Communications

Exploring and Reporting upon the Content and Diversity of Mathematicians' Views and Practices

LEONE BURTON

In FLM **18**(3), Jack Smith and Kedmon Hungwe presented some insights into the mathematical practices of three young mathematicians They wrote:

> Accounts of mathematicians' actual practices are few and limited to self-report [....] We hope our analysis encourages others to explore and report on the content and diversity of mathematicians' views and practices. (p. 41)

This piece is in direct response to their appeal. In 1997, I undertook an interview-based study of thirty-five female and thirty-five male career research mathematicians in twenty-two universities in England, Scotland, Northern Ireland and the Republic of Ireland The purpose of the study was to explore how robust an epistemological model which I had developed was (see Burton, 1995), as a way of describing how mathematicians come to know mathematics, and furthermore to ask what we, as mathematical educators, might learn from their research practices. The model has five components which describe knowing in mathematics. These are its person- and cultural/social-relatedness, the aesthetics of mathematical thinking invoked, intuition, styles of thinking and connectivities

The data were collected in single life-history interviews which lasted, on average, about one-and-a-half hours and were conducted face-to-face in a location of the interviewee's choice. (Six took place over the telephone) Details about the structure of the study, its conduct and the data analysis can be found elsewhere (see, for example, Burton, 1999a)

The interviews followed a similar pattern, beginning with what I called the 'historical trajectory' of their becoming mathematicians, then focusing upon a particular problem or problems on which they had recently been working, before moving on to questions about how, where and with whom they worked, how they saw their work in relation to both their particular research area and mathematics more generally, and finally enquiring about publication. While they dictated the direction of the interview, I made sure that, by its conclusion, we had addressed all of these aspects

Mathematics is a team sport

Five questions were asked by Smith and Hungwe (p 40) to some of which I have partial, and to others more complete, responses. With respect to how mathematicians go about their work and how social and interactive it is, one of the biggest surprises of the study, for me, was that mathematics is no longer seen, certainly by most of those I interviewed, as something you do on your own in an eyrie away from human contact (what I think of as the Andrew Wiles effect).

Of the seventy mathematicians, only three males and one female claimed never to work with others and most were extremely enthusiastic about the benefits of joint work (see Burton, 1999a). They identified two different forms of what they termed 'collaboration', one of which I am calling *co-operation* because the partners brought different disciplinary skills and knowledge to the team (for example, a statistician working with non-mathematicians). But even in these cases, by the end of the work, the paper was often a truly joint effort. In the cases of effective collaboration, they commented upon how individual sections contributed to the paper have often been incorporated into a seamless web of writing

Furthermore, they offered all the same reasons for collaborating on research that are to be found in the educational literature advocating collaborative work in classrooms. With what one of my participants called 'a cultural change' in mathematics towards this form of interactive working, we, as mathematics educators, could be much more effective in supporting a shift away from an individualistic and competitive emphasis with learners. However, that would require engaging with the distinction made by Smith and Hungwe (p 45) between *disciplined* and *undisciplined guessing*, itself requiring, in my view, a re-consideration of the 'objective knowledge' view of mathematics and, particularly of the ways in which it reduces the discipline, for learners, to the reception of 'inert knowledge' (Whitehead, 1962).

Keep the 's' on mathematics

My study certainly supports Smith and Hungwe's notion that there is no homogeneity amongst mathematicians' views, nor their practices. Elsewhere (1998), I have reported:

> Homogeneity was not revealed by my participants when they spoke about mathematics, how they understand mathematics, how they think about mathematics, how they work in mathematics Many public stereotypes were overthrown
>
> There is not one:
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> - mathematics – depending upon the research area, it was differently understood as:
>   - a 'rigorous' proof process;
>   - empirical;
>   - uncertain;
> - way of understanding mathematics:
>   - as well as the invention/discovery split, I found socio-culturalists and those who understand mathematics as a language;
>   - role of intuition/insight – there were those who denied and those who affirmed the importance of intuition and those who wanted to talk about the meaning of the different terms;

- way of thinking about mathematics - three, not the conventional two, different thinking styles were described;
- way of working in mathematics - individual/co-operative/collaborative were all described with an emphasis on the latter two

Only when it came to discussing ways of experiencing the world of mathematics, and the impact of sex, 'race' and class, was there any singularity of experience. A world of power was revealed and that power had all the overtones demonstrated in the literature.

The process of socialising mathematicians into the discipline, and the expectations and behaviour thereby encountered, permits all the heterogeneity described above, even within quite tightly defined mathematical areas. However, power operates to position some people such that their experiences do conform to a pattern. In the case of this study, this is noticeable with respect to sex: women (but not all) but no men reported incidents from verbal put-downs through to quite serious harassment (see Burton, 1999b).

But I also perceived a resistance by some of the mathematicians to using the word 'intuition' to label a phenomenon which they accepted but preferred to call 'insight' or 'instinct'. This caused me to wonder whether this was an intrusion of the social stereotyping female/intuition, male/cognition perceived as inappropriate for the supremely cognitive subject, mathematics. Nonetheless, there were mathematicians who were poetic in their descriptions of how their feelings and their cognitions interacted, and I would conjecture that there is a shift taking place within the discipline towards a more balanced, whole and certainly healthier view of the interaction between sense and sensibilities, which can only be to its and its practitioners' advantage.

It is important to point out, however, that although I interviewed equal numbers of females and males, I have no data on 'race' and only limited data on class. My participants were nearly all white and European and although many came from backgrounds which might be identified as 'lower' class, they themselves were now firmly embedded in a middle-class profession. I leave readers to draw their own conclusions on this

Pedagogical reflections

Smith and Hungwe also wrote:

> If guessing and the resulting cycle of inquiry does not become visible to students, they are left with only public mathematics – the carefully crafted propositions and polished arguments they see in their texts. They miss entirely the stumbling human process that created those results in the first place. When guessing is not exemplified and supported, most students find it more difficult to understand and enter the practice of doing mathematics. (p. 46)

Although the purpose of my study was *not* to discuss teaching and learning practices, inevitably, in some of the interviews, attitudes to students, expectations of students, experiences of the mathematicians as students, and even occasionally teaching practices, were explored. Indeed, I found it quite depressing how rare it was for these mathematicians to see any connection between what they did in their own researching and what they expected of, and received from, their students. This was noticeable when they talked about the role of intuition. Fewer than five mathematicians asserted that intuition had no place although, as mentioned above, some did not want to call it intuition.

However, while many agreed with Reuben Hersh (1998) that "intuition is an essential part of mathematics" (p. 61), at the same time they practised didactic habits which denied learners the same opportunities they felt to be so necessary for themselves and decrying their students' capacity for exercising intuition. An interviewee observed:

> One of the things I find about students, undergraduates in particular, is that they seem to have very little intuition. They are dependent upon being spoon-fed

I assume that such 'dependency' is not unconnected with habits established over a long learning history, especially where that experience has been 'successful' within the system.

In his important and unique book *Intuition in Science and Mathematics: an Educational Approach*, Efraim Fischbein (1987) explored mathematical intuitions, accorded them centrality in mathematical thinking and discussed their features. He also drew attention to the above contradiction amongst mathematicians, while also bemoaning our failure, as educators, to nurture, position and expand the making and using of mathematical intuition.

In conclusion, my study has been comfortingly supportive of the socio-cultural construction of coming to know mathematics to be found in the problem-solving and investigational literature (see, for example, Mason *et al.*, 1982; Resnick *et al.*, 1991; Steffe and Gale, 1995; von Glasersfeld, 1991). But it has also underlined the gap perceived by research mathematicians between their own enquiry processes and their didactic practices. However, I believe that it is in our interests, as mathematics educators, to draw out very clearly the links between these enquiry processes (and their epistemological implications) and the pedagogical practices which many of us have been advocating for a considerable time, sometimes against overwhelming social resistance!

Our colleagues who are teaching mathematics in higher education are facing diminishing numbers of students (see, for example, Seymour and Hewitt, 1997), but are often maintaining an atmosphere of student blame. This atmosphere can also be found in schools. On the contrary, we have research which demonstrates very clearly that students are more disaffected by the mathematics they encounter, and the pedagogic style in which it is presented, than they are disadvantaged by their 'ability' (see, for example, Boaler, 1997). I have believed for a long time that it is the perspective on mathematical knowledge which accounts for much of this. My study has confirmed this belief while, at the same time, offering the research practices of mathematicians themselves as a model for teaching and learning the discipline.

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A Four-Sided View of 'Function'

DIMITRIS GAVALAS

The term 'function' means the mutual connection and dependence of things, in any way, from a dynamic/operational perspective. Therefore, function must be viewed both from the aspect of things that are in mutual interdependence, that is structurally, spatially, statically, geometrically, and from the aspect of procedures, that is in operation, dynamically and algebraically.

The concept of the function is one of the two basic notions in modern mathematics, the other being that of set. The first has had an interesting evolution and gives us an example of the trend in mathematics to extend and generalize its concepts. Function is greatly involved in mathematics, which is why famous mathematicians (e.g. Mac Lane, 1986) support its use as a unifying and centralizing principle in organizing and teaching of the respective lesson. This concept seems to be the natural and main guide in selecting, organizing and developing mathematical subjects in general and particularly in mathematical texts. A very good example here is the *Conceptual Mathematics: a First Introduction to Categories* (Lawvere and Schanuel, 1993)

There are also problems, that this concept brings along with it, either when we want to conduct research or to teach it to students of mathematics. The realization of those difficulties out of the entire mathematical community has as a consequence the tremendous increase of the related research and publications, during recent years. Researchers such as Sierpinska (1991), Sfard (1989, 1991), Harel and Dubinsky (1992), Vinner (1983), Tall (1991) have expressed their views about the subject. In all these works, as well as in Barneveld and Verstappen (1982) and Even (1990), the subject of function is extensively discussed

The approach of this author differs from those mentioned above, mainly in two respects:

(i) by drawing on Jung's views and the corresponding ones of Rucker;

(ii) trying to classify all views of the function in a holistic schema, which will allow us to face up to and respect the individual differences of the students.

The necessary elements of Jung's theory that are used here are included in this essay when needed. For a further study of Jung's work the reader has recourse to Jung (1957-1977). Except for the well-known and widespread elements of his work, there is the aspect, which is based on his so-called *hypothesis of archetypes*. According to this hypothesis, the concept of the function may very well be of an archetypal nature. That means that human beings have built in the archetype/model of a 'good' relation between things, which projects onto the external world. In this way, they indirectly realise the existence of relations into the system of the universe, according to the Greek view *hen to pan* (everything is a unity). Consequently, we could consider the concept of function as the archetype of the healthy and regular relation to the psychophysical continuity of the one World. This view seems to be close to the idea of the arrow between objects in Category Theory (Mac Lane, 1986)

A four-sided view of function

Decisions, positive or negative, and questions regarding the teaching of function are known and considered as the common acceptable aspect which predominates today. But, it is natural to be aware of and curious about other aspects less well known. Nevertheless, as more aspects are expressed about a topic, the more hope we have of looking at it as a whole and not only from one side

Jung claims (1957-1977, vol. 6: Psychological Types) that in order to live in space-time and in the relations of the things of the world, in order to communicate and to exchange information with the environment, we need psychological functionings, of which here are four basic types:

(i) *sensation*, a psychological functioning which informs us that 'something' exists;

(ii) *thinking*, which defines what exactly is and what this 'something' that exists does;

(iii) *feeling*, which determines if this 'something' that exists has any value for us, if we like it or not; this psychological functioning, generally, binds us to the things of the world;