DYNAMIC MEASUREMENT REASONING FOR AREA AND VOLUME

NICOLE PANORKOU

Imagine an empty box *being packed* with cubic units. How do you measure the volume of the box? One way is to count every single cubic unit and find how many are needed to pack the entire box. A more sophisticated way is to count the cubic units in one layer and then multiply by the number of layers. Now imagine a second empty box *being filled* with cups of water. One way to measure the volume of this box is to find the number of cups of water required to fill the box. In both these approaches, *volume as packing* and *volume as filling,* volume is expressed as a relationship between a given volume unit of measure (cubic units, cups of water) and the three-dimensional (3-D) object that is measured.

Now imagine a two-dimensional (2-D) surface *being swept* over a particular height in a perpendicular direction to generate a 3-D object. The volume of the 3-D object generated depends on the area of the 2-D surface that was swept and the height of the sweep. In other words, volume is expressed as a multiplicative relationship between two lower-dimensional quantities that generate it: area of base and height. Indeed, space can be thought of as containing objects and as generated by the objects. In the packing and filling approaches, measurement is approached using the containment nature of space. In contrast, in the sweeping approach, what I refer to as *Dynamic Measurement*, the focus is on how space is generated, and thus measured, by the lower-dimensional objects that define it.

This article compares and contrasts the containment and generation perspectives of measurement and then discusses some forms of reasoning that are possible when students engage with dynamic measurement tasks for generating rectangles, right prisms, and right cylinders. I argue that these forms of reasoning can be productive for students for developing a more conceptual understanding of area and volume measurement.

The containment perspective

Extensive research on area measurement has described the importance of using square units to cover rectangular surfaces and quantify that covering. Although some studies suggest a developmental progression of understanding area, they also identify difficulties that students encounter, such as leaving gaps between units, overlapping units, double counting units, and even combining units of different size. Students also do not think of the unit as a dimensional object, rather they conceive it as simply an object (Thompson, 2000). As a result, shifting from the counting of square units to the multiplicative relationship of combining two linear measures in an area formula can be extremely difficult for them (Kamii & Kysh, 2006). Piaget, Inhelder & Szeminska (1960) explain that these difficulties arise because "the child thinks of the area as a space bounded by a line, that is why he cannot understand how lines produce areas" (p. 350).

Similarly, volume measurement in school mathematics focuses on *packing* using cubic units and quantifying that packing. Students may use a variety of strategies to measure volume, such as counting each individual cubic unit, finding the number of cubic units in one layer and skip counting or multiplying with the number of layers, or using the *length* × *width* × *height* formula. As with area, developmental progressions for volume have been proposed. However, studies (*e.g.,* Van Dine, Clements, Barrett, Sarama, Cullen & Kara, 2017) also show that students often count only the visible cubic units or the cube faces, ignoring the three-dimensionality of the object or its units.

Volume as filling is a second approach to volume measurement used in mathematics education. Compared to packing, this approach opens up the study of volume in non-rectangular prisms as the unit is not restricted to cubes. However, when Curry and Outhred (2005) asked students to find the volume of a jug by filling it using cupfuls of rice, they noticed that students focused only on the height of the cups and were not attending to the three-dimensionality of volume.

What is common in these difficulties with area and volume is that they emerge as students try to cover a 2-D shape or fill and pack a 3-D shape. In other words, students fail to understand the two-dimensionality of area and the three-dimensionality of volume when they experience them as space containment.

The generation perspective

Dynamic Measurement (DYME) is an alternative approach to geometric measurement that focuses on how space is measured by the lower-dimensional objects that generate it. An inductive approach to visualize this generation of area and volume attributes is by moving objects in space (Lehrer, Slovin & Dougherty, 2014). By imaginatively sweeping a line segment '*a*' in a perpendicular direction for a distance '*b*', we generate a rectangle with area '*ab*' (Figure 1a). Similarly, by sweeping a two-dimensional surface of area '*c*' in a perpendicular direction for a height of '*d*,' we generate a right prism or right cylinder of volume '*cd*' (Figure 1b).

While objects can be generated by dragging in any non-parallel direction, their distance/height of the drag is determined based on the distance perpendicular to the dragged object and not the distance travelled by the object.

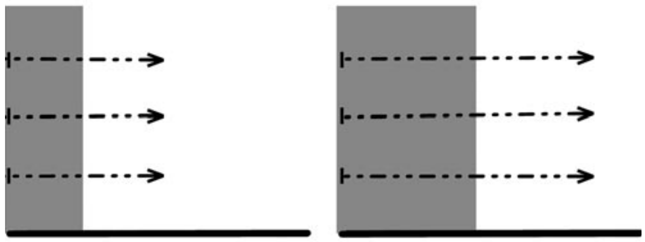

Figure 1a. Sweeping a segment over a distance.

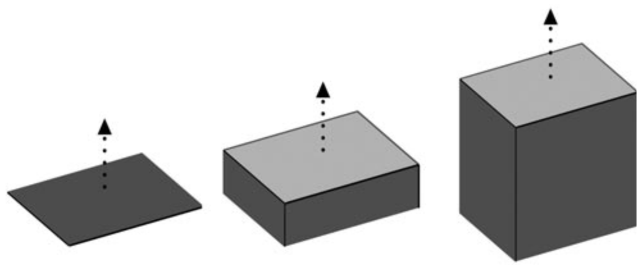

Figure 1b. Sweeping a surface over a height.

For instance, an oblique prism can be generated through a non-perpendicular dragging but its height is still measured perpendicularly. Consequently, perpendicular dragging can be seen as foundational for introducing the formula *base* × *height* as the distance travelled by the object and the height are equal. By generating right prisms and cylinders, students can conceptualize the multiplication of *base* × *height*, as a *quantitative operation* (Thompson, 2011) between two quantities that produce a new quantity (area or volume, respectively). This might be the kind of reasoning that would make this formula visible and meaningful to students.

In contrast to other studies on measurement that use an area unit to quantify the area of a rectangle and a volume unit to quantify the volume of a 3-D object, DYME focuses on how two length units combine to generate a unit of area, and how an area unit combines with a length unit to generate a unit of volume. It builds on Davydov's (1992) prior research on multiplication as a change of units "to larger-scaled, bigger units" (p. 17) to also include a dimensional change (composition) of units.

If students conceptualize area and volume as quantitative and multiplicative relationships, then we can engage them from an early age in reasoning about quantities as structures that can dynamically change. These dynamic experiences could help students construct *smooth* and *chunky* (Castillo-Garsow, Johnson & Moore, 2013) images of change about area or volume. For instance, a smooth image of change would involve envisioning both the distance dragged and area changing as the line segment is continuously dragged to generate the 2-D shape. An example of a chunky image of change would be to consider that if the height or the base of a 3-D object is split in half, its volume is also split in half.

Forms of Dynamic Measurement Reasoning

Here I report on a retrospective analysis of a series of design experiments (Cobb, Confrey, DiSessa, Lehrer & Schauble, 2003) with six pairs of fourth graders to discuss some forms of reasoning that are possible when students engage with DYME tasks. The tasks focused on the area of rectangles and the volume of right prisms and cylinders and utilized the dynamic dragging and tracing features of GeoGebra for generating spaces dynamically. The students had instruction in area as tiling the year prior to the experiment and had experiences of measuring volume in science by water displacement. They were also able to distinguish between 'flat' (2-D) shapes and 3-D shapes. I discuss three of those forms of reasoning that students exhibited: reasoning about the *quantities* involved in the 2-D and 3-D shapes' generation process, about the *dimensional change* of those quantities, and about their *coordinated change*.

Reasoning about the quantities involved in the generation of space

The early tasks in DYME for area involved dragging line segments of various lengths over various distances and reasoning about the size of the rectangle generated (Figure 2a). For volume, the tasks involved dragging surfaces of various shapes and sizes over different distances and reasoning about the size of the 3-D space generated (Figure 2b). To support students in focusing on the quantities involved and their relationships, I made the design decision not to include any numbers in the initial tasks.

By dragging a line segment over a distance to generate a rectangle, students conceived three quantities that are measurable—a line segment (described as a paint roller in our task design), a dragging distance, and a generated area—and began constructing relationships between those quantities. For example, Ben stated that, "the bigger the roller, the more space you can cover". Students also used compensation reasoning, for example, when Sophia was asked to create a

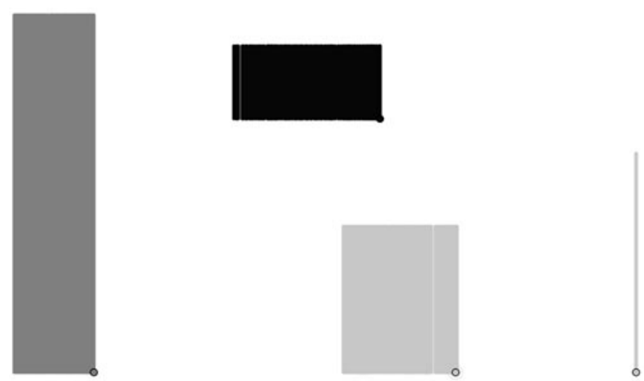

Figure 2a. Dragging segments over a distance.

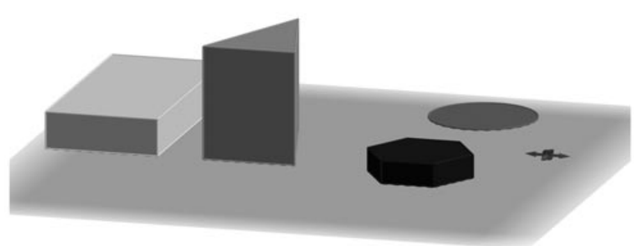

Figure 2b. Dragging surfaces over a height.

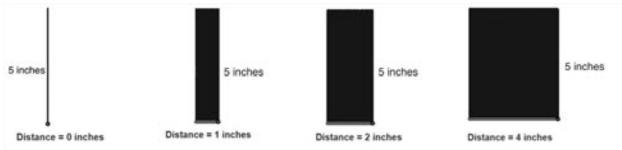

Figure 3. *Dragging a line segment of 5 inches over various distances.*

bigger rectangle than the one she just created, she claimed that, "you could use that roller and spread that wider [making the roller bigger] or take a smaller roller and drag it a lot". Her statement shows that she considered both the length of the roller and the dragging distance as quantities that influence the space generated and that she could compensate the length of the roller with the dragging distance to make a larger shape. In terms of volume, by dragging different bases over varied heights, students conceived three measurable quantities in the generation of right prisms and cylinders: area of base, dragging height, and 3-D shape created. For instance, Molly stated that the 3-D shape created "depends on how big the base is, and it also depends on how much you can drag it".

By engaging with these early non-numeric tasks, students experienced the generation perspective of space and conceived the area of surfaces and the volume of 3-D shapes as spaces that are generated, and thus defined, by these objects. Like Ben, Sophia, and Molly, all students conceived the quantities of length, area, and volume as measurable attributes. Although they did not reason multiplicatively, their articulations show that they began constructing quantitative relationships between those quantities.

Reasoning about the dimensional change of the quantities

While exploring the non-numeric tasks, students described the generation of the new quantities in terms of *dragging* a length for rectangles, and *dragging, expanding,* or *pulling* the base for prisms and cylinders. They also described the size of the new quantities in terms of those lower-dimensional quantities that generated them. In other words, they considered that there is some kind of dimensional change happening from 1-D to 2-D and from 2-D to 3-D. As Molly stated, you drag "a 2-D shape and you change it into a 3-D shape". The next tasks introduced numerical values to examine how students would reason numerically about those relationships of quantities that they constructed. For area, students were asked to drag a line segment of 5 inches over different distances and reason about the space covered (Figure 3).

As Molly increased the distance of the 5-inch roller from 1 inch to 2 inches, she argued that the generated space of the new rectangle would be 10 because "1 inch of that box, you would be covering half of it. In order to find the other half, you will have to multiply it by 2". One might argue that Molly constructed an area unit using the length of the roller and the 1-inch distance that she iterated to find the total space. Another interpretation is that she conceptualized the multiplication *length of roller* × *distance* as a quantitative operation that produces the rectangle space, and considering the length of the roller is constant, if the distance is doubled, then the space is doubled. Like Molly, all the students used multiplicative reasoning to talk about the generated space. To better understand their choice to use multiplication, I questioned them further:

Researcher	So, how much space does the first *[1-by-5-inch by rectangle in Figure 3]* cover and how much does the second *[2-by-5-inch rectangle in Figure 3]* cover?
Ben	1 inch and 5 inches, 2 inches and 5 inches.

Ben reasoned about the space of rectangles as a composite made by the measure of the line segment and the distance of the sweep. As his partner, Fernando explained, "We covered an inch of space going that way *[moving his finger across the distance of the 1-by-5-inch rectangle]* but going this way *[moving his finger across the height of the 1-by-5-inch rectangle]* we covered 5 inches". Both Ben's and Fernando's reasoning shows that they structured the space generated as a composite of two linear measures. When they were asked to explain why they reasoned multiplicatively, others similarly to Dan, who argued that because one quantity is fixed, he only compares the two quantities that are changing: "I am comparing the area and I am comparing the distance".

To examine if students were considering both the length of the roller and the distance of the sweep, the tasks that followed focused on painting rectangles using line segments of varied lengths dragged for different distances and asking students to reason about the space covered. Fernando dragged a 3-inch roller for a distance of 10 inches and stated, "Because the width is 10 inches going across and the length is 3 inches. So then 3 times 10 is 30". Other students generalized that the area "depends on the height and the width" and could find the area of any generated rectangle by combining those quantities multiplicatively. In other words, they conceptualized the multiplication of *length* × *width* as a quantitative operation between two quantities to produce area.

The tasks in volume measurement involved dragging surfaces of various areas over varied heights and reasoning about the space of the generated 3-D shape (Figure 4). As students increased the height of the extrusion from 1 inch to 2 inches, they reasoned multiplicatively about the relationship between the quantities:

Researcher	How much *[space]* is the yellow one *[rectangular prism in Figure 4]*?
Olympia	The yellow one is *[pause]* the yellow is *[pause]* 76. Because 38 times 2 is 76.
Researcher	Why times 2?
Olympia	Because the area is 38 and the height is 2.

Like Olympia, all students reasoned about the multiplicative relationship between area of base, height, and volume. When they were asked to explain why they reasoned multiplicatively, students talked about the 3-D space as consisting of copies of 1-inch drags of the specific base. For instance, Jayden argued that the size of the 3-D shape becomes two

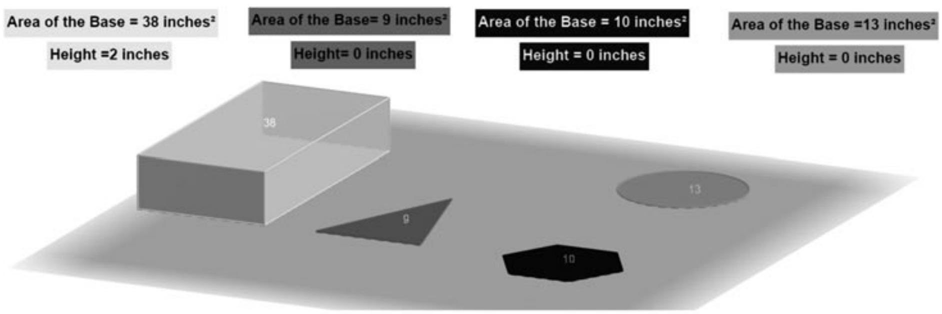

Figure 4. Students drag various surfaces to different heights.

times bigger if we increase the height to 2 inches:

Researcher How do you know it is two times?

Jayden Because it is kind like stacking this one on top of it because this one says it is 2 inches, so we are going to make this one another inch and it is going to be the same thing.

I interpret Jayden's reasoning to show that he perceived the change as making copies of 1-inch drags. He also described this iteration in terms of "make this one another inch and it is going to be the same thing" showing that he acknowledges that the area of the base is constant, and he coordinated the change in height with the change in volume. Jayden then started multiplying the base with the height to find the volume of other shapes. When I asked him why he was multiplying and not adding he stated, "Because the height is 5 inches so we are going to have to do the times 5 not 13 + 5, because if it was 13 + 5 it would probably be adding more to the base". This statement shows that he understands that multiplication transforms the space measured.

Like Jayden, students were able to conceptualize the multiplication of area of *base* × *height* as a quantitative operation between these two quantities to produce volume, a new quantity. This series of tasks also created the need for defining the unit of volume and distinguishing it from length and area units. Since students learn exponents later in the curriculum, I explored how students perceive these exponents on the area and volume units and asked them to explain what they think these might mean. I first asked them to explain what the number 2 meant on top of $inches^2$ in the area measure and Laila stated that "it stands for 2-D". When I asked her what she would call the inches in 3-D she stated, "inches three". Like Laila, students constructed a unit of volume as "inches three". All of them mentioned that the 2 stands for the space of a 2-D shape while 3 stands for the space of a 3-D shape. To them, the exponent on the unit was distinguishing the type of space they were measuring. This shows that they conceptualized this generation of quantities as involving a change of units (Davydov, 1992) and in particular a dimensional change of units.

Reasoning about the coordinated change of quantities

As students explored the generation of rectangles in the early non-numeric tasks, they coordinated the change in the length of the roller or dragging distance with the change in area, arguing for instance that "the bigger the roller the more space [area] we can cover". Likewise, for volume, students coordinated the change in the dragging distance with the change in volume, reasoning that "it is how much you drag it, how big you make it". These articulations show that they constructed smooth images of change because they described the quantities as if they are changing in progress.

In the numeric tasks, they reasoned about the change in 1-inch drags, showing that they transitioned from smooth to chunky images of change. I recognize that students' reasoning is deeply influenced by the design, hence, their construction of 1-inch drags might have been the result of increasing the height by 1-inch increments. Subsequently, in the next tasks, I provided two decimal points in the value of the height as students dragged the surfaces. None of the students commented on the decimals, so I asked them what these numbers mean. Jayden argued that decimals show "pieces of a whole number", showing that he understands that there are intermediate values between the 1-inch chunks of drags. Indeed, students' reasoning showed that they still had a smooth image of change although they also reasoned in 1-inch chunks, illustrating that they constructed a unified perception of space that included both smooth and chunky images of change. It is this unified perception that supported them in perceiving the generation of space both as a quantitative and a multiplicative relationship.

The subsequent area tasks asked students to double, triple, and halve lengths and reason numerically about the resulting changes in area. Students were able to multiplicatively relate the change of area and the change of one of its measures. For example, in splitting the width of a rectangle in half Liam argued that its area will now be half, stating, "Because this is like half of it. And 4 times 6 equals 24, but 2 times 6 equals 12 and it is a half". Similarly, the volume tasks that followed focused on changing the linear measures of rectangular prisms of a sculpture and reasoning about the change in volume (Figure 5). Aiming to help students focus on the quantities and the relationships among them, the values of the dimensions of the rectangular prisms were hidden at the beginning and students were asked to make a conjecture about the change of volume before looking at the values.

Students utilized the quantitative and multiplicative relationship they constructed to argue that if one of the linear measures becomes n times bigger, the volume becomes n times bigger, and thus reasoned in a way that Confrey and Smith (1995) call *covariational*. This way of reasoning is

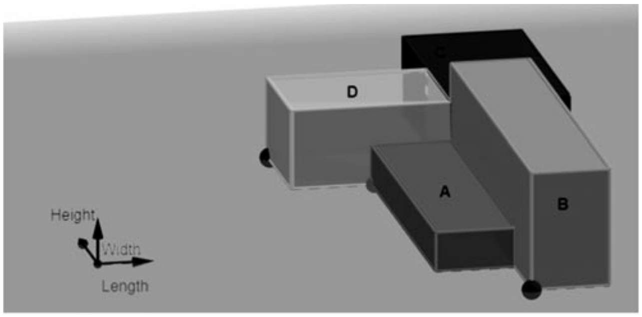

Figure 5. Students were asked to change the dimensions of the rectangular prisms.

foundational for functional, ratio, and proportional thinking. It is characterized by coordinating two quantities simultaneously, as the students did. For example Molly, when asked to anticipate what would happen to volume if the length of the Block A in Figure 5 is doubled, stated, "if you drag it *[the length],* the base will become bigger as well, so the volume will double". Molly coordinated the change in length with a simultaneous change in the area of base and also volume, reasoning in that way about the simultaneous change in multiple quantities. She was able to dimensionally deconstruct (Duval, 2005) the 2-D measure of area and the 3-D measure of volume by analytically breaking them down into their constituent 2-D and 1-D elements based on relationships.

Measurement for area and volume: pushing forward

This paper presented some forms of reasoning that are possible when young students engage with DYME tasks exploring the generation of rectangles, right prisms, and cylinders. Students exhibited advanced forms of reasoning about area and volume as they considered the measurable quantities and how these quantities work together for generating and changing the space created. By exploring what is changing and how it is changing during the generation of shapes, students conceptualized area and volume as quantities on their own and also as quantitative and multiplicative relationships. Students reasoned about the dimensional change of quantities during the generation of space and constructed a unit of measure that consisted of two length units for area, and of an area unit and a length unit for volume. By examining the generation of space non-numerically and then numerically through the sweeping motion, students constructed both smooth and chunky images of change and reasoned covariationally about the coordinated change in the quantities involved.

I argue that this focus on reasoning about the relationships between the quantities in measurement as well as their dimensional and coordinated change can merge with the construction of a dimensional unit to support the kind of conceptual development that could eventually make geometric measurement a purposeful tool for students' thinking and problem solving. Students' DYME reasoning can be foundational for understanding how the area and volume formulas for other shapes work through non-perpendicular dragging and shearing based on Cavalieri's principle, or through decomposition and rearrangement (Sinclair, Pimm & Skelin, 2012). These forms of reasoning also have the potential to connect students' thinking about measurement to advanced mathematical ideas in algebra and calculus, such as supporting students' understanding of the definite integral of differential calculus and perhaps even the limit concept in the later years of schooling, two concepts that are grounded to the notion of continuity.

I hope to initiate a discussion about dynamic measurement and the potential use of taking a generation perspective for developing students' reasoning about area and volume. In other words, that the discussions of students' reasoning here will elicit more questions about the nature of dynamic measurement and also provoke future studies to explore these questions. Studies are especially needed that examine further whether a conceptual understanding of area and volume constitutes a coordination or an integration of the generation and the containment perspectives of measurement.

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