A Theory of Intellectual Development [1] [2]

JERE CONFREY

PART 1*

In the last decade of the twentieth century, we in mathematics education continue to draw most heavily on the work of two scholars, Jean Piaget and Lev Vygotsky, for our theories of intellectual development. These theorists have offered us powerful insights into the human mind and its development, radically transforming our understanding of how children view the world and about how we understand ourselves as individuals within a cultural and historical setting. While these insights inform us about important ways to approach education, they also need revision in the light of the current cultural and historical conditions in North America. A variety of factors has created a critical need to revise such theories; these factors include: changing demographics, a reform climate in education, the creation of new technologies, the press of environment concerns, and issues of power and oppression.

In this paper, I propose to provide brief summaries of radical constructivism (as one interpretation of Piaget), and the socio-cultural perspective (as one interpretation of Vygotsky). The summaries will include major principles, primary contributions to mathematics education, and potential limitations. In a previous paper [Confrey, 1994b], I warned readers of combining these theories too simplistically. In this paper, I introduce a new theoretical perspective which integrates the two theories by means of feminist perspective.

RADICAL CONSTRUCTIVISM

Radical constructivism has one set of roots in the work in the philosophy of science. Starting with Karl Popper [1962], philosophers of science began to challenge the view of science as an accretion of information through careful application of "The Scientific Method." Popper argued that it was falsification rather than verification and accretion that directed the development of scientific knowledge, and he drew attention to the role of critical experiments in determining progress. Thomas Kuhn [1970] and Stephen Toulmin [1972] followed by presenting differing accounts. They suggested that scientific progress could not be explained adequately by the falsification of empirical results alone. They argued for the analysis of larger structures that would include individual knowledge claims, methodologies, standards, even the forms of proof themselves. Scientific truth began to lose its simple connections to reality. Stability rather than certainty could be achieved based on the robustness of the theoretical or paradigmatic framework within a scientific community. Kuhn proposed a revolutionary view, arguing for the incommensurability of paradigms—and for a process of replacement rather than of gradual evolution. Stephen Toulmin proposed an evolutionary view of the development of knowledge, in which it was during periods of change that one's fundamental commitments were revealed.

Philosophy of science

Lakatos [1970], virtually the only philosopher of the time to apply arguments from the philosophy of science to mathematical knowledge, proposed an alternative framework in which the theoretical hard-core of a research programme remained unassailable directly. It was surrounded by a protective belt of theories and a skin of empirical claims all of which could be abandoned under pressure, if only to ensure the continuation of the hard-core. In his well-known book, *Proofs and refutations* [1976], Lakatos demonstrated the fruitfulness of applying such a perspective to mathematics, producing a compelling "rational reconstruction" of the Euler conjecture concerning the edges, vertices and faces in a polyhedron. His goal was to challenge the formalists who portrayed mathematics as "authoritative, infallible, irrefutable" [p. 5], and to elaborate the point that "informal, quasi-empirical, mathematics does not grow through a monotonous increase of the number of indubitably established theorems but through the incessant improvement of guesses by speculation and criticism, by the logic of proofs and refutations" [p. 5]. Later, particularly with the introduction of computer implemented and enhanced proofs, philosophers like Tymoczko [1979, 1984], Kalmar [1967] and others begin to document that mathematics too is subject to challenge and is quasi-empirical in nature.

All of these theories coalesced to change the view of science, and to significant but lesser extent, mathematics, to make them vulnerable to systematic change, revision, debate, and rejection. All struggled to explain the twin processes of stability and change as they admitted relativism into the scientific and mathematical enterprise. And all of them challenged a simplistic view of objectivity; in each theory, the subjective, either as a psychological process or a sociological process, was inexorably involved.

In education, particularly in science education, these philosophical debates quickly influenced discussions of student learning. The classic paper by Karmiloff-Smith and Inhelder [1975], "If you want to get ahead, get a theo-

* This is the first part of a three-part article. The other parts will be published in consecutive issues of the journal

ry", documented that student approach and observation was also theory laden. Driver and Easley [1978] applied such theory-driven views to suggest that students' misconceptions might be better looked at in the light of conceptual frameworks in which there is an interplay between mental models and the sensory impressions of a phenomena From this developed a robust series of research studies documenting student conceptions and how these exhibited persistence, pervasiveness, rationality, originality, and intellectual integrity Hawkins, Apelman, Colton, and Flexner [1982] and Brousseau [1984] took the position that passage through these critical barriers, these epistemological obstacles, constituted an important part of learning. In particular, these pioneering researchers led their communities to recognize that much of what was being labeled as student errors and misconceptions actually represented legitimate alternative viewpoints. These researchers challenged traditional forms of assessment and learned to listen to students, to propose conditions for conceptual change, and to investigate larger systems of cognitive structures. [See Confrey, 1990a, for a review of conceptions research]

Piaget

The work of Piaget was exceptionally well-suited to form the conceptual basis for linking the philosophy of science to learning theories. First of all, Piaget was a biologist who became a child development specialist, hence he naturally incorporated an evolutionary perspective into his theories. Philosophers striving to make sense of the history of science and developmentalists examining the history of children's ideas formed an intellectual bond.

Piaget was also an excellent candidate for bridge-building because he examined the development in children of fundamental organizing concepts such as space, time, and number and identified the developmental changes that children proceed through given appropriate experience, time, and support. The effect of combining Piagetian work and the philosophy of science was to emphasize the importance of epistemological issues and to challenge the assumption that children's worlds were simply inadequate or incomplete representations of adult worlds Such contributions were necessary for the formation of a radical constructivist perspective in mathematics education

In this paper, I distinguish between constructivism and radical constructivism. Constructivists argue for the importance of children's active participation in the building up of concepts. They reject the view that children's minds are blank slates, and they believe that there must be significant discussion and interaction around the variety of strategies that students propose. However, for them, the endpoint of instruction, the character of mathematical knowledge, is seldom questioned Constructivists generally seek to reproduce in their students the same mathematical ideas that they themselves hold and that dominate modern mathematics. Little investigation is made of the meaning of the mathematical ideas through historical, cross-cultural, or cross-disciplinary methods. Generally, constructivism is replicative in its goals and only modestly re-visionary The methods of instruction are reformed, and the focus is more psychological then epistemological.

Radical constructivism is a theory whose roots lie in the rejection of illegitimate claims for epistemological certainty If one accepts the position that knowledge cannot be shown to represent reality in some iconic way, as a picture of the world, then one is left with a more subjective construction of reality, subjective in the sense that one abandons the effort to factor the human subject out of the process Although the radical constructivist is relativistic in contrast to the realist, that relativism is tempered by the stability which is achieved by the individual in relation to his or her experience Others exert a significant influence on those experiences The radical constructivist program assumes that the individual makes sense of experience in order to satisfy an essential need to gain predictability and control over one's environment Many of the efforts of researchers in this tradition have been devoted to describing how the individual builds up (rather than passively acquires) knowledge of the world

A framework for radical constructivism

In Confrey [in press], I argue that the **radical constructivist program** can be summarized by four "planks':

1. Genetic epistemology

The construction of knowledge occurs over time; to understand an idea, one needs to examine its construction, ontogenetically and phylogenetically Piaget explicitly rejected the view that "epistemology is the study of knowledge as it exists at the present moment; it is not the analysis of knowledge for its own sake and within its own framework without regard for development" [Piaget, 1970, p 1-2]. This claim commits an educator to "creating the need" [Confrey, 1993] for an idea, rather than informing someone of the contents of a knowledge claim According to Piaget [1970], "The fundamental hypothesis of genetic epistemology is that there is a parallelism between the progress made in the logical and rational organization of knowledge and the corresponding formative psychological processes" [p. 13]. This parallelism need not be assumed to argue for a recapitulation, that development follows historical routes, but argues only that historical routes are a rich source of diversity that can inform us about alternative developmental routes.

2. Radical epistemology

"[K]nowledge does not reflect an "objective" ontological reality, but exclusively an ordering and organization of a world constituted by our experience. The radical constructivist has relinquished "metaphysical realism" once and for all, and finds himself in full agreement with Piaget who says, "Intelligence organizes the world by organizing itself" [von Glasersfeld, 1984, p. 24]. Or, citing Vico, "Veum ipsum factum—the truth is the same as the made" [ibid., p. 27], and "Human knowledge is nothing else but the endeavor to make things correspond to one another in shapely proportion" [ibid., p. 29] By these quotes, we see that radical constructivism has two parts: 1) Constructivism rejects a picture theory of knowledge (that we are progressing towards an increasingly accurate view of the "way things really are"). 2) Constructivism entails a requirement that to know something is to act on it, so that knowledge consists of actions and reflection on those actions. "...[A]ll knowledge is necessarily a product of our own acts" [Confrey, 1990b, p 108].

Corollary 1: Recursive fidelity
Constructivism is subject to its own claims about the limits of knowledge. Thus, it is only true to the extent that it is shown to be useful or viable in allowing us to make sense of our experiences and to make predictions.

Corollary 2: Observer's presence
In every epistemological claim, an observer is present. Claims cannot be subjectless; a problem is defined by a proponent. When one accepts an epistemological claim, one is agreeing, or rather agreeing to agree, with the proponents. This means when we seek to speak of cognition, education, problem-solving, mathematics, or learning and teaching, we must take particular care to recognize the role of the observer in the description and analysis of the problem. In the radical constructivist research program, this has meant establishing clear methodological guidelines concerning the importance of "close listening" [Confrey, 1993], the careful conduct of clinical interviews and the articulation of models of student thinking [Cobb and Steffe, 1983]. More recently, I have further revised this discussion to include the importance of acknowledging the role of the observer's perspective in the development of student voice and of the importance of using voice to aid us in our understanding of our own epistemological beliefs and commitments [Confrey, 1994b].

3. Scheme theory
The first plank specifies that knowledge can only be understood by examining its genesis. The second plank rejects the view that what is eventually asserted to be knowledge can not be assured to be "the way the world really is." Both planks lead to the identification of learning, coming to know, as a critical site for investigation. As Piaget has said, "Nothing could be more accessible to study than the ontogenesis of these notions. There are children all around us. It is with children that we have the best chance of studying the development of logical knowledge, mathematical knowledge, physical knowledge and so forth" [Piaget, 1970, p. 14]. Furthermore, the genetic epistemological position (as stated by Piaget) is that .. "knowing an object does not mean copying it—it means acting on it. It means constructing systems of transformations that can be carried out on or with this object. Knowing reality means constructing systems of transformations that correspond, more or less adequately, to reality. These are more or less isomorphic to transformations of reality.[3] The transformational structures of which knowledge consists are not copies of the transformations in reality; they are simply possible isomorphic models among which experience can enable us to choose. Knowledge, then, is a system of transformations that become progressively adequate" [ibid., p. 15]. How can these transformations be understood?

For Piaget, "the operative aspect of thought deals not with states but with transformations from one state to another" [1970, p. 14]. This occurs first at the level of action, goal-directed activity. When these actions accomplish our goals, we abstract from it. There are two kinds of abstraction: simple abstraction, derived from the object, and reflective abstraction, derived from action on the object. This is reflective in that it moves from action to operation, and in that it involves a "reorganization at the level of thought itself" [ibid., p 18]. "Reflective abstraction is not based on individual actions but on the coordination of actions" [ibid., p 18]. Operations are the result of reflectively abstracting actions. Operations possess four qualities: 1) they are internalized actions, 2) they are reversible, 3) they suppose some invariant, and 4) they exist within a system of operations. Scheme theory is a way to discuss the development of stable and predictable courses of action. For Piaget, a scheme is "whatever is repeated or generalizable in an action" [ibid., p 34].

Schemes involve the anticipation and/or recognition of a situation. For the constructivist, a primary role is assigned to "differences." First, there was a difference, a perturbation, which is noticed. For the constructivist, the child must "emerge from embeddedness" [Kegan, 1982, p 78] in that the newborn is considered to live in an objectless, continuous, timeless "world in which everything sensed is taken to be an extension of the infant" [ibid., p 78]. That is, there is no distinction between what is the infant and what is not the infant; for the infant, there is no boundary. The infant learns to create distinctions which lead to his/her "hatching out" [ibid., p 80]. Without distinction, there is no pattern. A difference creates a perturbation that is a call to action. This perturbation and action to resolve the perturbation are internalized through the process of reflective abstraction. The overall structure that is created, if the sequence of perturbation, action, reflective abstraction is repeated until the action is formed into an operation, is labeled a scheme (Figure 1). For the radical constructivist, the unit of analysis is a scheme with its genesis and modification. a scheme for the constructivist provides that Vygotsky called the "investigable microcosm" [Wertsch, 1985, p. 193]. In contrast to Piaget, for Vygotsky the investigable microcosm is the word.

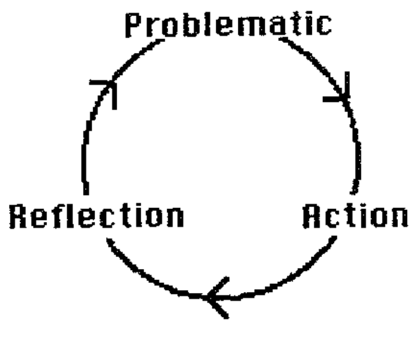

Figure 1

It follows from scheme theory that the child proceeds through stages of development wherein the constructs may not mirror those of adults. Children's views are not miniature adults' views; nor are their views merely missing pieces; nor are they inadequate for the purposes for which they have been built. Children's views are built differently, because the entire task situation may be viewed differently, and because a child's sensory perceptual world for building concepts is different from adults A child's sensory and perceptual world responds to the world that the child has built, the experiences she or he has had, and the theories she or he has created The implication of this is that when examining a child's performances, utterances, preferences, or ways of talking, one must not presume that one's own views of knowledge provide sufficient or adequate preparation for understanding that child. It is also suggested by this that the knowledge of the child is epistemologically intriguing, for it provides legitimate (and useful) alternatives to adult knowing.

The importance of scheme theory lies not only in the identification of schemes but in the recursive building potential created by knowing about schemes:

> ...We build this world for the most part unawares, simply because we do not know how we do it This ignorance is quite unnecessary Radical constructivism maintains... that the operations by means of which we assemble our experiential world can be explored and that an awareness of this operating... can help us do it differently, and, perhaps, better [von Glasersfeld, 1984, p. 18]

4. Model building and the construction of others

The discussion of "others" in von Glasersfeld [1982] evolves from a discussion of young children who imbue objects with life, and later give it up, because it does not add prediction and control.[4] He argues for the viability of creating "*models* of Others who...come to be considered as perceivers, knowers, and intentioned actors, because such an investment does, indeed, make them more predictable" [p 631]. The emphasis on models is essential for it emphasizes that no privileged access is accorded to our knowledge of others We remain within our subjective confines, building viable models of the others in our environment through experience. As von Glasersfeld emphasizes, " .. when a subject feels or says that it *understands* an Other, this implies no more than that the cognitive structures which the subject has attributed to its *model* of the Other have so far, or once more, turned out to be viable in the interpretation of the subject's experience of the Other" [p 632] Researchers in the constructivist tradition have stressed that such models are the only possible product of investigations of children and that they are "the mathematics of children, even though they are not taken to characterize how mathematical knowledge of children really is [Cobb and Steffe, 1983, p 13].

I have argued that a constructivist view of instruction recognizes the key role for "reflection, communication/interpretation, and the use of resources" [Confrey, 1985]. Rejecting the view that communication is the transfer of information, I stressed the importance of interpretative acts More recently, constructivist researchers [Cobb, Wood, and Yackel, 1991] have proposed that in a community of learners, there develops a form of knowledge they describe as "taken as shared knowledge," a phrase used to indicate the tentativeness of communication In our own work, we have preferred to use the term "agree to agree" to emphasize that it is the participants, and not some external observer, who agree to assume successful communication, until such time as those agreements are called into question by either party.

The contributions of radical constructivism

The radical constructivist perspective has contributed significantly to reform in mathematics education Its primary contribution has been to challenge the stark evaluative climate of the mathematics classroom Instead of the quick labeling of student answers as right or wrong and the dismissal of differences, the radical constructivist viewpoint has legitimated diversity among individuals as a fundamental part of learning It has also elevated the discussion of epistemological issues as central and problematic, rather than making the assumption that established knowledge is unassailable As stated by Susan Jo Russell, constructivism has enabled us to recognize "the complexity of apparently simple ideas" [1993, p 7]. In part, this is because constructivism has demonstrated that understanding what children do requires one to decenter from an adult perspective and to imagine how a child's actions and utterances make sense from the child's perspective. And in doing so, mathematics has become an issue of communication and interpretation, and not just the documentation of logical necessity Finally, constructivists have been significantly effective in challenging the passive mode of learning, putting into practice the use of manipulatives, contextualized problems, the use of small group work, and the coordination of actions, operations and representations

To demonstrate these ideas, consider the following example from earlier work by our research group [Confrey, 1991a] We demonstrated how a college freshman named Suzanne built a number line to display a set of historical events from the Big Bang to the present She did so by building a scale that had two different kinds of units (Figure 2). Her large units were marked in powers of 10 (multiplicatively), and her small units were in additive increments counting from the lower power of ten to the higher one (for example, from 10^7 (10,000,000) to 20,000,000, 30,000,000, 40,000,000 ..90,000,000 and 100,000,000 (10^8). This gave her nine additive intervals between her powers of ten At first, we evaluated her display as inadequate as it incorporated a change of units. However, on reflection, we realized that it had a great deal of legitimacy and that it was we who were responding inflexibly Our reasons for changing our opinion of the epistemological validity of her approach included: 1) her facility in working with intervals which cut across her multiplicative intervals, 2) her flexibility in moving among the representational forms (from scientific notation to extended decimal notation) and 3) the insights her display provided about scientific notation Due to her work, we recognized that her visual display paralleled the arrangement of scientific notation which is segmented into the familiar counting numbers between 1 and 10 (like her additive units) and orders of magnitude (like her multiplicative units). Finally, her work set us an appropriate challenge: how to devise a task that would encourage a student to want to create a purely logarithmic scale.

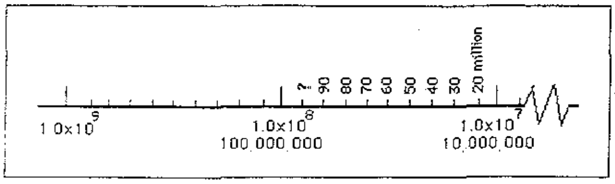

Figure 2

A second example comes from research conducted by Russell and Corwin [1991][5] and demonstrates the role of communication, interpretation and mathematical reasoning in the constructivist program In a project entitled "Talking Math," project teachers worked on the construction of pyramids. They cut out and built pyramids with regular bases, triangular, square, hexagonal, octagonal from two dimensional networks They were asked to formulate conjectures about the relationship between the number of corners and the number of edges and sides. In form, this does not appear to be a very different lesson than what would be advocated by a proponent of discovery learning; however, at this juncture, the instructors departed from the discovery learning format They encountered the following claim by teachers: "The top of a pyramid," claimed some teachers, "should not be counted as a corner It is a point. Only those at the base count as corners."

The instructors encouraged the teachers to develop their arguments. Some of the arguments which evolved included:

1. A corner should only have three planes coming into it (which is true only of the base corners and the point of the tetrahedron where, as the teachers pointed out, any side could function as the base)
2. The handout defined pyramids as "having a point," thus legitimating their distinction
3. Corners could be defined as formed by two lines (as in the corner of a sheet of paper) and thus each "corner" of the pyramid is really three (or more) corners.
4. Street corners, corners in a room, all have three planes coming together. However, from the outside, they look like points.
5. Before the pyramids were constructed, they were represented by two-dimensional networks. In each of these networks, the corners at the base appear quite different than did the "point" which appeared multiple times at the top of each triangle which protruded from the base

Now, the instructors could have simply restricted the debates by introducing the formal term, vertex, and using it to apply to both points and corners. However, by not doing so, they witnessed teachers engaging in spirited debates which were thoroughly mathematical in their character and taught the teachers a great deal about forms of argument. Numerous teachers in the group followed up using the sheet with students and reported similar discussions in those settings

Limitations of the constructivist perspective

It is difficult with a powerful theoretical perspective to differentiate between the qualities of the theory which are limited because of their failure to be successfully implemented and those which are limitations inherent in the perspective.

I want to suggest three limitations of the constructivist program:

1. *Many constructivists assume an incremental view of knowledge construction.* As a result, most proponents of constructivism have focused attention on the elementary grades, to the neglect of secondary and post-secondary instruction. That this result is partially inherent in the theoretical perspective wherein a) Piagetian stages imply that there is a movement from concrete operations to formal, abstract operations with developmental level and, thus, leave intact the beliefs that at the higher cognitive levels, contextual influences recede and reliance on formal manipulation of symbols is judged to be a more sophisticated way of thinking; and b) a rationalist view of knowledge development is assumed which includes the assumption that there are well-defined paths to complex forms of thinking, and that the movement is always from simple to complex. Thus, for instance, in the research on the counting types to additive structures to multiplicative structures [Steffe, 1994], we see evidence of a systematic path through counting types to unit types that is of increasing scope but seldom acknowledges competing structures and alternative approaches I have argues against a singular approach in my own work on "splitting" in which I have posited complementary but independent roots of multiplication/division and ratio to those of counting [Confrey, 1994a]. In this argument, geometry plays a significant role as a divergent form of thought from number. An alternative view of knowledge development that makes it equally necessary to consider the secondary level is to assume a less incremental view of knowledge development in which complexity can be lived in and comprehended with increasing depth Within such a view, context does not simply create the purpose for the goal-directed activity, but creates participation structures that encourage increasing awareness of complexity [Sabelli, 1993; Lave and Wenger, 1991; Smith, 1993].

2. *Constructivist approaches can be criticized as positing a universalist or essentialist view of cognition across classifications except age.* Viability typically seems to explain how the character of knowledge changes as a function of age, but less attention is paid to how viability must also lead to differences among children of different cultural background, race, or gender. Constructivism has resulted in the documentation of diversity in student method, but little or no discussion exists in the literature to explain systematic differences among classifications of student participants according to culture, race, or gender One possible explanation lies in the tendency for the constructivist program to assert a heavy dependence on the autonomy of the individual As I have written myself, autonomy is the backbone of constructivism. However, emphasizing autonomy can lead one to devalue or misjudge individuals who resist exhibiting independent judgment, preferring perhaps to gain group consensus, to avoid attention, or to show respect for authority Constructivism needs to be able to account for differences in performance, behavior, or opportunity in relationship to group membership.

3. *Constructivism may lack an adequate theory of instruction.* In constructivist classrooms, the students are encourages superbly to articulate their views and to explain their reasoning. However, the teachers, when required to make use of the diversity of ideas, find themselves at a loss. Teachers express a fear of telling—the belief that all constructivism commits them to refusing to inform the discussions with expert opinion or to bring the discussion to closure. Constructivism seems to assume that a theory of learning provides an adequate theory of teaching. As result, constructivist classrooms can lack direction and progress, and have been critically described by some as "sharing ignorance." Furthermore, constructivism seems to imply that all imitative behaviors imply rote learning, and are therefore bad. Such a perspective ignores the ways in which imitative behaviors can be transformed into meaningful behaviors, and how the act of imitation is a form of adult-child, expert-novice initiation. A theory of instruction might help to differentiate those occasions on which imitation becomes a charade and in which it is transformed into deeper understanding.

I pose these criticisms as challenges to us as constructivist theorists. And I propose that elements of the socio-cultural perspective help to form responses to these criticisms.

Notes

[1] A paper presented to the Canadian Mathematics Education Study Group, May 28-31, 1993, in Toronto, Canada

[2] This research was supported by a grant from the National Science Foundation MDR 9053590. Opinions expressed are those of the author and not necessarily those of the Foundation

[3] The phrase "more or less adequately" or "more or less isomorphic" could be interpreted to mean that Piaget intends that the adequacy or isomorphism can be "objectively" checked; a denial of the radical constructivist position. It could be also the case that he intends to indicate the distinction that von Glasersfeld has promoted between fit and match. Objective reality may bound our potential solutions in the way that a lock bounds the possible keys that "fit." This does not imply a unique and absolute correspondence in the way that the term "match" does. More or less, then, could refer to "fit," an interpretation supported by his next statement that the transformational structures are not copies, but isomorphic models that are selected from experience

[4] This description ignores the viable beliefs of many cultures including Native Americans who derive a cogent view of reality from the assumptions of life forces in many objects which most Anglo-Americans deny. It reinforces the importance of integrating socio-cultural and radical constructivist theories, to place an analysis such as von Glasersfeld's within a Eurocentric perspective as a socio-historical event

[5] The author of the paper attended one of the teacher sessions at IERC considering this problem, hence this report may include details not discussed in the bibliographic reference

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[School] is charged with [the] intellectual and moral development [of adolescents]. In a culture like ours, in which tragedy is regarded as a problem and problems are assumed to have solutions, the school is held responsible for observable deficiencies in the adolescent much as a department store is held responsible for the quality of its merchandise

For the most part the school accepts this responsibility It tries to meet it professionally; that is, by means of a program planned to meet stated objectives through techniques derived from empirical research. The statements of objectives are often so naive philosophically, and derived from so vulgar a conception of what life will demand of its students as to be indefensible; the research is often so stupidly planned and executed to be irrelevant to the conclusions drawn from it. But the school is seldom frivolous or irresponsible in its attitude toward youngsters; it tries to understand its job and do it as conscientiously as the quality of its staff permits

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