REGULAR NUMBERS AND MATHEMATICAL WORLDS

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We take the perspective that people inhabit distinct, subjective mathematical worlds. Over time, people learn to inhabit different mathematical worlds, many of which remain accessible while they mature and progress in their study of mathematics. In our view, mathematically literate adults are comfortable (to varying degrees) inhabiting multiple mathematical worlds. In this article, we articulate a distinction between regular numbers and signed numbers, which we ground in the framework of mathematical worlds. We provide examples of student thinking that illustrate these ideas, and we discuss implications for both teaching and research.

Over time, what students know or believe to be true about numbers and operations changes. Significant transitions occur when one learns about fractions, integers, and so on. Some authors have characterized these transitions in terms of students extending their numerical domains (Bruno & Martinón, 1999) or their mental number lines (Peled, Mukhopadhyay & Resnick, 1989). In this article, we argue that people's understanding of distinct numerical domains may instead be productively viewed in terms of inhabiting mathematical worlds. While students learn and grow, they do not permanently cease to inhabit one mathematical world and begin to inhabit another. Rather, such worlds persist, and people metaphorically transport from one world to another in moments of mathematical activity.

Our view is distinct from that of authors who write about *natural-number bias*: the tendency of students to overgeneralize from their knowledge of natural numbers and as a result err or experience difficulties with tasks involving integers or rational numbers (*e.g.,* Van Hoof, Vandewalle, Verschaffel & Dooren, 2015). Rather than describe people as being biased toward reasoning in terms of natural numbers, we would say that people sometimes inhabit worlds of regular numbers. At other times, the same person may inhabit a signed-number world.

For us, the term *signed numbers* refers to a conception: a person operates with signed numbers when that person conceives of the numbers involved as having the property (or quality) of sign. This conception contrasts with *regular numbers*: a person operates with regular numbers when that person conceives of the numbers involved as having only magnitude. Our use of the term *signed numbers* departs from that of Peled and Carraher (2008), who use it to refer to objectively defined sets of numbers, such as the integers. In our terminology, signed numbers are conceptual entities. When people evoke such a conception, they inhabit a signed-number world [1].

We believe that distinguishing between objectively defined domains and conceptions of numbers is important because people have different ways of thinking about numbers. Objectively defined domains do not describe people's conceptions. When two people work on the same task, one may think of the numbers involved as signed, while the other may think of them as regular. Furthermore, one person may have different ways of thinking about the sets of numbers involved in the same mathematical task. Therefore, to make sense of learners' experiences, we need to recognize their conceptions of numbers.

Across the literature on the teaching and learning of integers, negative integers feature prominently. Students' difficulties with integers are discussed in terms of difficulties making sense of and operating with negative numbers (Bruno & Martinón, 1999; Vlassis, 2002). Students' thinking about integers is discussed in terms of their thinking about negative integers (Chiu, 2001; Hativa & Cohen, 1995). The history of integers is discussed in terms of the history of negative numbers (Gallardo, 2002; Hefendehl-Hebeker, 1991). The notion of sign itself gets relatively little attention in this literature, and natural numbers are sometimes equated with positive integers. Our purpose here is to highlight a phenomenon that has not been emphasized in the literature and that may contribute to the development of theory that will help mathematics educators to better understand students' thinking and learning in this area. In contrast to the emphasis in the literature on students learning to deal with negative numbers, we suggest that an important transition to consider is that from conceptions of regular numbers to signed numbers (both positively and negatively signed).

Mathematical worlds

We build on Greeno's (1991) environment metaphor to analyze mathematical thinking in terms of the mathematical worlds that people inhabit. Greeno developed a perspective of situated knowing in a conceptual domain—the domain of numbers and quantities. From this perspective, a person's knowledge and activities are seen metaphorically as situated within a physical environment. Knowing in an environment consists of knowing how to get around, where to find things, and how to use them. In any conceptual domain, knowing one's way around requires relating concepts and solving problems. Greeno's metaphor relates mathematical properties to features of a physical environment. Objects in an environment have certain constraints and affordances that represent (whether accurately or not) properties of the domain. As an example from our data (discussed in the next section), students may perceive 3 – 5 as possible or impossible to evaluate, depending on features of the environment that they inhabit. In the environment metaphor, the situated, subjective nature of reasoning is emphasized. The environment in which a person operates is a feature of the person in interaction with a situation; it is not merely analogous to an objectively specified domain.

Mathematical worlds are related to number domains in the sense that we can describe the numbers that belong to the worlds that people appear to inhabit during moments of mathematical activity. The construct of mathematical worlds is an attempt to convey the phenomenological nature of mathematical thinking in a way that is true to human experience across various social contexts. For example, when someone says, "Pick a number between 1 and 10," this instruction is often taken to mean that one should select a whole number. Many people in everyday settings will choose 5 or 7. However, depending on the audience and context, the instruction may be interpreted differently. Many mathematics enthusiasts will choose π.

One author of this paper is fond of saying "I have a number of motorcycles in my garage. That number is zero." The punch line—zero—is surprising to people because they tend to assume that he would not speak of motorcycles in his garage unless he had at least one. This assumption in no way reflects a deficiency in a person's knowledge of number. On the contrary, it reflects an understanding of the way in which the phrase "a number of" is used in context. Such understanding is useful because it facilitates communication in social situations. Our point here is that people, including mathematically literate adults, often inhabit regular-number worlds.

Mathematical worlds characterize the environments in which people act and interact mathematically. Our concern in this article lies in the kinds of numbers of which people conceive during their mathematical activities. We focus specifically on the contrast between regular numbers and signed numbers. We regard the following as key characteristics of the framework of mathematical worlds:

1. Mathematical worlds are specific to mathematical thinking and activity; however, we conceive of mathematical thinking and activity quite broadly (*e.g.,* including the motorcycle joke above).
2. Different people may inhabit different mathematical worlds.
3. The same person may inhabit multiple, distinct mathematical worlds over the course of his or her development.
4. The same person may inhabit different mathematical worlds from moment to moment.

Working from this framework, we consider cases of students inhabiting different mathematical worlds, focusing on the kinds of numbers that belong to those worlds.

Cases of student thinking to illustrate different mathematical worlds

We identify four major categories of mathematical worlds. There are certainly more fine-grained distinctions that could be made between students' mathematical worlds. We chose our categories in order to emphasize the contrast between regular numbers and signed numbers. We illustrate each of these four types of mathematical worlds with examples from interviews with K–12 students. Note that the examples concern moments of mathematical activity. We do not intend to make restrictive claims about the mathematical worlds to which the children had access. Our purpose is to describe the mathematical worlds that the students seemed to inhabit *in these moments of activity.*

Students inhabiting worlds that consist exclusively of regular numbers

Many children in the primary grades consistently inhabit regular-number worlds. These are worlds in which numbers represent magnitudes. In the worlds that Danny and Sam inhabited in the episodes described below, zero was the smallest number—not only the smallest number known, but the smallest number conceivable. In such a world, addition makes larger and subtraction makes smaller (except in the special cases of adding or subtracting zero). The responses of Danny and Sam, both second graders (aged 7–8), exemplify the reasoning of children inhabiting such worlds when they are posed problems that (from a more expert perspective) invite the use of negative numbers.

> *Danny*: After reading the problem $3 - 5 = \square$, Danny attempted to act out the operation by taking 5 fingers away from 3 fingers. He tried twice and then asked the interviewer, "How come there's 3 and take away 5? I don't have enough." Danny explained that he could not perform the operation because 3 is less than 5. He concluded that solving the problem was "not possible."
>
> *Sam*: Sam read $6 + \square = 4$ and said that he could not answer unless the plus were changed to a minus. Sam explained that when adding a number to 6, the result should be larger, not smaller. He pointed out that even when adding 0 to 6, the result would be larger than 4. In the world that Sam inhabited, 0 was the smallest number, and so adding to 6 and obtaining a result less than 6 was impossible.

The worlds that Danny and Sam inhabited [2] in these episodes consisted exclusively of regular numbers. In a regular-number world, minuends cannot be smaller than subtrahends, and a sum cannot be smaller than either of the addends. These generalizations hold because numbers that behave otherwise do not exist in such a world. Danny's and Sam's reasoning make perfect sense, given the worlds in which they were operating. In a regular-number world, addition affords making larger, and subtraction affords making smaller. Numbers represent amounts, which must be at least zero. The constraints inherent in these features become apparent only when students are posed unusual tasks, such as $3 - 5 = \square$ and $6 + \square = 4$.

Students inhabiting separate worlds of regular numbers and negative numbers

Many children in the elementary grades can inhabit negative-number worlds that exist independently of regular-number worlds, such that these types of numbers cannot be combined or coordinated meaningfully. The responses of Jake (second grade) and Jamie (first grade) illustrate activity involving these separate number worlds.

> *Jake*: Jake was asked to compare pairs of numbers represented in printed form on a sheet of paper. He was asked to circle the larger of the two numbers or to write an equal sign if they were equal. Jake was familiar with negative numbers and had ideas about comparisons involving them. For example, he correctly compared –5 and –6 by relating these numbers to locations on a number line. Jake was then asked to read and compare +20 and 20. Jake responded, "Plus 20? And 20. What? I don't get the plus part. That's kind of strange." Jake explained, "Well, I know there's no plus on either side [points to the two sides of the number line], so that [plus sign] can't be there." He decided to cross out +20 and circle 20. When asked why he crossed out +20, Jake said, "Because it has a plus, and there's only supposed to be negative and just regular numbers."

In this episode, Jake inhabited a mathematical world in which 20 was not a signed number. There was no such thing as "Plus 20" in the worlds that seemed to be available to him. Rather, numbers could be either regular (signless) or negative (signed). This is in contrast to a signed-number world, in which all non-zero numbers are either positive or negative.

> *Jamie*: When asked to complete –5 + –2 = ☐, Jamie answered –7. He explained, "Because like 5 plus 2 equals 7. So, like if you, if you're doing negatives, it's like the same as regulars." When inhabiting a separate negative-number world, Jamie could solve problems like –5 + –2 = ☐ and –7 – ☐ = –5 by thinking of negative numbers as being "like the same as regulars." However, this negative-number world did not allow for solutions to problems that involved both types of numbers. Jamie said that the regular numbers and negative numbers behaved like magnets that would repel one another. To inhabit a negative-number world, Jamie applied an analogy between negative numbers and regular numbers, which enabled him to solve certain types of problems. If the given numbers were both negative, he thought about the problem as he would if the given numbers were both regular, and then he simply wrote a minus sign in front of his answer and called the number "negative."

Many students who have some familiarity with negative numbers demonstrate the ability to inhabit two different worlds that are analogous. In particular, subtraction can be performed only if the absolute value of the subtrahend is less than or equal to the absolute value of the minuend (*e.g.*, –5 – –3 = –2, whereas –3 – –5 is not possible). From the perspective of students who inhabit these worlds, the only apparent difference between the regular-number world and the negative-number world is that negative numbers are written with a minus sign. For example, Jamie could take five blocks and decide to "pretend this is a negative 5." Then he would proceed just as he would in dealing with regular numbers, except that every amount would be called negative. So, –5 + –2 would equal –7, because 5 + 2 equals 7.

When students inhabit separate worlds of regular numbers and negative numbers, these two types of numbers are incompatible. They recognize a distinction between two types of numbers (*e.g.*, that –2 is different from 2), and deal with them similarly but separately. To mix regular numbers and negative numbers involves a clash between these separate worlds. For Jamie, 5 + –2 was not sensible. It would not make 7 regulars or 7 negatives, and those would be the only possibilities. On one hand, Jamie's reasoning in a negative-number world was limited. On the other hand, inhabiting such a world enabled Jamie to solve problems such as –5 – –3 = ☐ and –7 – ☐ = –5, which can be difficult for some middle-school students.

Students inhabiting connected worlds of regular numbers and negative number-locations

Some students inhabit worlds that include both regular numbers and negative number-locations. In such worlds, in contrast to a separate negative-number world, one can cross zero; however, adding or subtracting a negative remains impossible. Thus, tasks such as 3 – 5 = ☐ are sensible in a connected world of regular numbers and negative number-locations, but tasks such as 6 + ☐ = 4 are impossible. The reasoning of Violet, a second grader, exemplifies activity in such a world.

> *Violet*: When solving problems such as –5 – 4 = ☐, 3 – ☐ = –2, and ☐ + 5 = 3, Violet inhabited a connected world of regular numbers and negative number-locations. In each case, she made use of a drawing of a number line that included negative number-locations. For example, she acted out –5 – 4 by starting at –5 on the number line, moving 4 spaces to the left, and ending at –9. To solve 3 – ☐ = –2, Violet started at 3 and counted the number of spaces that it took for her to get to –2. She wrote 5 in the box. To solve ☐ + 5 = 3, Violet used a guess-and-check method, starting from a location to the left of zero, moving 5 spaces to the right, and checking the ending location. She soon figured out that she should start at –2.

In her approach to each of these tasks, Violet showed that she inhabited a world of number-locations that could be negative, zero, or regular. Addition and subtraction behaved as usual with regular numbers. Addition required moving to the right, and subtraction required moving to the left. These operations behaved consistently from one side of zero to the other.

For a child in a world in which negative numbers exist only as locations, many problems involving addition or subtraction of integers remain impossible to solve. How to interpret expressions that explicitly involve addition or subtraction of a negative number (*e.g.*, 3 + –5) is unclear in such a world. Likewise, problems that require adding but moving to the left or subtracting but moving to the right are not solvable. For example, when inhabiting this type of number world, Violet was unable to solve 5 + ☐ = 2. She said, "I'm just thinking that this [pointing to the +] has to be a minus for that to be possible."

Students inhabiting regular-number worlds and signed-number worlds

Students who have had formal instruction or special opportunities to learn about integers are familiar with both positively and negatively signed numbers. They may say that positive numbers are the same as regular numbers—likely because they are taught to do so. However, we see evidence that these two types of numbers continue to be phenomenologically distinct for students. Positive numbers belong to a world of signed numbers, whereas regular numbers do not. In their conceptual development, students do not transition from a regular-number world to a signed-number world, leaving the former behind. Rather, we see them moving back and forth between number worlds. The following example of an 11th-grade student (aged 16–17) illustrates this point.

> *Sarah*: When presented with a story about borrowing money from a friend ($8 for a school T-shirt and then $5 for lunch), Sarah wrote 8 + 5 = 13 to represent the situation. She explained the meaning of her equation simply in terms of adding amounts of money. She articulated orally who owed money to whom, but she had not used signs to represent that aspect of the situation. When Sarah was shown –8 + –5 = –13, she said that it could also describe the story. She then addressed sign in contrasting this equation with hers:
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>> Because this [pointing to −8] could be like negative of my money, [points to −5] negative of my money, and [points to −13] negative of my money. Whereas, this one [points to 8 in her original equation] is positive of her money [points to 5], positive of her money, and [points to 13] positive of her money.
>
> Although initially Sarah had used her equation only to represent amounts of money, when she was shown an equation involving negative numbers, she was able to make sense of it. Furthermore, she reinterpreted her original equation as involving positive, rather than regular, numbers by offering contrasting meanings: the money that she had borrowed from her friend would be "negative of my money" but "positive of her money."

In this episode, Sarah initially operated in a world of regular numbers, which was sufficient to solve the problem and to describe the situation with an equation. Her equation initially consisted of one-dimensional, signless numbers, which represented amounts of money. When presented with an equation involving negative numbers, Sarah transported to a signed-number world. Now the numbers in *both* –8 + –5 = –13 and 8 + 5 = 13 were two-dimensional, and she interpreted the signs meaningfully in relation to the situation of borrowing money. We believe that Sarah was reasoning sensibly when she inhabited a regular-number world and moments later when she inhabited a signed-number world. Our point is to highlight the fact that Sarah had access to both of these worlds. Sarah's initial response was like that of most of the students we interviewed. In fact, the majority of students who had access to signed-number worlds nonetheless initially inhabited regular-number worlds in response to the story problem [3].

Summary: students' mathematical worlds

We have identified four distinct mathematical worlds that emerged from our analyses. Some students consistently inhabited regular-number worlds. Other students, who had some degree of familiarity with negative numbers, inhabited additional mathematical worlds. Our interest lies in understanding the nature of these mathematical worlds. We recognize that in different moments of activity, students may inhabit different mathematical worlds. In keeping with our perspective, the key distinctions lie in the mathematical worlds to which students have access and in their abilities to navigate such worlds.

Figure 1 illustrates the mathematical worlds that the students we discussed above appeared to inhabit during the episodes presented. Danny and Sam consistently inhabited regular-number worlds. They appeared to be unfamiliar with negative numbers and viewed many of our tasks (*e.g.*, 3 – 5 = ☐) as impossible to solve. Jake and Jamie's responses showed that they had access to a regular-number world and a separate negative-number world. In the examples presented, Violet inhabited a world of regular numbers and negative number-locations [4]. Sarah showed that she had access to both a regular-number world and a signed-number world. She may also have had access to other worlds that included negative numbers. Likewise, any of these students may have had access to other mathematical worlds.

The universe of mathematical worlds is not limited to those we have presented. Other mathematical worlds could be added to Figure 1, and mathematical worlds could be categorized differently. We find this picture useful for highlighting distinctions between regular numbers, negative numbers, and signed numbers.

Discussion

We have borrowed the term *regular numbers* from many students who have used this and similar terms to communicate an important distinction. Other researchers have also observed students using this term (*e.g.,* Bofferding, 2014, p. 194). The particular term *regular numbers* is less important than the distinction that it reveals. Children's use of such a term indicates that they have recognized an important distinction between a familiar type of number and another, more complicated, type of number.

It is commonplace to conflate positive numbers and regular numbers. When they learn about integers in middle school, students may be told that the whole numbers are actually positive. Such a statement is mathematically imprecise. More to the point, to say that a number like 5 is now "positive 5" is inconsistent with people's everyday experiences of number. If there are five apples in a basket in your kitchen, do you have positive five apples or just five apples? Introducing sign brings with it the need to interpret that sign meaningfully. If one is concerned only with the number of apples, there is simply no need to invoke sign.

Regular numbers precede signed numbers in both ontogeny and phylogeny. Historically, the ancient Greeks considered

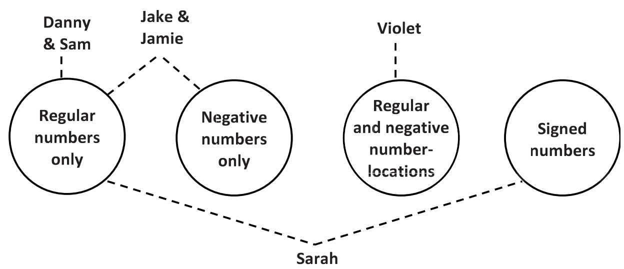

Figure 1. Students inhabited at least *these types of mathematical worlds.*

numbers to represent counts or measures. Zero was controversial because (as in the example of having zero motorcycles) using a number to represent nothing seemed strange (Seife, 2000). Ancient Greek mathematicians had no notion of negative—or positive—numbers. Similarly, for most young children, numbers are not signed. Children learn to count with natural numbers. At some point, they learn about zero. Later, they encounter (regular) fractions, decimals, and percentages. Typically, a child's formal introduction to the notion of sign comes after all this experience. Interestingly, we have found that many students in the elementary grades have some familiarity with negative numbers but have never heard of positive numbers. These children inhabit intermediate worlds that consist of regular numbers and negative numbers before they begin to (intermittently) inhabit worlds of positively and negatively signed numbers.

Although mathematical tasks may specify a domain, the people reasoning about such tasks do not necessarily inhabit the intended mathematical world. For example, Zazkis and Mamolo (2012) found that high school mathematics teachers tended to imagine a finite sample space when the domain in a story was specified to be real numbers between 1 and 10. Similarly, when we tell students that "x can be any number" or "any number can go in the box," they may (quite reasonably) assume a finite domain. In fact, when we posed equations involving integers in interviews, the numbers involved were usually between −10 and 20. To say that the domain for these tasks was the set of integers seems unrealistic. In practice, we used a small subset of the integers.

The research literature on natural-number bias is focused on errors resulting from overgeneralizations. *Natural-number bias* has a decidedly unfavorable connotation. In our view, by contrast, it is not that people have a regular-number bias. Rather, people often inhabit regular-number worlds, and doing so is practical and unproblematic in many situations. To go searching for evidence of bias is to highlight exactly those instances in which people inhabit a number world that may not be the best fit for the task or situation at hand (Van Hoof *et al.*, 2015). The result is that the tendency to think in terms of natural numbers is cast as dysfunctional, despite the fact that this tendency is found even in expert mathematicians (Obersteiner, Van Dooren, Van Hoof & Verschaffel, 2013). We suggest a more balanced view that acknowledges that inhabiting simpler number worlds is viable and even advantageous in many everyday situations.

Our examples show students thinking reasonably, given the mathematical worlds that they inhabited in the moment. In some cases, students inhabit relatively stable mathematical worlds. In other cases, students may reason in multiple mathematical worlds and transport quickly from one to another. Depending on the questions that teachers or researchers ask, together with the social context or framing of such questions, they can invite students to inhabit different mathematical worlds. We believe that awareness of this phenomenon is important to avoid jumping to conclusions about the limits of people's mathematical understandings or abilities based on moments of activity in one world or another.

As a result of our investigation, we have come to see people as inhabiting distinct mathematical worlds. These worlds characterize individuals' ways of viewing and understanding mathematics, and they differ depending on the features and possibilities for action that an individual perceives (Greeno, 1991). For young children, these worlds may be relatively stable. Regular-number worlds come to include bigger numbers over time, but the numbers continue to behave in reliable ways. When students get older, they come to know different mathematical worlds. We observe that instead of the new worlds replacing the old, these worlds coexist. In particular, adults often inhabit regular-number worlds in the course of their daily activities. In many everyday interactions, people assume a domain of regular numbers, or they must assume such a domain to make sense of the utterances of others. If someone says, "I have a number of motorcycles in my garage," and then proceeds to reveal that the number is zero, the initial statement seems disingenuous. Just as one assumes that the number is not a fraction, one also assumes that it is non-zero and certainly non-negative. Our point here is not to say that we cannot identify real-world contexts in which zero or signed numbers make sense; it is to illustrate that everyday interactions often evoke simple number worlds, which are adequate for the purpose.

Reflecting on the mathematics education literature related to integers, we find that the distinction between regular numbers and signed numbers tends to go overlooked. Often in the literature the terms *integers* and *negative numbers* are used more or less interchangeably: teaching and learning of integers is synonymous with teaching and learning of negative integers. Students' understanding of positive integers seems to be taken as a given. The transition from whole numbers to integers does not consist merely of the introduction of negative numbers. Students whose worlds once included only regular numbers must now make sense of regular numbers, negative numbers, *and* positive numbers; in other words, they must now coordinate a world of regular numbers with a world of signed numbers. Students may be told that regular numbers and positive numbers are the same or that the regular numbers were actually positive all along. However, we find that students who have had substantial integer instruction often reason in terms of regular numbers. Being told that these numbers are really positive may make communicating the nuances of their reasoning difficult for students. They learn to say "positive" because they are supposed to, and this language masks a profound conceptual distinction.

Ideally, students would learn to inhabit signed-number worlds when doing so makes sense (*e.g.*, to think of 5 as +5 in contexts that involve directionality), while also being able to recognize and reason about the distinction between regular numbers and signed numbers, and to sometimes inhabit regular-number worlds. We believe that the kind of flexibility of reasoning that Sarah exhibited is desirable. To help students to develop that flexibility, we must be sensitive to these distinctions and precise in writing about the challenges associated with the teaching and learning of integers. Greeno described the development of number sense in terms of learners becoming increasingly familiar with a mathematical environment and attuned to constraints and affordances that correspond to the properties of the domain of numbers and quantities. This perspective applies within a given mathematical world. Our emphasis on different mathematical worlds leads to the further implication that people must learn to navigate between worlds—to recognize which mathematical world fits a given situational context.

Thus, the distinctions discussed in this article have the potential to be powerful tools for teachers, giving them a lens and a language to aid in communicating with students about challenging ideas that arise in integer instruction. Sensitivity to the distinction between regular numbers and positive numbers can inform the language that we use with students and the care that we take in introducing them to the notion of signed numbers. If teachers are unaware of students' distinct mathematical worlds, their abilities to support student learning are limited. A teacher and her students may use the word *positive* to talk about two kinds of numbers, and class members may talk past one another, not realizing that they are interpreting numbers in different ways.

Furthermore, the notions of regular numbers versus signed numbers and the framework of mathematical worlds apply more broadly than to the topic of integers. In the elementary grades, fractions and decimals are typically treated as *regular* [5]. More broadly, the notion of regular numbers may be taken to describe other contrasts that students see between more and less complicated (or familiar) types of numbers. In ongoing work, we are exploring the mathematical worlds that college students inhabit when working with real versus complex numbers. The construct of *regular numbers* highlights a crucial distinction in people's conceptions of number. At the same time, the construct enables us to recognize common ground, because everyone inhabits worlds of regular numbers at times.

Acknowledgements

This material is based upon work supported by the National Science Foundation under grant number DRL-0918780. Any opinions, findings, conclusions, and recommendations expressed in this material are those of the authors and do not necessarily reflect the views of NSF.

Notes

[1] Peled (1991) referred to "worlds" of positive and negative numbers. The meaning of our language is different. She referred to a "positive world" to the right of zero and a "negative world" to the left of zero.

[2] Note that there may well have been nuanced differences between the mathematical worlds that Danny and Sam inhabited during their interviews. There are also differences between the two of tasks given above. Our point here is to illustrate what it means to inhabit a world of regular numbers.

[3] We discuss the responses of 7th and 11th graders in more detail in an article focused on students' thinking about the relationship between arithmetic equations and the Money Problem (Whitacre *et al.*, 2015).

[4] In additional interviews, Violet began to access and gain familiarity with a signed-number world. Bishop, Lamb, Philipp, Whitacre and Schappelle (2014) examined Violet's reasoning in detail.

[5] This distinction arises in curricula and standards documents. For example, in the US Common Core State Standards for Mathematics, the standards for the elementary grades refer to fractions and decimals, which are exclusively non-negative. Fractions are related primarily to amounts in terms of parts of a whole, and the notion of sign is not mentioned. By contrast, beginning in Grade 6, the standards refer to rational numbers, which may be either positive or negative. Thus, the terms *fraction* and *decimal* seem to denote the regular-number analogues of the rational numbers.

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