWalker Press in 2020. For more information, see http://www.brownwalker.com/book/1627347488.



See p. 54