Problems in the Development of Mathematical Knowledge in the Classroom: the Case of a Calculus Lesson*

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1. Introduction: is it possible to learn something about teaching from transcripts of lessons?

Transcripts of mathematics lessons are difficult texts to digest, especially when they comprise whole teaching units. A good deal of assiduous effort is required to find a way to their understanding and to uncover their structure and meaning. To a large extent the transcript of the instructional dialogue is itself a cause of this difficulty. However, this is not the single reason why these texts are both difficult and troublesome to decode.

To a certain degree all lessons are unique events. While the multiplicity of aspects that emerge here is very much context-dependent, the written representation petrifies everything into a fixed form. This fixation by the transcript, however, brings to a unique specific lesson the possibility of infinite repetition. One single lesson can be viewed in many different ways; it can be interpreted and analysed. This then can lead to the discovery of more general structures and relationships. And if one puts aside the painfully deciphered transcript for a while and comes back to it some three or six weeks later, it often happens that one faces the same problem again and has to puzzle out the same text with no less trouble than the first time.

This experience with transcripts shows that they cannot be read and analyzed in the same way as scientific texts which are constructed according to a certain general and conventional pattern. The contents and the course of a lesson cannot be read directly from the transcript. Quite a few means have to be called on for the structuring, for the analysis, and for the representation, which can provide a unique teaching event with relationships and meaning.

The actual transcript (see Appendix 1) mixes up individual aspects with general basic tendencies of development in a manner that makes it difficult to explore their mutual interplay and to discriminate between the trivial and the important. Individual students' speech events might have had an influence on the fundamental course of the teaching. Has the question "Will there ever be an end to these functions?" (79; Note: in what follows, numbers in brackets refer to statements in the transcript) had some effect on the further course of the teaching or was this just another of those jocular interjections that this student is well known for? And the students' inquiries and misunderstandings—do they change the teaching discourse as prepared and intended by the teacher or are they estimated to be trivial aspects which will automatically be corrected with the solution of the mathematical problem?

From the perspective of a teacher, the search for the essential thread in an interaction involving both individual and general aspects is reduced to the discussion of its mathematical contents. As teachers, and especially as teacher educators, we often feel that it is possible to make an adequate evaluation of a teaching process just on the basis of the knowledge of the most important concepts and problems implied by it. For instance, key expressions such as "domain of a variable", "extrema", "function", "graph of function", "shape of a box" and "determining which box has the greatest possible volume", etc., evoke associations not only on the level of their contents but also on the level of possible teaching and methodological variants.

But even from the perspective of experience with content and teaching one is not free of surprises when going through transcripts of lessons. Here, likewise, it is necessary to find one's way: what and in what sequence is established in a teaching unit? One has to clarify for oneself the relation between the particularities of this lesson in *this* classroom with *these* students and *this* teacher and the pedagogical and didactical approaches which seem to be meaningful for an introduction into the domain of extrema. Perhaps the following questions would arise upon a preliminary perusal of the transcript: Is the concrete construction of the box taken seriously by the students? What is, for the students and the teacher, the meaning of the construction of the box, and, later, of drawing a "rough" graphical representation of the function under consideration?

The question is how to make transcripts of teaching more readable. Polishing the verbal statements by correcting all the slips of the tongue and grammatical mistakes is certainly not a solution. Neither is it useful to reduce the description of a lesson to a short sketch of the core contents, recounting only what has been mediated by the teacher and learned by the students. In this case there is the risk of leaving aside important aspects which, on the surface, seem to be trivial. As the transcripts of a real teaching episode contain at the same time different individual and

*This paper was originally published in German in *Der Mathematikunterricht* 36(3) [1990] 4-28 general aspects, it seems that the requirement to make transcripts more readable can be met only by the elaboration of several "retranscriptions" in a number of different contrasting forms of description, representation, and analysis.

We shall be using this approach later, going through the transcript of a certain calculus lesson several times, considering it from several points of view. First we give a short overall description of the teaching episode; then its detailed structure will be presented. Further, we analyse the various instructions of the teacher to the students, the questions she posed with their subsequent modifications, the gradual increase in the use of technical concepts and terms, etc. We shall also take a look at the lesson from the point of view of the students, their difficulties or problems of understanding. An analysis of these problems will reveal that the students' interventions contained openings into mathematically rich avenues which were, however, voluntarily ignored by the teacher and not taken up in the course of her teaching.

Besides the usual form of transcript, two "retranscriptions" of the lesson will be constructed: a more concise verbal-tabular representation highlighting the overall structure of the lesson, and graphical representation. The graphical representation focuses on the distribution, in the course of the lesson, of three levels of discourse: the "empirical", the "geometrical", and the "computational".

The discussion of the analysis as presented in the following sections is certainly arranged according to a linear order, starting with structuring the follow-up to the different phases of the lesson and ending in a global conceptual characterization of the whole lesson. When making the actual analysis one cannot simply use such a step-by-step procedure, passing along the chronological course, but one always has to take a global perspective to identify the local bounds of the different phases and their hierarchy, and one has to analyze carefully the epistemological status of the mathematical knowledge from phase to phase (for instance, via the change of teacher's questions and via the change of mathematical problems) to explore the development and shifts of knowledge interpretation and understanding in the classroom interaction. The method of analysis is a dialectical one reflecting global and local aspects simultaneously.

2. The contents and the course of the teaching episode

2.1 An overall description of the lesson

The lesson that we shall discuss hers is the first of a block of two lessons meant to introduce the students to the domain of extrema. This topic is treated in grade 12 (as part of a basic course), the next to the last grade of the Gymnasium; the group consisted of 13 girls and 8 boys aged between 17 and 19 years. The students already knew the basic rules for the derivation of a function. The introduction to questions and problems about extreme values of a function and the optimization of volume is based on an example widely known in the practice of teaching [see, e.g. Glatfeld, 1982, p. 100]: the construction of a box by cutting off four equal square corners from a rectangular sheet of paper.

The students construct boxes, make measurements, calculate the volumes of the boxes and write the values they have found in a table. On the basis of these concrete activities and the data obtained, the biggest possible box is searched for. This activity is strongly guided by the teacher and her questions. The "empirical" question: which of these boxes has the greatest volume (18)? is promptly answered. It serves as a transition to the theoretical question: "Do you know for sure that the fourth box is definitely the biggest box that I can build from this sheet of paper? (33)". During the main phase of the lessen, this question, under changing wordings, leads to the construction of the functional equation that is aimed at, to a statement in which this equation is defined relative to the construction constraints on the box, to the drawing of a graphical representation of the function, and finally to a possible approach to a solution.

During the discussion, initiated by the teacher's questions as well as by the splitting and reduction of questions into certain subquestions, the students are provided with the preconditions for "recognizing" the initial problem as an application of the already-known rules of differentiating a function. From now on the students are able to use their previous knowledge in a relatively assured and automatic manner to work on the volume function of the box and to make their way towards the mathematical solution of the problem.

2.2 The "phase structure" of the lesson

Three phases can be distinguished in the lesson: a first (1-25), during which the initial problem is presented and the boxes are constructed, a second phase (26-133) during which the problem is dissected, discussed, and finally a mathematical approach to a solution is found for the central question "what is the biggest possible box?", and finally a third phase from (133 on), which only starts in this lesson and is continued after the break, during which an algorithmic procedure of solution is elaborated step by step (see Appendix 3. For a better understanding of the rest of the article, the reader is invited to read the transcript of the lesson at this point).

The division into three main phases can be refined into subphases which are framed according to the teacher's specific questions or instructions for the students. For example, a subphase can be started by an instruction from the teacher and ended when the instructions have been more or less carried out.

2.2.1 Analysis of the first main phase

The first main phase can be divided into two subphases. The first comprises the presentation of the initial problem and the discussion of how to build the concrete boxes (1-13). During the second (14-25), the students are putting together the boxes, measuring or calculating mathematically the volumes of these boxes and putting the values into a table. The leading question of "which box has the largest volume?" is asked by the teacher twice during this main phase (6,18).

2.2.2 Analysis of the second phase

The second main phase can be subdivided into three sections, each starting with the same question by the teacher:

"what is the biggest box that it is possible to construct with this rectangular sheet of paper?" Before any of these three subsections is started, the empirical problem, inherent in the concrete construction of boxes, is transformed into a theoretical question—at the beginning of the second main phase. What the students are supposed to look at now is not merely the biggest "empirical" or actually-constructed box. The students are no longer asked to sort the constructed boxes according to their volume; they are expected to find a potentially *biggest* box. This is how the problem is worded now: "Do you know for sure that the fourth is definitely the biggest box I can build from this sheet of paper? (33)".

During the second main phase it becomes obvious that the question posed at the beginning implies too many presuppositions. During each of the three subphases (II.1-II.2-II.3) it is split up into subquestions. For instance in section II.1 the teacher focuses the discussion with the students on the questions: "What does the volume depend on?" (40) and: "How do the cut-off squares determine the volume of the box?" (in the sense of 44). The students are supposed to conceive of the length of the side of the squares as the (independent) variable x and also find out the bounds ($x = 0$ and $x = 10$) within which, in this example, the volume function exists.

Then the teacher comes back to the basic question and starts a new phase (II.2, 70-101) during which the graphical representation of the volume function is worked on and the first, visual, idea of the structural connection between the possible values of this function is forgotten. Taking, now, the discussion about the functional dependence of the volume and its graphical representation as a new background, the teacher again asks the basic question about the biggest possible box. Thus she initiates the third subphase.

This subphase is concerned with interpretating the expression of the volume as representing the function $V(x)$. The teacher expects this to allow the students to recognize the initial problem as an application of the known derivative rules and to find "the" mathematical approach to the solution. She is successful, indeed, and this allows her to pass to a next step (133, third main phase) by asking the question: "Precisely. So, come on. We want to find the position where the volume is maximal.... What should I do now?"

2.2.2.1 Section one of the second main phase

The subphases can be structured in an even finer way (see Appendix 3). The subphase II.1 can be split into six short sections. During the first (II.1.1, 34-39) the students' intuitions show a misunderstanding of the basic question. Yes, it is correct that the quantity of the volume "depends on" the quantity of the surface of the box. But the students are unaware of the fact that, in this example, the volume of the box first increases with the decreasing surface, but later it decreases too. The contradiction between the students' intuitions and the actual situation causes the teacher to make her question more concrete.

At the beginning of subphase II.1.2 (40-46) the teacher directly poses the question "What does the volume depend on?". During the discussion elementary "functional" arguments are given and the constructed boxes are sometimes referred to to support statements or elaborate new arguments.

The third subphase II.1.3 (47-53) is again opened by a concrete question from the teacher: "What is the volume determined by, taking account of the way in which we have constructed the boxes?" Again, some students claim that volumes with more surface are bigger, or the less surface that is cut off the bigger the volume of the box has to be. The teacher comments that this contradicts our empirical knowledge and the phase is interrupted at this point by the intervention of two "hasty" students who have found a formula for the volume and have used it in calculations on the pocket calculator (this starts the subphase II.1.4, 54-55). A discussion starts concerning the formula for the volume of a rectangular parallelepiped: "length times width times height". This formula is then used to find the specific arithmetical representation of the volume in this example by putting x as the height, $(25 - 2x)$ as the length, $(20 - 2x)$ as the width of the box.

This intermediate result allows the teacher to directly put the question in the following subphase II.1.5 (60-62): "So what does the volume depend upon? Now it becomes very clear". The teacher then translates a student's answer into a more technical form; she says that the volume can be represented as dependent on x. In this way x is introduced as an independent variable into the discussion.

The sixth subphase II.1.6 (62-70) opens with the teacher's question about the domain of this variable. The result of this discussion is an agreement that, for the construction of a box from the given sheet of paper, the variable x, i.e. the length of the side of the square, can vary between 0 and 10. This closes phase II.

2.2.2.2 Section two of the second main phase

The subphase II.2 is again opened with the basic question. During a first subphase II.2.1 (71-77) mathematical keywords such as "strictly monotonically increasing", "strictly monotonically decreasing", "curve", "turning point", etc., start emerging from the answers of some students. At the same time, first qualitative and approximate solutions are proposed: some students assume the solution will lie between 3 and 5, some others that the middle point between 0 and 10 is a good value.

Reference to isolated points on the curve, to the increasing and decreasing, to turning points, etc., lead the teacher to ask her students directly to draw the graph of the function. For this purpose the values from the table are used. The teacher comments that some values in the table cannot be regarded as precise because they were not calculated but measured. During the construction of an approximate drawing of the graph of the function the problem of precision is raised and the question is tackled of where the maximum of the function could be.

Phase II.2.2 is interrupted by a short episode (79-84) marked by an interjection from a student who has noticed that, in principle, the objective of this lesson is nothing but traditional mathematics: "Will there ever be an end to these functions?" (79). And he adds that he hopes he will not have to do any more mathematics later on.

2.2.2.3 Section three of the second main phase

Using the sketch of the graph of the function as a basis, the teacher again takes up the fundamental question about the box with maximum volume. This starts subphase II.3. During a first intermediate phase II.3.1 (102-109) one student provides content-related key words in answers to the teacher's question: "equation", "x", "unknown", "maximal volume", "Vmax", "second equation", "unknown", "maximum". The discussion that follows makes it obvious that what the student has in mind are two equations with two unknowns, and his problem is that here there is only one equation at hand. Another student resolves this problem by pointing to the fact that the volume and its maximum are dependent only on the one observed variable x.

At the beginning of the second intermediate phase II.3.2 (110-114) the teacher is provoked to ask for the "type of equation" which is given here. "What do we call it then? When one magnitude permanently depends upon some other magnitude?" (110). As the students do not come with an answer very promptly, the teacher reinforces her question by pointing to the mathematical symbolic way of writing: "In principle you should see it from $V(x)$" (112); this makes one student try a guess: "Function?". The teacher confirms this, emphasising it, at the same time: "Yes, that's right. We have already drawn the graph of the function. We have observed that the function is defined on the interval (0,10). Yes, when we now know that this is a function... Look at the graph once again" (114).

The intermediate phase II.3.3 (115-120) starts at once with a new rewording of the initial question: "I want to find an x for which $V(x)$ is biggest" (114). To this question the students are now able to give the right answers as provoked and expected by the teacher. The modification directed at least one student's thoughts towards an approach to a solution based on the rules for finding derivatives of functions. The proposed solution is to determine the slope by calculating the derivative of the function and to look for the point at which the slope will be 0.

In the intermediate phase II.3.4 (121-132) this way of solving the problem is reinforced by the teacher who gives it a kind of geometrical interpretation. The teacher begins by asking: "What does it mean graphically that the slope is nothing?" (121). Parallels are then drawn in the sketch of the function in order to visualise the geometric solution. At this point one student expresses his doubts concerning the generality of such a graphical visualization: "....I cannot draw one parallel and then conclude that at 4 it is really the highest" (123). The teacher agrees with the remark by saying: "You are perfectly right, we do not know how this curve behaves in the interval (3,5)" (124). However, she then puts the remark aside as irrelevant, in a way, by commenting that "this does not disturb the plan for calculating [the maximum of the function]" (124). Thus the proposed strategy of solving the problem is accepted and the graphical representation provides it with a justification. The subphase II.3 ends with this last intermediate phase. This also ends the second main phase.

The propedeutical discussions regarding a possible solution, regulated by the teacher's questions, are now finished. Now the given problem can be solved by using the rules of derivatives. A beginning to the calculation is what marks the transition to the third phase. The solution is indeed found quickly at the beginning of the next lesson.

2.2.3 Analysis of the third main phase

Phase III comprises statements 133-164. No finer structure seems to be necessary here because, on the technical level reached now, every question seems to lead directly to an answer and there is no need for further splitting into subquestions (and, therefore, subphases). Here the thread enabling the teacher and her students to disentangle the problem has finally been found and the solution is now being developed, step by step (even though there are some minor mistakes at the level of calculation). This third working phase shows a familiar image of mathematics teaching: a mathematical procedure is used and worked on technically and finally a solution is reached according to "the principle of small and smallest steps" [Wittmann, 1988] without any major discussion or inquiry into the problem itself.

This first "phase-oriented" analysis of the lesson provides us with an overview of the general structure of the lesson. Besides the well known three steps: problem posing, problem discussing, and problem solving, phase II in particular shows how the ever-changing initial question concerning the box with maximal volume indeed structures the course of the teaching. During the lesson the routinized patterns of communication between the teacher and her students become more and more visible [Voigt, 1984]. With the intended solution of the mathematical problem in mind, the teacher again and again puts the same basic question. The question takes on different forms as, during the interaction with her students, the teacher herself has to adapt it within the frame of an interplay between a methodological concretization and an intended or assumed generalization [Steinbring, 1988]. Further, this description gives some insight into the specific culture which is dominant in this classroom and how the teacher's traditional understanding of mathematics induces her to make preparations and arrangements for transmitting the mathematical knowledge to her students [Nickson, 1992].

A characteristic feature of this manner of teaching is a kind of movement from the concrete level of instructions for constructing the box to the level of calculation of the solution wherein the mathematical "theory" (in this case the elementary rules of calculus) is applied, dissolved from any empirical or visual idea.

3. A specific "logic" of the course of teaching

Which of the contributions and statements during this lesson are of an individual or random nature? Which, on the other had, seem to be essential? In the course of teaching, is there something like a "logic of development"? Is the development of knowledge subject to a certain type of "causality"? What kind of cause-effect relationships are to be found in this teaching?

3.1 The pattern of the teacher's questions

The sequential structure of this lesson as presented in the previous section (see also Appendix 3) was strongly organised by the questions of the teacher. This shows a first kind of "logic of teaching". With regard to the new subject matter to be learnt, one can tentatively interpret the questions of the teacher as "causes" while the answers and solutions given by the students can be regarded as their "effects" in the temporal evolution of the lesson; on the level of interaction between teacher and students, the questions posed by the teacher also seem to be strongly determined by the answers given by the students [Kieren & Pirie, 1992].

The teacher's questions play a central role in structuring the content-related contributions observable on the surface of the interactions. The main questions are variants of the mathematical theme of this lesson, i.e. the quest for the biggest possible volume; the subquestions always refer to particular aspects of it.

The teacher's questions:

1) "When does the box have the biggest volume?" (6)
2) " ... Which box has the greatest volume? Look for the box with the greatest volume" (18)
3) "Which box is probably the biggest?" (26)
4) "Do you know for sure that the fourth is definitely the biggest box I can build from this sheet of paper?" (33)

4.1) "What does the volume depend on?" (40)

4.2) "How does the volume behave after what we tried out here?" (44)

4.3) "What is the volume determined by, taking account of the way we have constructed the box?" (47)

4.4) "So what does the volume depend upon? Now it becomes very clear." (60)

4.5) "What are the values that x can take?" (62)

5) "From the boxes you have constructed we know that the one with $x = 4$ is roughly the biggest in volume. Do I know that this is really the biggest box I can construct from this paper?" (71)
6) "Are we obtaining more precise values now? What does the biggest box look like? How can we determine the value x for which the volume will be maximal?" (102)

6.1) "What type of equation is this?" (108)

6.2) "What do we call it then? When one magnitude permanently depends upon another magnitude?" (110)

6.3) "In principle you should see it from $V(x)$." (112)

7) "I want to find an x for which $V(x)$ is biggest." (114)

7.1) "What does it mean graphically that the slope is nothing?" (121)

8) "So, come on. We want to find the position where the volume is maximal. What should I do now?" (133)

The eight central questions (together with the chosen nine examples of subquestions), disconnected as they are, here, from their interactional context, represent a kind of "logical system" for reaching the objective of the lesson. If one considers the sequence of teacher questions from an *a priori* point of view, i.e. from the point of view of the goal to be reached (in particular, the knowledge of the function giving the volume of the box in terms of x, x being the length of the side of the square that has been cut off from the rectangular sheet of paper), and if one knows how to apply the well known technique for finding derivatives of functions, then one could indeed attribute to this system of questions something like a specific logic and a consistent construction. With the help of the questions and subquestions presented by the teacher all the knowledge elements are put together step by step: those for the specific equation of the volume, its interpretation as a function (with variable x and the domain of x) and for the insight that this is a special case for applying the derivative. [For the role of teacher questions from the point of view of methods of mathematics teaching, see e.g. Wallrabenstein, 1978, 156, and from the point of view of pedagogy, see, e.g. Aschersleben, 1989, 23]

This more or less logical arrangement of the teacher's questions represents only one characteristic, even if it dominates and constitutes the specific type of interactive teaching patterns. According to widespread opinion, the teacher is the person who precisely determines and regulates what has to happen and when. But however well the lesson plan follows the casual structure of its mathematical contents, the teacher must take into account the interference from many other factors.

The modifications to the teacher's questions show, in part, that the course of teaching cannot simply be performed step by step on a single level, fixed once and for all.

The eight central questions can be divided into four types. The first type (questions 1, 2) refer to the boxes which have to be concretely constructed; one could call these "*empirical* questions". Questions of the second type (3, 4) have the intention of putting the problem into a more *theoretical* perspective; here one can find empirical generalising questions, formulated without mathematical symbols or concepts. Questions of a third type (5, 6) connect contextual conditions with mathematical concepts and use symbols in their presentation; this type could be called *empirical-mathematical*. Finally, the fourth type of question (7, 8) expresses the problem in purely mathematical terms with no reference to the context of concrete boxes. These can be called the *technical-mathematical* questions.

The characterisation of these four different types of questions shows that the epistemological status of the technical questions changes substantially in spite of the unchanged goal; they mark the movement of the teaching from the empirical problem situation to the mathematical and technical calculation of the result.

3.2 The pattern of the problems posed by the teacher

The observed way of "methodical abstraction" is accompanied by and reinforced through an increasing use of technical mathematical terms. This appears not only in the teacher's questions but also in the students' contributions. Looked at carefully, this is not simply an increase of additional descriptions expressing a mathematical point of view, but rather a continuous substitution: the concrete elements of the initial problem situation are gradually replaced by mathematical terms. It is only at the end, when the value of x is finally determined, that a single reference back to the concrete box is made

That in this lesson there are indeed substitutions for the object of discourse can be detected in the instructions from the teacher to her students. These can be interpreted, within the frame of the respective problems, as a kind of "hidden descriptions by definition" of the actual objects of teaching and learning.

Here is a sample of the teacher's instructions:

1) "As an introductory example you will have to solve the following exercise: with a 20 by 25 cm sheet of cardboard one can build a box, open on top. Have you any idea how this can be done?" (1)

1.1) "What you should do is this: you have to construct such boxes first of all. For this you will need your manipulative abilities because all you will get is paper. Each pair of you please take a sheet of paper and, by gluing the corners, building the boxes, compare them with your neighbours'. Then compute the volumes of your boxes and write the values in a table of values which I shall give you later. Always work in pairs" (6)

2) "Draw the graph of the function that we have obtained" (78)

2.1 "Who will come to the board and complete the graph? .....would you please do it" (98)

3) ".....What should I do now?" (133)
"Yes, first we have to calculate the slope at any point of the curve, so—find the the derivative" (134)
"Do that, please!" (135)

Alongside the system of teacher questions, the course of the teaching is structured by the above-mentioned instructions. This is like a second logical system. "Instructions" and "teacher questions" complement each other. The object of learning is presented by the teacher with the help of instructions, in a context-dependent and still imprecise manner at first. Later, if necessary, this object may be replaced by another object (in particular: "construction of a box" is replaced by "drawing the graph of a function") At the same time, the implicitly accepted mathematical frame within which this object will be elaborated is furnished. (In this situation Krummheuer speaks of "framing": Krummheuer, [1982]). The questions (with refinements) on the other hand serve to decode and clarify the mathematical problem. The instructions given to the students, the possibility of hands-on experience, the teacher's questions developed in interaction, more and more emphasize the "mathematical content" of the object of learning. In this lesson, as well as in her other lessons, one could observe that there existed, for the teacher and her students, a jointly-assumed attitude of expectation, a kind of tacit agreement; the real goal of the lesson, the intended problem and level of activity, was the technical terms and the calculation procedures. In this respect, for instance, the problem of "constructing boxes" was not really taken seriously (6). and the "drawing the graph of the function" does not fit the requirements of mathematical rigour because there are still imperfections in it (88). On the other hand, it is not necessary for the teacher to explain in detail the objective of "finding the derivative of a function" shortly before reaching the level of calculation; all the participants promptly accept and understand this type of a mathematical problem.

3.3 The epistemological pattern of the lesson

The movement from an empirical concrete problem to its formal mathematical solution can be described qualitatively with the help of a graphical diagram (see Appendix 2). In this diagram three levels are distinguished as marked by the instructions of the teacher:

- the *empirical* level: construction of concrete boxes with the help of the given rectangular sheets of paper
- the *geometrical* level: drawing an approximate graphical sketch of the function (at discrete values) aiming at the graph of the function;
- the *computational* level: differentiating the function and finding the maximum.

The content-related contributions (of teacher and students) are represented as black stripes according to the different levels and according to the particular phases and subphases of the teaching as structured by the teacher's questions. (This visual diagram for representing the qualitative development of knowledge in everyday mathematics teaching is a modified version of an empirical procedure for analysing the interactive negotiation of meaning of mathematical knowledge in teaching; see Bromme & Steinbring [1990]. In inspecting this graphical display along the different phases one can observe that while during phase 1 the classroom interaction remains on the empirical level, in phase 3 the interaction concentrates on the computational level alone. Phase 2 (with all its many subphases) shows alternating transitions between the empirical and the computational levels. During subphase II.1 discussions are kept mainly on the empirical level; however, there are a couple of shifts to the computational level. One exception is the intermediate phase II.1.4 during which the two "hasty" students present an algebraic formula for calculating the volume. Their formula already contains the independent variable x. At the end of this intermediate phase the teacher emphasises the importance of this variable x at the computational level and the constraints on this variable are discussed. During subphase II.2 the above-mentioned substitution of the object of study takes place: "box" is replaces by "graphical representation of the function". This subphase starts with key words such as "monotonically increasing", "monotonically decreasing", "turning point", etc. In this phase we find the "complaint" by a student. Phase II.3 runs on the computational level nearly all the time. This means, in particular, that no connection between "equation for determining the volume" and "equation of function of the dependent variable x" is discussed, and an "optimal" version of the question is formulated in a manner that forces the "correct" solving approach onto the students. This approach to solving the problem is justified on the level of geometrical interpretations (II.3.4). In the last phase (main phase II) all interactions are reduced to the computational level.

A perusal of the structure of the lesson as represented in this diagram may bring us to question the idea that teaching can be organised by the teacher on the basis of the *a priori* systematic structure of the mathematical knowledge It undermines the deterministic idea that a teacher can successfully carry out his or her plan of questions in order to gradually reach the objectives of the lesson. It is true that, in the analysed lesson, the teaching starts from the empirical level and finishes strictly on the computational level (without any systematic feedback to the context) It is also true that, through her instructions, the teacher in some way "instigates" the three levels of interaction and pushes, by her more and more pointed questions, the observed form of a "methodical mathematisation" of the teaching-learning object. However, the alternations and the changes of interaction on all three levels during phase II give no reason to believe that, from the beginning of the teaching, there is only one strictly organised deterministic structure (whether it follows the intention of the teacher or follows the hierarchical structure of the knowledge). What we observe here, in fact, is a most difficult epistemological rupture: the only clear thing about it is that, starting from the empirical level, in some way the computational level has to be reached. From a macroscopic perspective, the preconditions posed by the teacher and the content-related structure (questions, instructions, technical terms, procedures for solutions) are "causes" that lead to the desired effects. On the other hand, though, the microscopic perspective, especially in local subphases, shows detailed deviations that put into question the supposed strictly causal and logical construction. Evidence is provided by the context-dependent teacher's modifications of questions which either emphasize or neglect the students' ideas. It is these techniques that allow the teacher to pick her way through to reach the global objective of her teaching in this lesson (from a theoretical communications perspective this phenomenon is known as the "funnel pattern", Bauersfeld, [1978]. The specific logic of a lesson is interactively constructed and regulated according to the content related objective of teaching.

The two observable "movements" in the course of this lesson—that is, the more strict and linear interactive pattern in accordance with the mathematical structure on the macro level, and the situation-dependent mutual influences and ruptures on the micro level—can be characterised both from the teacher's, as well as from the students' perspective. If one takes the teacher's perspective, as is done here, based on the system of questions and instructions as well as on the technical terms and computational procedures (which the teacher, unlike her students, already knows) then a conception of teaching as a "completely consequent system" [Andelfinger & Jahnke, 1987] dominates. For the teacher, phase II does not simply represent a rupture in mathematical knowledge because she has already decided what kind of know-hows the students will have to acquire. However, the alternation and changes of interactional level may appear to her to depend on trial and error, and thus the teaching course might seem not always consequent. As far as the students are concerned, they are faced with additional problems: they do not yet have the knowledge necessary for the solution of the problem. This is why they are really in a phase of rupture and reorganisation of their knowledge during this part of the lesson.

4. What kind of school mathematical knowledge is elaborated and mediated in the teaching process?

4.1 The concealed ruptures in the understanding of the students

Let us now try to analyse the development of knowledge in the lesson from the students' perspective. At some point in the main phase II, underneath the structured surface of the observed teaching, one can detect some ruptures and misunderstandings of the students which cannot be simply dismissed by saying that, after all, at the end of the lesson these misunderstandings are dissolved by a generally-accepted solution on the technical mathematical level. These emerging problems of understanding are symptoms of the students' difficulties in the acquisition of new concepts. These difficulties arise in the tension between the concrete context (constructing boxes, graph of function, specific type of equation, etc.) and its conceptual generalisation. For the teacher the students' problems of understanding seem to be just disturbances within the system of her questions and instructions, but they are legitimate and true problems occurring in the development of comprehension in the students.

Here are the possible "ruptures" in the students' comprehension:

1) what kind of (functional) relationship is there between the surface and the volume of the box? (34 ff.)
2) how does measurement relate to computation when determining the volume of a box? (84ff)
3) is it possible to determine the value of x for which the volume is maximal by using symmetry conditions? (77)
4) how does the concept of "equation with two unknowns" relate to the concept of function? (105ff)
5) how is it possible to obtain the idea of a "universal" parallel with the help of a very specific drawn parallel? (124ff)

These five problems of understanding are instances that show the difficult balance between unique and concrete cases and their theoretical generalisations, between the contextualisation and the decontextualisation of theoretical knowledge. These are problems that the students have to face in their learning process because they are still "ignorant" and not yet able to dissolve this tension from the perspective of a more comprehensive view of knowledge [Seeger, 1990].

4.1.1 The first problem of understanding

In what ways does the classroom interaction deal with these emerging problems of understanding? The students approach the first problem (34-35, 37, 39, 50) with the apparently correct idea that the more surface or paper is at hand the bigger will be the volume of the box. But this relation has to be linked to the specific conditions of construction by means of cutting off squares at the corners (and neglecting these surfaces) as well as with the folding of the rectangular sides. The intuitive starting idea then no longer holds: first, the volume of the box increases when the paper at hand decreases, and then the volume becomes smaller and approaches towards zero. A "solution" of this problem of understanding is given by pointing to the constructed boxes—that is, by empirical examples and counterexamples—and afterwards by the transition to a formula describing the connection between the surface at hand (or the squares cut off) and the volume. This "solution" strongly fits the strict objectives of the specific problem. The problem is not treated for its own sake: there is nor real and serious discussion about the practical adequacy, particularities, or modifications of the conditions of construction within the practical context. Is it really true that when building containers, about 10% of the material will not be used? What are the practical conditions of construction in reality? The construction procedure given in this example seems to be simply a procedure to be justified *a posteriori* from the perspective of the mathematics developed from it. One has the impression that this problem of understanding has been merely put aside and not properly treated in this lesson.

4.1.2 The second problem of understanding

The second problem of understanding (84,88,89) shows that the concrete situation is not really taken seriously but is already evaluated from the point of view of the intended solution strategy. Indeed one can observe that the students cut and glue, but the measurement of the edges for determining the concrete volume remains imprecise; here the teacher expects the students to calculate. Again, the conditions of the practical context are in opposition to the mathematical interests. Within the concrete practice of constructing containers the exactness of mathematical results to many decimal places cannot be used directly; measurements will matter if, for instance, very thick cardboard is used. During the classroom interaction there is a tendency to functionalise this problem in favour of the pure mathematics: a solution is elaborated with the implicit implication that the practice of constructing containers simply follows the solution.

4.1.3 The third problem of understanding

The third problem of understanding (73, 74, 76, 77) is, in fact, an avenue to a different solving strategy. When constructing and measuring the concrete boxes, the students spontaneously express presumptions such as, for instance: [the maximum volume is when the side of the cut-off square is] "between 4 and 5", "between 3 and 4", or "between 3 and 5". Further, probably by an implicit assumption of symmetry, someone proposes midway between 0 and 10—namely 5—as a solution. These contributions of the students could be quite a good starting point for an approximate calculation of the mathematical problem. It would be possible to obtain very quickly a practically useful solution by means of smaller and smaller intervals and calculations with a pocket calculator. But the teacher does not accept this direction; she wants to solve this problem with the help of the rules of differentiation (this is how she planned it). Using such a non-theoretical, direct, and approximate way of working could eventually make the mathematical theory which has to be developed here superfluous. With her question "it must be somewhere near 5?" (78) the teacher emphasizes the imprecision (the inadmissibility?) of the students' proposal and asks them to draw the graph of the function in order to be able to pursue the "correct" solution approach in the frame of the mathematical theory.

4.1.4 The fourth problem of understanding

The fourth problem of understanding (105-107) shows an insufficient understanding of the notion of dependent and independent variables at the level of formula and equation. The assumption of one student that what is given is an equation with two unknowns is"cleared away" by evoking the concept of "function". The student's problem is thus dissolved by this abstract mathematical term, there is no meaningful relation to the context of the mathematical problem.

4.1.5 The fifth problem of understanding

The fifth problem of understanding (124,125) emerges in the context of the geometric representation of the function (which is then only given a small finite set of values) and during the attempt to draw parallels to the x-axis touching this graph to illustrate the idea of a solution. Because the graphical representation is not complete but only given at a few discrete points the problem arises how this very specific parallel could represent a general solution. One possibility for discussing this difficulty might be to "complete" this graphical representation by means of a kind of graphical approximation (as is possible for instance with plotting functions on a computer) in order to get a solution. Again, this way to a solution is not pursued but the solution planned beforehand is aimed at: the definition of the general parallel can be thought of as the "slope equals 0" calculated from the derivative of the function. This problem of understanding is again used to reach the goal assumed by the teacher and the specific inherent difficulties of this problem are not treated.

The problems of understanding the students express are not valued in their own right. Step by step they are put aside by developing and clarifying the correct technical procedure for a solution. Each of these problems of understanding could induce sufficient possibilities to solve the problem within an appropriate frame of means and procedures: on the practical level, by trials and measurements; on the numerical level by computational approximations; on the graphical level by approximate drawings of the graph. The ruptures of understanding are not really solved or explained (what in part is not possible in a direct way) but the classroom interaction and the work of teacher and students are progressively shifted more and more towards the technical level.

Our analysis has shown from a different perspective problems of understanding that students have with the concept of function as they have been investigated by Sierpinska. "Students' difficulties with the notion of function are widely reported and well known. ... Students have trouble in making the link between different representations of functions: formulas, graphs, diagrams, word descriptions of relationships; in interpreting graphs; in manipulating symbols related to functions ... etc." [Sierpinska, 1992, 25]. In our example, a source of these problems is the teacher's ignorance of the students' implicit frames of interpretation and of justification; the students seem to be bound to very close domains of experience—which also seem quite sufficient for solving the given problem—whereas the teacher always aims at the "general" theory she has in mind.

4.2 The conflict between the students' growing ideas and the teacher's fixed knowledge structure

What are the reasons for this? For one, such forms of mediating mathematical knowledge in teaching dominate because teaching and learning processes are considered, by teachers, from a ready-made perspective: the teaching is goal-oriented. The teacher already knows the elementary theory still to be learned by the students and she (not always consciously) orients her students to this level by her choice of questions and instructions. This is why she evaluates ruptures, misunderstandings, questioning of meanings, primarily from the point of view of her ready-made knowledge and lets these problems appear superfluous, or to be solved by switching to a more formal mathematical level. One could say that, in this way, the problem of justifying knowledge [Jahnke, 1978] is, in principle, evacuated from this context of interaction. For example, one can suppose that, at the beginning of the lesson, in engaging the students in the activity of constructing and comparing the volumes of the boxes, the teacher does not consider the task as having a value in itself but understands it as a "disguised" introduction to functions, derivatives, extreme values, etc. This seems to be the attitude of the students as well; the fact that they do not take this seriously is expressed by laughter and smiles during the interaction between the students and the teacher when speaking about boxes.

I believe that, in order to break with such a structure of classroom communication, it is necessary for the students to gather more experience with real objects. A more serious reflection on the practical construction of boxes would be worthwhile. Also the question could be posed of how the mathematical theory of extreme values (as a simple form of optimalisation) could contribute to the practical problem of constructing boxes; what practical constraints (like the thickness of the cardboard) should be given serious consideration in the mathematical model? This would help to emphasise the relative autonomy of the two domains, the domain of reality and the domain of mathematical theory, and to clarify the possibilities and limitations of mathematical models.

In the lesson, the actual objects of study are always too quickly removed from the students, they are defined away, they are not taken seriously. So students get the hint that these are not really the objects to be analysed and they orient themselves towards the mathematical procedures and methods of computation. Later on it is only these that they will accept as the proper important objects of mathematics teaching and learning

The teacher considers the box and the first graphical representation of the function (for the role of "objects" of investigation in the mathematical classroom, cf. Seefer & Steinbring [1992]) essentially as methodological means for skilfully reaching the goal of the lesson, namely the mathematical analytical approach to the solution and computation of the problem of optimal volume. Neither the boxes nor the plotting of the graph represent any valuable or meaningful objective of understanding in this lesson. One might get the impression that during this lesson the teacher's main intention is not to teach for developing and improving the personal understanding of the students but to influence the students to adopt the teacher's understanding. [Hiebert & Carpenter, 1992]

4.3 The course of the mathematical lesson as a complex, self-organizing system

A posteriori the actual teaching course in this lesson could be characterised as a "complex system" in which manifold, partially counteracting, mechanisms of regulation are functioning. Besides the mathematical content, with its seemingly logical structure, one can observe the influence of "teacher questions" and "teacher instructions"; a certain counteraction to the tendency to a consequent linear organisation of this teaching course is observable in the emerging problems of the students' understanding.

How can one describe the structural type of a such system? Is it primarily a completely consequent system in which the teaching process is organised according to the logic of the subject-matter structure? Or is it a self-organising, living system evolving in interaction between teacher and students and driven by their common intention to understand the mathematical contents? It seems that both of these opposite positions have been observed in the lesson. From the teacher's perspective and intentions, with her ready-made knowledge about the problem in question, a rather deterministic and causal structure dominates (for a critique see Wittmann [1988]). From the students' perspective this structure appears different and is less unambiguous. Mainly in phase II, during the transition from the empirical level of the problem to the level of the formal computation, severe problems of understanding emerge that influence and disturb the linearly-ordered teaching sequence at some local points. For the students the lesson may often appear unsystematic and chaotic; the problems of understanding and the way to their "resolution" by "methodical abstraction and mathematisation" of the context of the problem testify to a search for the "correct" level of work, sanctioned by the teacher. Partly, the students are unsure whether they really have to work on the empirical, constructive, or geometrically-imprecise level. These problems of understanding are also test cases, signalling to the students in their interaction with their teacher whether and how they should work at this juncture in an accepted mathematical way. This is why the students do not persist in asking their meaningful questions and readily accept the "solutions" of the teacher which they know will lead to the correct approach. During many mathematical lessons students have unconsciously acquired the patterns of mathematics teaching and know when the level of "correct" mathematical activity is reached. This shows in a exemplary manner that, for the students, there is more to understanding in mathematics lessons than simply establishing a relation to already acquired knowledge. Students develop an understanding of what kinds of justification and terminology will be "officially" admitted and legitimized. This sometimes becomes even more important than the understanding of the mathematical contents [Maier, 1988].

Globally considered, every teaching unit can be characterised as an interactive and self-organising system [Seeger, 1990]. The manifold influences which this system is subject to can in some cases lead to the reduction of this living system to a causal and linear sequence. A dominant logical and hierarchical structure in the subject matter and a teaching method strictly organised accordingly (through question and instructions) will especially tend to form the sequence again and again in a "logical manner"; if in these cases the students also very quickly abandon their problems of understanding because of their former school experiences and simply follow the required "methodological ways of abstraction", the symmetrical situation of a mechanical system becomes more and more rigid. During negotiations of the meaning of mathematical concepts, problems and solutions, necessarily done from different positions for the teacher and the students, some of the emerging problems of understanding could provide important means for trying to make the teaching system more interactive and self-organising and to counteract the tendencies of reduction to a linear sequence.

This kind of "normal state" mathematics teaching produces obstacles to the development of a comprehensive understanding of mathematical ways of thinking and working. I hope it has become clear in the above analysis that it is impossible for the students to (seemingly) effectively learn step by step single elements of knowledge and procedures of computation because, together with the adjustments to a linear teaching sequence, a decisive change in the epistemological status of the mathematical knowledge has taken place: in principle, mathematical knowledge loses its content-related richness when it is gradually reduced to some formal ways of speaking, and to techniques of calculation and manipulation. Instead of a serious *evolution* of new mathematical concepts and knowledge for students not yet in this developed position, there is a seemingly *mechanical adaptation* of new verbal descriptions to an old frame of vocabulary.

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Appendix 1

Audio tape protocol of 15th October first lesson (shortened version)
(In the following "S" stands for "Student", "T" stands for "Teacher")

"... what is ahead of us is the topic of extrema"

I. The initial problem:
Constructing boxes (1-25)
I.1 The problem is presented and the guidelines for constructing boxes are discussed (1-12)

1 T: What is ahead of us is the topic of extrema As an introductory example you will have to solve the exercise: With a 20 by 25 cm sheet of cardboard paper one can build a box, open on top. Have you any idea how this can be done?
2 S: A box?
3 T: Yes, 20 by 25 cm.
4 S1: Yes, one can cut this sheet into little squares...5 by 5 ..or maybe it doesn't work? Or rather into 4 by 5, let's say
5 S2: Yes, I would first take 25 as the length and then something like 5 as the bottom ...or cut a bit from each corner and then glue it all together.
6 T: By cutting off little squares at the corners (blackboard) and then folding the sides The question is then: when does the box have the biggest volume? What you should do is this: you have to construct such boxes first of all (smiles). For this you will need your manipulative abilities because all you will get is plain paper. Each pair of you please take a sheet of paper and, by gluing the corners, build the boxes, compare them with your neighbours'. Then compute the volumes of your boxes and write the values in the tables of values which I shall give you later. Always work in pairs.
7 S: We make four squares here?
8 T: Yes, build the boxes from the prepared sheets, is this clear?
9 Ss: No.
10 T: You have all got sheets of the same dimensions but the squares that you have to cut away from the corners are different for each sheet. How can I build a box from this?
11 S3: Yes, when we cut away the corners, then we shall get practically a cross, and now we can cut off big corners like this and then fold up these arms of the cross
12 T: Yes, that's right. This is both the simplest possibility and an optimal utilisation of the paper.

I.2 Boxes are constructed, volumes are measured or calculated and the values are noted down in a table (13-25)

13 S1: Yes, sure, this is clear.
[Students construct boxes]
[Laughter, Teacher walks from table to table]
14 T: S13 has almost finished already The box looks quite good.
15 T: You have the most difficult box to construct because you are only allowed to cut off very little pieces.
16 S: (sarcastic) I'm glad.
[Murmuring]
17 T: The first example is already in shape.
[Voices]
18 T: When you are finished with the boxes, compare them with what your neighbours have built. Which box has the greatest volume? Look for the box with the greatest volume.
[Students construct, compute, talk]
19 T: When you have finished calculating the volumes please write the values into the table If you have time you can think how to calculate the volume in the general case.
20 S11: How to calculate this?
21 S12: Length times width times height.
22 S5: Yes, good question
23 T: Who has already computed the volume? S9?
[Voices. Students write values into the table]
24 S8: Here, you miscalculated, S1! this does not make sense!
25 S1: No I have not miscalculated at all, it makes sense. You can measure it for yourself, you are welcome... please.

II. Development and treatment of the question "which box is the biggest?" or "which is the biggest box I can construct with the given sheet of paper?" (26-132)

26 T: We have all but the first value. Take a look at these. Which box is probably the biggest?
27 S: Probably the first one.
28 S1: What do you mean by biggest? In height or what?
29 T: What were we looking for?
30 S17: We were looking for the box with the biggest volume. We have clearly computed the volumes.
31 S5: Oh, so Then it must be the fourth.

Transition from the empirical to the theoretical posing of the question/problem (32,33)

32 S7: It seems to be the smallest
33 T: Do you know for sure that the fourth is definitely the biggest box that I can build from this sheet of paper?

...

II.1 How do the cut-off squares determine the volume of the box? Understanding the side of the squares as the independent variable x (34-70)

34 S6: I wouldn't be sure. One can state it a different way. It's only in our case the biggest But if one would now... , you always have to cut away something The narrower the pieces we cut away, the bigger box we get.
35 S1: Yes, this sounds funny somehow, because with the first box, for instance, we have cut off only 1 cm on the sides and 4 cm from this one.
36 T: Could you repeat it, please, S1, I didn't understand?

II.1.1 (36-39)

37 S1: The comparison Anyhow, one cut off 16 square centimeters at each corner. And in the first only 1 square centimeter at each corner So it somehow does not fit.
38 T: What does not fit?
39 S1: The comparison. I think that this has a bigger volume, after all.

II.1.2 (40-46)

40 T: What does the volume depend on? Maybe we should think about this for a while.
41 S2: The volume depends on the height of the edge And when it is only one centimeter, it is not a big deal.
42 T: Where is the box with 1 cm cut off? Perhaps you can show it.
[Laughter]
43 S7: This isn't a box!
44 T: How does this volume behave after what we tried out there? When we cut off a square with side 1 cm, and then a square with side 2 cm?
45 S11: It strictly monotonically increases and then strictly monotonically decreases.
46 S8: The values grow in each case until 4 and then they decrease again

II.1.3 (47-53)

47 T: What is the volume determined by, taking account of the way in which we have constructed the boxes?
48 S9: Yes, I would say, on paper..
49 T: The size of the paper was the same in each case.
50 S9: Yes, but something was cut off, I mean, bigger parts. So, now, the less paper is lost, the bigger the volume should be.
51 S5: Then the biggest volume should be with 1 or 2 cm because then I have cut less away.

52 S9: Ach, so.

53 T: This is what contradicts our empirical knowledge. S10!

II.1.4 (54-59)

54 S10: Yes, I have just calculated a little bit on the calculator. I would say: $(20 - 2x)(25 - 2x)x$.

55 T: What is this?

56 S: He probably doesn't know himself.

57 T: Who will explain what S10 did?

58 S: In general! Yes, exactly, the general formula for calculating the volume.

59 S11: For the volume you have a times b times c. In our box the height is x, from each of the sides we cut off $2x$, so there remains the length of $25 - 2x$, and for the width $20 - 2x$.

II.1.5 (60-62)

60 T: So what does the volume depend on? Now it becomes very clear.

61 S3: Yes, the volume depends on the cut-off lengths.

62 T: Yes, this means we can represent our volume as depending on x.

II.1.6 (62-70)

Let us think about if for a moment. What are the values that x can take?

63 S12: Certainly less than 20, otherwise we would run out of the length of the sheet

[Laughter]

No, wait! 10! 10! 10!

64 T: What happens if $x = 10$? S3?

65 S3: It's just right.

66 S7: Hoopla! In this case you get a double wall

[Teacher shows it]

And then we can no longer speak of volume

67 T: So we have a bound. S4!

68 S4: And greater than 0. Because then we would have no edges So this we have no longer

69 T: What is obtained then?

70 S4: A surface

II 2 Drawing the graph of the volume function and attaining a rough idea about the course of the volume function (71-101)

71 T: So now the question once more: from the boxes that you have constructed we know that the one with $x = 4$ is roughly the biggest in volume. Do I know that this is really the biggest box I can construct from this paper?

72 S: No!

73 Ss: Between 4 and 5! Between 3 and 4!

II.2.2 (78)

78 T: It must be somewhere near 5? Well. S6 just said that we obtain a curve; let us draw this curve. Maybe this will give us a basis to judge whether the biggest volume really lies around 5. Draw the graph of the function that we have obtained.

Intermezzo (79-83)

79 S2: Will there ever be an end to these functions?

80 T: In grade 12.2! Is that OK?

81 Tt (the observing teacher): In class 13.2 we shall repeat it all once again

82 S: Then with luck I shall have no more math

83 S: By that time math will have killed me.

84 T: [draws on the board] You can draw it a little better in your copybooks. Here one value is not exact, it is not 755 cubic centimeters but 750

85 S13: I take this to be gossip.

86 T: Why, S13?

87 S7: Because he's got the biggest mouth.

88 S13: We have not yet calculated correctly. We have only measured. What we have probably measured with precision are the edges.

90 S13: Yes, that's right

91 T: And then you worked with decimals.

92 S13: Yes, precisely.

93 T: In your copybooks if you take 1 cm as a unit you should be able to draw a good graph.

[T draws on the board, students draw in their books]

94 T: S8!

95 S8: This value, we already know..

96 S11: But we have excluded these. We are considering the function on an open interval.

97 S8: The right bracket is by 10?

98 T: Yes, it should be by 10. Who will come to the board and complete the graph? S14! would you please do it?

99 S12: Yes, up to here because at 10 we cannot obtain any value Yes, and in the range between 0 and 1, what do we get?

100 S: That is a good question!

101 S8: These values will be, I don't know, between 3 and 400. One value lower. Then the curve will be on the x-axis and so the volume will tend to zero.

II 3 Interpreting the volume equation as the function $V(x)$, recognizing the problem posed as an application of the known rules for finding the derivative. (102-132)

102 T: Are we obtaining more precise values now? What does the biggest box look like? How can we determine the value x for which the volume will be maximal?

103 S6: I think we need another equation. When we solve the equation for x, then we shall have two unknowns

II.3.1 (103-109)

One unknown is Vmax. So we need yet another equation.

104 T: Once more, what is unknown here, S6?

105 S6: The x is unknown. This is what we have to find out. And the maximal volume depends on x—it is therefore also unknown.

106 S: This means that if we had x we would also have the volume

107 S6: Yes, but we have only one equation and two unknowns and therefore we cannot calculate.

108 T: What type of equation is this?

109 S1: I don't know what type it is. But one variable depends on the other. So if I have x, I automatically have the other one. This way we don't have two unknowns, only one is depending upon the other.

II.3.2 (110-1114)

110 T: What do we call it then? When one magnitude permanently depends upon another magnitude?

...

111 S: Proportional?

112 T: In principle you should see it from $V(x)$. S6!

113 S6: Function?

114 T: Yes, that's right. We have already drawn the graph of the function. We have observed that the function is defined on the interval (0,10).

II.3.3 (114-120)

Yes, when we know that this is a function, look at the graph once again. I want to find an x for which $V(x)$ is biggest.

115 S9: Oh, boy!

116 S6: Yes, one can calculate the slope and then read from the slope when the point has the highest value. Or maybe it wouldn't work? Yes, when the slope is equal to zero, then we have the highest value.

117 T: S19, do you know what S6 just said? Please continue.

...

118 S7: I didn't hear.

119 T: S15!

120 S15: Yes, S6 says that the slope always becomes nothing at the highest point, where it falls flat. Somewhere it must be zero. And we have to determine the point where the slope is zero. Then we have the highest point.

II.3.4 (121-132)

121 T: What does it mean graphically that the slope is nothing? S1?

122 S1: A parallel to the x-axis.

[T draws a parallel at $(4,V(4))$]

123 S12: Yes, so when I say, for example, that I want the graph... namely, that here once again it goes up and then down to 5. So I cannot draw one parallel and then conclude that at 4 it is really the highest.

124 T: You are perfectly right, we do not know how this curve behaves in the interval (3,5). [T draws an alternative behaviour of the curve] But does this disturb the plan for the calculation?

125 S15: We do not know how the curve behaves in this range and now you simply draw a parallel to the x-axis through the point $(4,V(x))$.

126 T: What S6 just said was: I find the maximal volume where the parallel to the x-axis touches the curve. Now the graph is drawn so that it admits of a maximum at $x = 4$, and this is why I have drawn the tangent here. If this is not the case and the highest point lies somewhere else, then is the proposal of S6 wrong? S3!

127 S3: Since we have found only some position, if at 4 or near 4 there is no highest point, I think it doesn't matter, practically, we might just as well try 4

128 S6: I think we are going in another direction now—making things precise So I propose that we calculate the slope of the curve and when the slope is zero then we have the highest point. But from that we need not conclude that the highest point is at 4. This is still to be found I am not saying that the highest point is at 4.

129 S15: But this is how it is drawn.

130 S6: Yes, but only for illustration.

[T draws tangents in alternative drawings of the curve]

131 S: For visualization

132 S3: And precisely where it lies we obtain from S6's approach

III Computation of the mathematical solution of the problem (133-164)

133 T: Precisely. So, come on. We want to find the position where the volume is maximal. S5, what should I do now?

134 S5: Yes, first we have to calculate the slope at any point of the curve, so—find the derivative.

135 T: Do that, please!

136 S5: Which function do I have to take?

137 T: Who can help S5? Which function do I take? S13!

138 S13: I would take the function that is on the board.

139 S: So would I.

140 S: Because there is only one on the board.

141 S: You have to calculate the derivative first.

142 T: How is the derivative found?

143 S: x,x squared, x cube, plus 500?

144 S: Hey, I have forgotten everything!

145 S: x squared?

146 S: We can go on calculating this $V(x)$ first

147 S: S17, do it please!

148 S: What?

149 T: S16, do it.

150 S: Multiply?

151 T: S15!

152 S: Wrong!

153 T: S17!

154 S17: Yes, I would naturally multiply first and then calculate the derivative.

155 T: Yes.

156 S17: Multiplication gives us the $4x^3 - 90x^2 + 500x$ and the $V'(x)$ is then equal to $12x^2 - 180x + 500$.

157 T: What did S6 propose to do next? S8?

158 S8: We put it equal to zero and solve for x.

159 S14: $0 = 12x^2 - 180x + 500$.

160 S: Divide by 12 and use brakets.

161 T: This is a quadratic equation. S1!

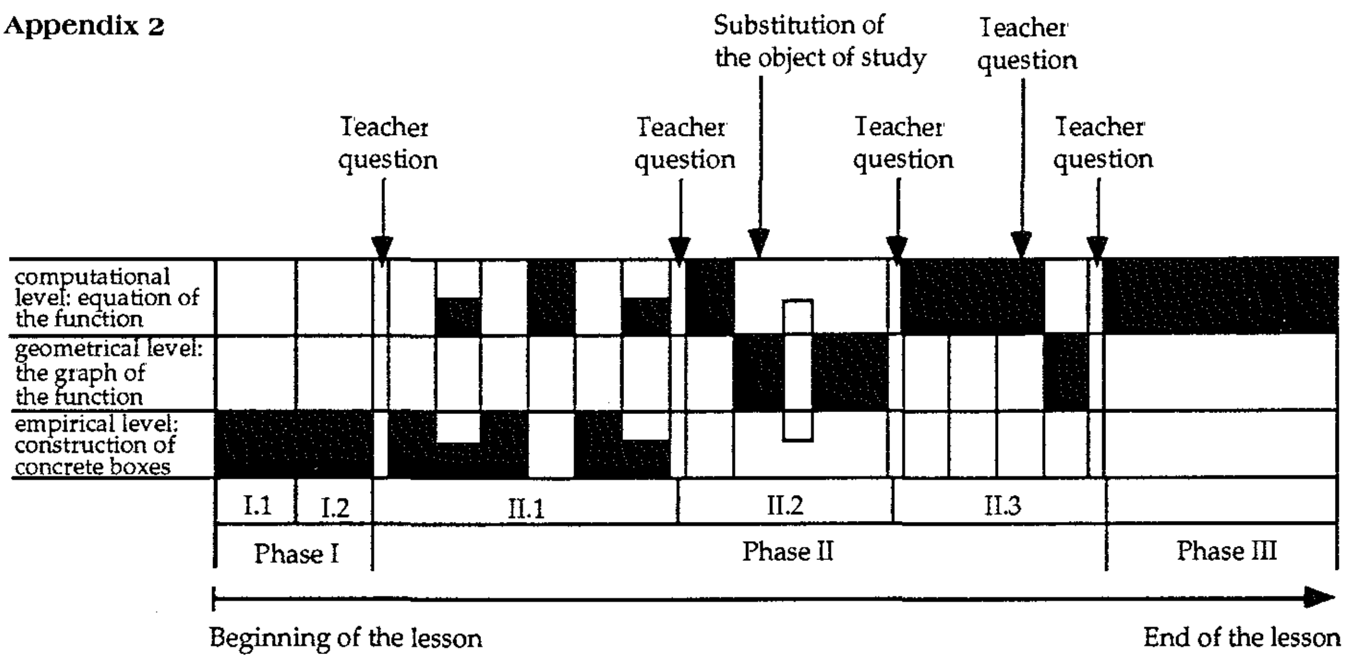
162 S1: I divide by 12: $x^2 - 180/12x + 500$

163 T: Plus 500?

164 S1: Ah, no, a moment, ...500

[Bell for break]

Appendix 2



Appendix 3

Structure of the lesson

Lines	Phase	Topic	Lines	Episode	Description	Sub-episode	Description
1	I	The initial problem	1				
					The problem is presented and the guidelines for constructing boxes are discussed		
		Constructing boxes	13				
			14		Boxes are constructed, volumes are measured or calculated and the values are noted down in a table		
25			25				
26			26		Transition from the empirical to the theoretical posing of question/problem		
	II	Development and treatment of the question 'which box is the biggest?' or "which is the biggest box that I can construct with the given sheet of paper?"	33	II.1	How the cut-off squares determine the volume of the box. Understanding the side of the squares as the independent variable x	II 1 1	Which box is definitely the biggest? Wrong intuitions of students with regard to the box
						II 1 2	What does the volume depend on? Functional arguments: monotonic decrease, monotonic increase.
						II.1 3	What is the volume determined by? Contradiction with empirical knowledge: large surface implies big volume
						II 1 4	Hasty students, formula and pocket calculator
						II 1 5	What does the volume depend on? It depends on the length of the side of the square cut off, it depends on x
						II 1.6	What are the bounds for x? $x = 0$ and $x = 10$
			70	II 2	Drawing the graph of the volume function attaining a rough idea about the course of the volume function	II 2 1	Taking the basic question again; key words: strictly monotonically decreasing, increasing, approximate solution.
						II 2 2	Drawing the curve; key words: graph of the function, graph, function Intermezzo: *Is there no end to these functions!* Drawing measuring calculating vs mathematical rigor
			100	II 3	Interpreting the volume equation as the function $V(x)$, recognizing the problem posed as the application of known rules of differentiation	II 3 1	Taking the basic question again; key words: equation, x, unknown, Vmax, second equation, etc
						II 3 2	Transition from equations to functions (provoked by the teacher's question)
						II 3 3	The basic question is now given in the conceptual, correct manner and provokes the formulation of a proposal for the solution approach
						II 3 4	Geometrical representation: "What does it mean graphically that the slope is equal to 0?" The discussion shows the proposed approach is accepted
132			132				
133	III	Computation of the mathematical solution of the problem			Single procedures: - calculating the slope at every point - which function? - calculating the function - multiplying out $V(x)$ - calculating the derivative, equating it to 0 - quadratic equation		
164							