Communications

Equivalent fractions and natural numbers bias

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Learning the concept of fractions is among the most challenging topics in school mathematics. One of the main sources of difficulties in learning fractions is related to *natural number bias* (Van Hoof, Verschaffel & Van Dooren, 2015). Applying properties of the natural numbers incorrectly in situations involving rational numbers can lead individuals to make systematic errors. But natural number bias does not occur only when students first encounter fractions. It occurs again in different forms when students encounter fractions in different contexts. Here I argue that it is not enough to address natural number bias when fractions are first taught, it must be addressed each time fractions are used in a new way.

Natural number bias

Natural number bias has generally two different aspects, *congruent* and *incongruent*. According to Van Hoof, Verschaffel and Van Dooren, "where reasoning in terms of natural numbers leads to an error, [it] is called incongruent. [...] where reasoning in terms of natural numbers leads to a correct answer [it is] called congruent" (2015, p. 40). For example,Obersteiner, Van Dooren, Van Hoof and Verschaffel (2013) report that, when comparing two fractions, some students think that $1/4$ is larger than $1/3$ since 4 is larger than 3 (incongruent aspect). Students' incorrect claim that $6/8 > 3/4$ can also be considered as an example of incongruent natural numbers bias. In contrast, selecting 0.89 as the largest number between 0.7 and 0.89 for the reason that 0.89 has more digits is an example of the congruent aspect of natural number bias (Van Hoof, Verschaffel & Van Dooren, 2015).

Based on the literature, the main incongruent aspects of natural number bias concerning fractions are the following:

1. Density: Every natural number has a successor, but it is impossible to name a successor of a rational number, as there are infinity many numbers between any two rational numbers. This property becomes even more confusing at higher levels when cardinality of infinite sets is introduced, as we tell learners that both sets, the natural and the rational numbers, have the same number of elements.
2. Representation: "While a natural number has a single symbolic representation, each rational number has an infinite number of possible symbolic representations" (Van Hoof, Vandewalle, Verschaffel & Van Dooren, 2015, p. 31).
3. Numerical size of rational numbers in comparison tasks: this aspect occurs with both decimal and fractional representations. With decimal representations some learners believe that a number with more digits is bigger. With fractional representation, "erroneous answers occur because learners commonly wrongly assume that a fraction's numerical value increases when its denominator, numerator, or both increase" (Van Hoof, Verschaffel & Van Dooren, 2015, p. 41).
4. Arithmetic operations on rational numbers: this aspect concerns both the learners' thinking about the outcome of an operation and the learners' ways of performing an operation. For example, believing that multiplication of two numbers always results in a larger one, or adding the denominators of two fractions rather than finding a common denominator between them.

Valuable studies have been done on these aspects when fractions are first introduced, but the use of fractions in later contexts has received less attention.

Many mathematical concepts are extensions of other concepts. As new concepts are learned based on understandings of old ones, congruent aspects of the old concepts can be used as tools supporting new learning. Making connections between what students already know and what they need to learn is important for teaching and learning mathematics. But attention must be paid to the incongruent aspects, which may form obstacles to learning. Failure to identify and pay attention to them can lead learners to make errors. Seemingly sensible extensions of old concepts can fail in a new context. Learners may generalise some well-known attributes that are not generalisable to a new situation. Below I offer examples of natural number bias that occur when negative numbers, irrational number and rational exponents are introduced. My examples focus on the impact of incongruent natural number bias on the concept of the equivalent fractions.

Natural number bias and equivalent fractions

The procedure for finding equivalent fractions usually taught involves multiplying the numerator a and denominator b (both natural numbers) by the same natural number n:

$$\frac{a}{b} = \frac{na}{nb}$$

This procedure is related to models such as the number line (Figure 1) or an area model such as a circle or rectangle (Figure 2).

Such representations help learners to connect fractions to their prior knowledge of natural numbers, and to give meaning to the computational procedure. According to Streefland,

> equivalence of fractions is almost exclusively dealt with in an algorithmic way. However the equivalence of fractional parts of pancakes or rectangles… [it] is so convincing, so evident, one hardly needs to give expression to it, even to primary pupils; they simply can observe this equivalence. (1978, pp. 51–52)

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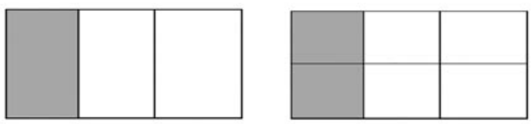
$$\frac{1}{3}=\frac{2\times 1}{2\times 3}=\frac{2}{6}$$

Figure 1. Number line representation.



$$\frac{1}{3}=\frac{2\times 1}{2\times 3}=\frac{2}{6}$$

Figure 2. Geometrical representation.

But this means that the concept of equivalent fractions is taught with the help of natural numbers. Of course there is no alternative, since students have not yet learned about other types of numbers. Thus, a justifiable natural number bias is introduced in connection to equivalent fractions. Later, however, incongruent aspects of this bias create obstacles to understanding equivalent fractions in some contexts. Below I explore incongruent aspects of natural number bias on equivalent fractions in the contexts of negative and irrational numbers, and rational exponents. In each section I begin with a problem, which I encourage you to consider before reading further.

The context of negative numbers

> **Problem:** Is the equality $a/b = c/d$ always incorrect, where $a < b$ and $c > d$ ($b \neq 0$, $d \neq 0$)?

With a small amount of mathematical knowledge, one may argue that since $a < b$ and $c > d$, then $a/b < 1$ and $c/d > 1$ so the above equality cannot be true. This argument is justified by the warrant that a number more than 1 cannot be equal to a number less than 1. This reasoning is affected by natural number bias, as there are no natural numbers *a, b, c, d* that satisfy the equality. But a look at the following correct equality reveals this bias:

$$\frac{-3}{3}=\frac{2}{-2}$$

Here, the fraction $2/{-2}$ appears as the ratio of a bigger number to a smaller one, while $-3/3$ indicates the ratio of a smaller number to a bigger one. In general, in the context of negative numbers, the following special relation can be concluded:

$$\frac{-a}{b}=\frac{a}{-b}=-\frac{a}{b},\ (a,b>0,\ b\neq 0)$$

It should be noted that in the context of negative numbers, the number line and geometric representations do not work well. Where should $2/{-2}$ appear on the number line, or how can a division of a shape into −2 parts be depicted?

The context of irrational numbers

> **Problem:** The reciprocal of a number equals half of the number. What is the number?

One might try to solve this problem using trial and error. One first tries a few small natural numbers. It becomes clear that 1 and 2 are not correct answers and also that no natural number more than 2 can be the correct answer. This is because the reciprocal of such a number must be less than one, while its half is more than one. At this stage, under the influence of natural number bias, one may even think that there is no such number.

Writing an equation, however, reveals a new possibility:

$$\frac{1}{x}=\frac{x}{2}$$

The reciprocal of a number can equal half of the number only in the context of irrational numbers, because the equation $1/x = x/2$ implies that $x = \pm\sqrt{2}$. Under the influence of natural number bias, one searches among the natural numbers instead of thinking of irrational numbers. Here the well-known form of equivalent fractions, $a/b = na/nb$ still applies. The correct equality $1/\sqrt{2} = \sqrt{2}/2$ is obtained by multiplying the numerator and denominator of $1/\sqrt{2}$ by the same number $\sqrt{2}$:

$$\frac{1}{\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}=\frac{\sqrt{2}}{2}$$

The context of rational exponents

> **Problem:** Is the equality $a^{1/3} = a^{2/6}$ always correct?

Here, it is clear that the equality $1/3 = 2/6$ is correct and a is fixed on both sides. Therefore, one might accept that the equality $a^{1/3} = a^{2/6}$ is correct. It is difficult to provide another reason for the acceptance of the equality. But here it becomes important to realise that while equivalent fractions are equal rational numbers, they might not be equivalent rational exponents. Here the natural number bias is two-fold, drawing on experiences with natural number exponents and with fractions with natural number numerators and denominators. The equality $a^{1/3} = a^{2/6}$ is not always correct, not because of the incorrectness of the equality $1/3 = 2/6$, but rather due to the meaning of rational exponents.

Suppose one wants to examine the validity of the equality

$$(-8)^{1/3}=(-8)^{2/6}$$

based on one's previous knowledge of equivalent fractions. One can get the desired result without even having to do any calculation under the influence of natural number bias. This is despite the fact that accepting the equality will result in this contradiction:

$$-2=(-8)^{1/3}=(-8)^{2/6}=[(-8)^2]^{1/6}=2$$

(Goel & Robillard, 1997, p. 319)

This contradiction occurs although there is no doubt about the correctness of the equality $1/3 = 2/6$ in the context of equivalent fractions.

The interpretation of rational exponents can also run into a 'real number bias'. If a rational exponent is defined in this way:

$$x^{\frac{p}{q}} = (\sqrt[q]{x})^p = \sqrt[q]{x^p}$$

then one sees that $(-8)^{1/6}$ is a complex number, because in general if q is even and x is negative, $\sqrt[q]{x}$ is a complex number.

The usual definition of equivalent fractions cannot resolve the contradiction as it occurs beyond the context of equivalent fractions. This problem is specific to the context of rational exponents and is not seen in the context of natural or negative numbers. Here, one can conclude that another criterion is needed to deal with the correctness or falsehood of the equality. This point can be considered as a didactic tool for teachers to demonstrate that neglecting the context of discussion may cause individuals to commit to contradictions. Unlike the fractions students have encountered previously, rational exponents cannot exist in many equivalent forms. In the context of rational exponents, according to Dugopolski (1995) "if $r = p/q$ is a rational number and the qth root of 'a' is a real number, then one defines a^r by $a^r = a^{r'}$ where r' is r reduced to its lowest terms" (as cited in Goel & Robillard, 1997, p. 319, notation modified slightly), therefore, $(-8)^{2/6} = (-8)^{1/3} = (-2)$.

These examples indicate that the concept of equivalent fractions does not always have the same meaning as the context changes from one number system to another. Below, I discuss a context *within the natural numbers* in which equivalent fractions take on different meanings, to further illustrate the impact of context on the understanding of a concept.

The context of probability

> **Problem:** Give an example of an occurrence with the probability of $3/6$.

For one with a minimum knowledge of probability, this will not be a difficult problem. Recall that the probability of an event A is defined as a fraction:

$$P(A) = \frac{n(A)}{n(S)}$$

where $n(A)$ is the number of elements in the set A, which is a subset of the possible occurrences in the sample space S. Both numerator and denominator of the fraction belong to the set of natural numbers. Perhaps the most familiar probability contexts are throwing a coin, $n(S) = 2$, and rolling a dice, $n(S) = 6$. In answering the problem posed above, you are likely to have thought of a dice, and perhaps the probability of rolling an odd number. But we know that $3/6 = 1/2$, so why is it strange to say the probability of getting heads on a coin is $3/6$? In the context of probability, the denominator of the fraction shows the size of the specified sample space, and if changed, it will indicate another sample space. There is no doubt that the equality $1/2 = 3/6$ is numerically correct, but in the context of probability, equivalent fractions are not interchangeable.

Implications for teaching and learning

The examples above show that understanding of equivalent fractions can be affected by natural number bias, and that this bias must be faced anew with each new context. In some contexts, congruent aspects of natural number bias can help teachers and learners to link prior knowledge to new concepts, and in others, incongruent aspects lead to obstacles. According to Ni and Zhou,

> A bias, regardless of whether it is innate or learned, provides both cognitive efficiency and inflexibility. When certain adaptive advantage of a bias is exploited, at the same time inflexibility brought about by the bias is reinforced. It is a challenge to manage the trade-off in instruction (2005, p. 40)

Paying more attention to both congruent and incongruent aspects of natural number bias in each new context can help teachers to predict and prepare appropriate instruction, and learners to be aware of obstacles they may encounter. My examples, and the special case of probability, reveal the importance of the context of discussion, especially with regard to equivalent fractions. I hope these examples can open up new horizons for both teachers and learners of mathematics in considering the concept of equivalent fractions beyond its formal definition in the context of natural numbers.

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$$-2 = (-8)^{1/3} = (-8)^{2/6} = [(-8)^2]^{1/6} = 2$$

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Some considerations towards the concept of money in elementary school: money as measurement

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In this short communication, we describe how money represents a unit of measurement with a distinct nature. Our goal is to argue for money to be considered a domain of mathematics which belongs to measurement and financial education. We conceptualize this as a first step toward a domain of financial numeracy.

What is measurement?

When measuring, we assign a *quantity* to a property or an attribute of a given *object*. Different properties can be measured, such as length, time, mass, volume, cost, etc. Learning to measure is more than simply reading a measurement tool, it is about comparing properties of an object with a unit, standardized or not. It requires a dialogic process of establishing properties to be measured and techniques to measure. When measuring, we move from direct comparison of two objects to indirect comparison using a tool of measurement such as a ruler or a scale, to compare an object to a *unit of measurement.*

The concept of mapping properties of objects from the physical to the numerical world by using units of measurement is what Hand (2016) calls 'representational measurement'. Such mapping must conserve the relationships that are observed in the physical world. Therefore, representational measurement translates attributes of objects and relationships between them onto numbers. We assign numbers to different attributes so that the relationship between the numbers corresponds to the relationship between the attributes. These relationships can still be verified without the practice of measuring, just by making a direct comparison between two objects.

Representational measurement is rooted in the physical sciences and provides the means to measure physical properties (mass, time, length, *etc.*). It has been extensively researched in mathematics education and is present in most elementary school mathematics curricula. Typically, students learn a set of related concepts about measuring a specific attribute of an object. These include direct comparison, the established units for the attribute, how to conduct measurements, how to make estimations and approximations, and how to use and transform conventional units.

In our daily lives, however, we deal with properties that are not only related to a single object, but to an aggregate of different objects. Furthermore, not all measurements stem from physical properties, some emerge in abstract or social contexts. When measuring abstract aggregate phenomena the property being measured is constructed by the process used to measure it. This is what Hand (2016) calls 'pragmatic measurement', in which properties come into existence through being measured. Examples include the statistical mean and inflation. This kind of measurement is widely utilized for aggregate phenomena, *i.e.,* when more than one quantity is taken into account to define an object.

In representational measurement, one can compare properties of two objects without the use of a measurement instrument (like comparing the lengths of two different pencils). However, in pragmatic measurement of aggregate phenomena, not only do we need to define the properties to be measured, but also how measurement will take place. Defining the economic value of a product, *e.g.,* a loaf of bread, is therefore an activity rooted in pragmatic measurement. To know how much to charge for bread or how much to pay for it, one must take into consideration many different measurable variables. The amount of bread, the kind of bread and the place it was sold are all important in defining its economic value. Additionally, knowing how to balance each variable is a calculation that needs to be defined and agreed upon between both parties in the economic activity. The economic value is, therefore, the result of the aggregate of various other measures, even if their units of measurement are not the same.

Pragmatic measurement may include multiple types of measurement (of different properties) put together to form one. Data modeling is one kind of pragmatic measurement, one that requires multiple measures of the same property (repetition), and the study of the data's variability and distribution. However, other forms of pragmatic measurement are possible: defining the cost of a product, negotiating a salary or budgeting a mortgage are a few examples.

The use of units in representational measurement can create an unfortunate illusion, that what is being measured is discrete, because the limitations of the measurement tools used put a lower limit on the size of a difference that can be measured. Take, for example, the case of a ruler. Although length and its measurement are continuous, every ruler is limited in its capacity to measure length. Consequently, the units meter, centimeter, millimeter, *etc.* seem to measure a discrete attribute. Measuring time also shows the same illusion. A clock gives a reading of time in hours, minutes and seconds. Other instruments give more precisions but every instrument has a limit.

Pragmatic measurements can suffer from the same illusion, because they are defined by the units used to measure them. However, pragmatic measurements can also be the result of calculations done on other measurements, making their continuous nature more visible. For example, height of people is unlikely to be measured more precisely than to the nearest millimeter, but the mean height of a group of people might be calculated to any number of decimal places. In many contexts, however, some level of precision is considered 'normal' and can be perceived as making the measurement discrete.

Money as a measurement concept

Money is an unusual, but uniquely accessible, case of pragmatic measurement. What is being measured is an aggregate of many factors (as in the example of the price of bread), but the units used are often represented by physical objects (coins and bills). This can give rise to an illusion of discreteness. Dollars and euros are divided into cents, but below one cent, there is no unit name anymore. That does not mean, however, that there is no measurable value below one cent. Currency exchange rates, for example, have more than two digits after the decimal point (usually four), and that can be extremely important when large sums of money are being exchanged. What money measures is abstract, aggregate and continuous, even though the coins that represent money are physical, individual and discrete. Moving from physical bills and coins to digital money representations in cryptocurrency or online banking systems may make the continuous aspect of money more visible as they allow the use of an arbitrary number of digits.

Functions of money

Money was invented, among other reasons, to establish a standard measurement for the exchange of goods and services (Ifrah, 2000). To this end, Hill (2010) points out that money serves three different functions in our modern society:

a store of value, a medium of exchange and a unit of account. The two last functions are related to measurement.

Money can be used as a medium of exchange, which is a standard tool that helps facilitate a transaction between parties (like a ruler or a scale). This tool must clearly represent a standardized value to be used as a medium of exchange. Bills and coins provide representations of money that all parties to a transaction can agree on. In earlier times the value of a coin was tied to the value of the metal is was made of. The value of a bill was linked to a national reserve of gold. The value of present day bills and coins is a complex aggregate of relationships to other currencies and to commodities.

A unit of account, according to Hill "can be used to measure and compare the value of goods and service in relation to one another" (p. 30). The idea of unit of account is used in the daily lives of consumers when comparing the value of different goods and services. Consumer decisions are guided by a variety of different factors. Those factors might include, among others, the quality of the service, the service itself, the cost and the trustworthiness toward the company. This is pragmatic measurement.

Because the value of money fluctuates over time it provides a clear example of the arbitrariness of human measurement. The history of measurement shows us that units of measurement have changed over time until scientific conventions brought stable definitions to some physical units of measurement. Measurement of mass progressed from every market having its own balance with its own set of local accepted weights, to a definition based on the mass of a liter of water, to a single reference mass kept in Paris, to a redefinition of the Planck constant that defines mass in terms of time and length. In the case of money, we notice that a similar progression towards stability has been developing over the course of the last century (with the establishment of international reference currencies), and the rise of cryptocurrencies might lead to more standardization in the future. The stability of money as a unit of measurement highlights the conventional and social aspect of measurement as a mathematical activity.

Discussion and implications for teaching mathematics

In many mathematics curricula, money is approached from an arithmetic perspective. This means that money is treated as a quantity which students operate on, in order to learn basic operations and percentage. They learn the names and the values of coins and bills and perform operations in solving word problems involving calculating costs or pocket change. Teaching money from an arithmetic point of view can be seen as using financial contexts to develop mathematical understanding (Savard, 2018). But if money is simply a context for arithmetic, not studied as a system of measurement in its own right, students miss an opportunity to develop a conceptual understanding of economic value in terms of pragmatic measurement, which is a key intersection of mathematics and financial education. We conceptualize financial numeracy as a field in mathematics education at this intersection. Financial numeracy refers to the numerical/mathematical/quantitative aspect of financial education. It is an important domain to theorize the ways in which mathematics contributes to financial education.

Developing a conceptual understanding of money as a pragmatic measurement would have strong implications for individuals and the society. Students can develop their knowledge by deepening their understanding of what money represents and how the economic value of products and services can be defined and put into question. To do so, they need to develop various mathematical concepts, including measurement as a mathematical activity (units, instruments, processes). They will engage in multiple mathematical practices such as comparing, measuring, quantifying, *etc.* Finally, students can also develop motivation to investigate the reasons behind the establishment of such prices, costs and values.

It is the combination of these three aspects that encompasses financial numeracy and approaching money as a measurement allows students to succeed. At this end, Caprioara, Savard & Cavalcante (2020) showed that it is possible to teach financial numeracy in elementary school using representational and pragmatic measurement. However, there is an urgent need for more research in this area.

Concluding remarks

Understanding the structure and dynamics of money in contemporary societies is at the core of developing essential competencies for citizenship. We argue that money can only be fully understood once students have developed their sense of measurement both in representational and pragmatic terms. The complex relations involved in assigning the economic value of a good or service must come to the surface, and mathematical practices can help us conceptualize such relations.

This cannot be done if money is simply a context in which to learn arithmetic. Measurement should serve as a framework to approach money in elementary and secondary schools. More generally, we believe financial numeracy can be incorporated in all strands of mathematics education, whether as a context for developing mathematical knowledge, or as financial concepts which can be conceptualized through mathematics.

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