INVESTIGATING THE IMPLEMENTED MATHEMATICS CURRICULUM OF NEW ENGLAND NAVIGATION CYPHERING BOOKS

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The foundation of public high schools in the nineteenth century has been a notable starting place for the study of the teaching and learning of mathematics within the United States. This choice is a practical one, given the surviving printed documents (*e.g.,* textbooks, syllabi) and many published histories (*e.g.,* Cajori, 1890; Smith & Ginsburg, 1934). However, this choice overlooks evidence from handwritten mathematics manuscripts known as cyphering books [1]. These historical documents shed light on an educational tradition that was common long before public schools.

Just what is a cyphering book? Ellerton and Clements (2012) describe a cyphering book as a handwritten manuscript with four properties:

1. The contents were written by a student for the purpose of learning specific mathematical content or by a teacher for use in instruction of students;
2. All handwriting and illustrations were done in ink with a calligraphic style typically used for headings and subheadings;
3. The focus of the book was on laying out rules and cases for a sequence of mathematical topics;
4. The sequence of mathematical topics within the book was arranged so that the content became progressively more difficult and mirrored an expectation that no child under 10 years old would prepare a cyphering book.

The preparation of a cyphering book was done with great care and considerable effort. The books themselves were created under a range of conditions (for example, night classes, seasonal schools, but also, especially in the south, local "private" schools). Each book was prepared page-by-page under the guidance of a teacher, with the student first practicing problems on scraps of paper or slates and then transcribing each problem into the cyphering book after approval. Once completed, a cyphering book served as a mathematical reference for the student who created it. Students typically used cotton (rag) paper, homemade ink, and wrote with homemade quills. It is because of the durability of these construction materials that a large number of books have survived to the modern day.

The cyphering approach, which is relatively unknown today, was the primary means by which students in North America learned mathematics between 1607 and 1840 (Ellerton & Clements, 2012). The approach did not originate in North America but was derived from a Western European abbaco tradition, which, as Ellerton and Clements (2012) discuss, can be traced back to India and Arabia. The abbaco tradition is largely regarded as being first introduced by Leonardo of Pisa (Fibonacci) in Italy around 1200, and from there it gradually spread across Europe [2]. European settlers who came to North America in the 1600s brought this tradition with them and adopted it as a means for teaching mathematics.

Although the majority of cyphering books were created to teach the basic mathematical ideas needed for commerce, such as single and double rules of three, a subset of manuscripts focused on the specialized mathematical knowledge needed for specific professions. In particular, as the industry of shipping grew during the sixteenth and seventeenth centuries, a number of mathematical techniques, including geometry and trigonometry, became important for the day-to-day tasks of navigation (Waters, 1958). The growth of shipping created a demand for individuals trained in these techniques and many turned to the cyphering tradition as a means to do so. Thus, the surviving cyphering books provide a first-hand account of the mathematical instruction that these individuals experienced. For example, Figure 1 shows a page from William Hale's cyphering book. William, who lived in Newbury, Massachusetts, prepared his book in 1837. The page shown is the first from his section on plane sailing. Here he sets out a definition of plane sailing and solves a common problem using two different methods, one based on geometric construction and the other using logarithms.

For most of the twentieth century, the evidence of cyphering books, like the manuscripts themselves, has been ignored. Instead, historians have written from a perspective dominated by surviving textbooks and related documents. Although this perspective is valuable, it ignores the influence of educators and the realities of classroom practice. Research on the implemented curriculum [3] of cyphering books offers an opportunity to reexamine the accepted history of the development of the mathematics curriculum, shed light on early practice, and draw contrasts with the intended curriculum (see, for example, Wessman-Enzinger, 2014). Additionally, as others have demonstrated (*e.g.,* Ellerton & Clements, 2012, 2014; Herbst, 2002), historical work can offer new perspectives on "established" practices

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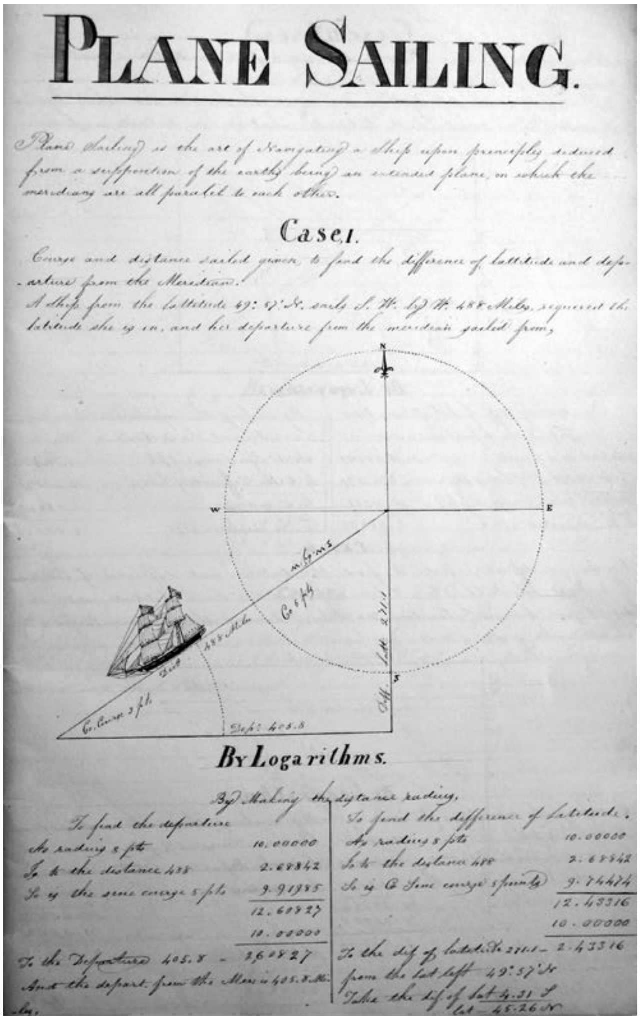


Figure 1. A page from William Hale's cyphering book (photo courtesy of N. F. Ellerton & M. A. Clements).

and provide valuable insight into how dominant perspectives came to be accepted.

Understanding the larger cyphering tradition requires a shift away from individual books and a move towards examining collections of manuscripts. Although individual cyphering books provide a rich example of a unique educational journey, they reveal little about broad trends across the cyphering tradition. For example, George Washington's cyphering books provide information about the mathematics he learned related to navigation, but they do not reveal how his training compared to others at the time or the extent to which his instruction had been tailored. Consequently, understanding the implemented curriculum of cyphering books requires analysis of collections of manuscripts.

Unfortunately, large collections of cyphering books are rare. Although cyphering was the primary means of education in school mathematics in North America before 1840 (Ellerton & Clements, 2012), the tradition itself mostly served white males with financial means (particularly for advanced mathematics). Additionally, misunderstanding of what the manuscripts are has played a role in their destruction, with some cyphering books having been repurposed as scrapbooks, others having been treated as memorabilia and cut apart for sale or as gifts. Abraham Lincoln's cyphering book is likely the most famous example of a manuscript that was taken apart in this way and scholars are still working to piece it back together (Ellerton & Clements, 2014). Finally, for those manuscripts that were donated to libraries or other institutions, lack of knowledge about the cyphering tradition has resulted in misidentification. As a result, many cyphering books are tucked away with family papers, cataloged as copybooks, or listed as mathematical notebooks.

After examining several individual cyphering books in a collection gathered by Ellerton and Clements (see Ellerton & Clements, 2012, for details of their collection), I became more interested in cyphering books that dealt with navigation content, which I call navigation cyphering books. I wondered whether there were commonalities that could be used to characterize navigation cyphering books. Was it just that they had a section on sailing or were there more general characteristics that they shared? For example, did right-triangle trigonometry and geometry typically appear? To what extent did the manuscripts share a common curriculum? How similar (or dissimilar) were the manuscripts from seventeenth and eighteenth century printed texts?

Fortunately, previous work by Ellerton and Clements had identified two places to seek answers to my questions: Phillips Library in the Peabody Essex Museum in Salem, Massachusetts, and Houghton Library at Harvard University. From a historical lens, it makes sense that both of these locations would likely contain surviving cyphering books dealing with navigation. During the eighteenth and early nineteenth centuries, New England prospered as a center of commerce and trade. In particular, Salem, Massachusetts, which was the sixth largest US city and the richest per capita in 1790, had a lucrative shipping trade with vessels traveling to India, the East Indies and China. Thus, the combination of social and economic forces created a demand for training in navigation, which is reflected by the large number of surviving cyphering books.

In Fall of 2011, I organized a trip to New England in order to examine the manuscripts held at Phillips Library and Houghton Library. The Phillips Library has 201 cyphering books organized into three series: mathematics, navigation, and textbooks. The size of the collection at Harvard is unknown, but I estimate it to be about 100 manuscripts with less than a fourth focused on navigation. In my examination, I first conducted a review of manuscripts that had sailing content. Using a word processing program, I then recorded descriptive information (*e.g.,* details about the origin of the manuscript, student information, approximate page length) and copied selected content information verbatim (*e.g.,* chapter headings, section headings, section text, definitions of mathematical terms, solutions) with a focus on identifying similar sections between books (*e.g.,* the first plane geometry problem, the definition of trigonometry). Due to library policies, photographs were used sparingly to capture representative problems and document unique sections. I then analyzed the data I had collected to identify patterns and themes using grounded theory (Glaser & Strauss, 1999). Codes were developed based on content within the cyphering books and a secondary analysis was done on these codes to identify themes across the data set.

Surveying the landscape of navigation cyphering books

My examination of the various navigation cyphering books allowed me to identify a number of common features among manuscripts. Rather than view these features as walls that define the boundaries of a manuscript, I instead see them as distinctive points in the landscape of navigation cyphering books. It is important to remember that these manuscripts were created by hand and, as such, naturally had variation in content and construction.

Related to content organization, I identified two commonalities across the cyphering books: the selection of topics within each book and the order in which the topics were presented. I use the word "section" to describe pages of a cyphering book that dealt with particular content. In most manuscripts, students used headings to distinguish sections. Plane (sometimes spelled "plain") sailing, which treats the surface of the Earth as a plane and uses methods of right-angled trigonometry to determine a change in latitude, was the most prevalent section occurring in almost every cyphering book I examined. The next most common sections were traverse sailing, middle latitude sailing, Mercator's sailing, and right-angled trigonometry. Traverse sailing applies the principles of plane sailing to situations when there are two or more courses. Middle latitude sailing is used when a ship's course is not due east or west and relies on finding a mean latitude to convert a departure from the meridian into a difference in longitude. Mercator's sailing is a method used when sailing at high latitudes and relies on use of a Mercator's chart (see National Imagery and Mapping Agency, 2002, and Silverberg, 2005, for a more detailed discussion of types of sailing). Table 1 provides a summary of the sections.

Typically, navigation cyphering books began with an introductory mathematical sequence focusing on topics from Euclidean geometry, such as definitions and constructions. Often, this section was followed by right-angled trigonometry and, in some cases, oblique trigonometry. Following these mathematical topics was a sailing sequence that usually contained sections on plane sailing, traverse sailing, parallel sailing, middle latitude sailing, and Mercator's sailing.

Across the books, most student mathematical work focused on exercises with little justification. For example, although geometry sections focused on methods of geometric construction using a straight edge and compass, students' work tended to show only the final construction and omitted relevant definitions, postulates, or theorems. Similarly, trigonometry sections focused on methods for solving triangles and presented little (if any) discussion of the trigonometric functions themselves. Likewise, although there was frequent use of logarithms in calculations, almost no attention was given to their definition or properties. Problems within the sailing sequence showed a similar pattern.

Three solution methods appeared within the sailing sequence: the calculation method, the projection method, and the method using Gunter's scale. The calculation method and projection method were most prevalent with the first appearing in roughly 80% of the manuscripts and the second in 50%. The method using Gunter's scale appeared in less than 25% of manuscripts. Although different, each of these methods focused on the same outcome: finding an unknown distance. The calculation method relied on the use of the product rule for logarithms and trigonometric reference tables. The projection method and the method using Gunter's scale, in contrast, used specialized calculation devices to determine the unknown distance through geometric construction.

	Frequency (*n*=64)	% of Books
Navigation		
Plane Sailing	59	92%
Mercator's Sailing	49	77%
Traverse Sailing	46	72%
Middle Latitude Sailing	46	72%
Parallel Sailing	33	52%
Variation of the Compass	22	34%
Log of a Journey	20	31%
Geometry	34	53%
Trigonometry		
Right Angled	47	73%
Oblique	32	50%

Table 1. Cyphering book sections.

A date of preparation was included in 56 of the 64 manuscripts that I analyzed. These dates ranged from 1692 to 1864 with the majority of books prepared in the period from 1760 to 1840. Fifty-four books also had identification or a phrase of ownership such as “John Smith, his book.” In every case the student was male. No manuscript indicated the age of the student at the time of preparation.

I also used Google Book Search to link cyphering books to texts from the period. These connections were made by linking verbatim transcriptions of cyphering book writing to the words within surviving texts. I refer to these transcriptions as seeds. A typical seed contained a few sentences and numbers in a mathematical problem. Identifying a seed within a published textbook indicates that the seed within the cyphering book and the words within the text share a common ancestor. For example, several cyphering books had the following for their first problem in the trigonometry section:

> Given the hypotenuse AC 250 leagues and the angle C opposite to the side AB = 35°30′ to find the base CB and perpendicular AB.

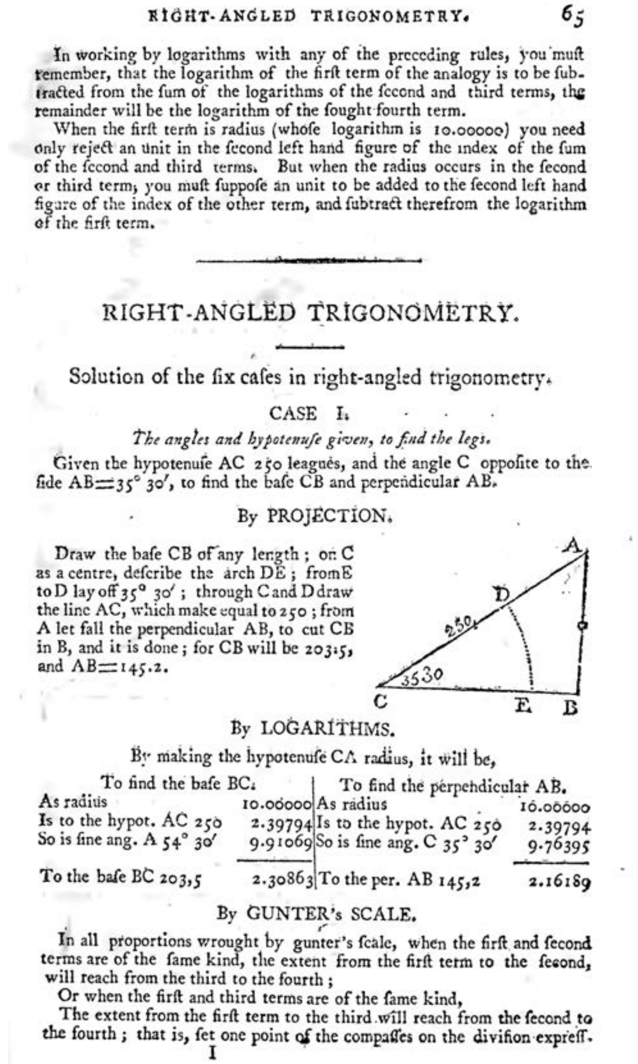
This problem corresponds closely to the first trigonometry problem in *The American Practical Navigator* by Nathaniel Bowditch (1802). This seed has particularly strong links to the source material since it appears to be reproduced verbatim in terms of wording and numbers used in the problem (see Figure 2).

For each cyphering book, I gathered seeds from several sections in order to try and identify source material. In a few cases, the student indicated the source of the manuscript material, but this was not typical. Overall, I was able to link 50 of the manuscripts to at least one text from the period. This included 9 of the 13 manuscripts from Harvard and 41 of the 51 manuscripts from Salem.

Using the seeds, I then created a network showing connections between manuscripts as well as published texts. I refer to this network of connections as a *cyphering book ancestry network* for the navigation cyphering books (Figure 3). Manuscripts are referenced using an S or H (Salem or Harvard) and an identification number (*e.g.,* S22 denotes Salem manuscript 22). For simplicity, the network only shows the title of a published text and the first author. Multiple editions of a work were treated as a single text.

As is shown in Figure 3 (overleaf), three texts were most prominent: Nathaniel Bowditch’s *The New American Practical Navigator;* John Hamilton Moore’s *The Practical Navigator and Seaman’s New Daily Assistant;* and James Atkinson’s *Epitome of the Art of Navigation.* A few books had seeds from more than one published text. For example, seeds from S42 linked to Bowditch’s *The New American Practical Navigator* and Flint’s *A System of Geometry and Trigonometry Together.* One Salem manuscript, S19, had seeds that linked to three sources: Ward’s *The Young Mathematician’s Guide;* Pike’s *A New and Complete System of Arithmetic;* and Mackay’s *The Complete Navigator.* I identified common seeds for S02 and S09 as well as S03, S06, and S08 but could not identify possible ancestor texts.

Building on the connections I found in the cyphering book ancestry network, I created a timeline showing when books linking to Bowditch, Moore, or Atkinson were prepared (Figure 4, overleaf). The majority of manuscripts after 1800 had seeds linking them to Bowditch’s text, which was first published in 1799. Manuscripts with seeds linking to Atkinson’s text were scattered from the 1730s to the 1820s. Similarly, manuscripts with seeds linking to Moore’s were also scattered, but the range was smaller from the 1770s to the 1810s.


RIGHT-ANGLED TRIGONOMETRY. 65

In working by logarithms with any of the preceding rules, you muſt remember, that the logarithm of the firſt term of the analogy is to be ſubtracted from the ſum of the logarithms of the ſecond and third terms, the remainder will be the logarithm of the ſought fourth term.

When the firſt term is radius (whoſe logarithm is 10.00000) you need only reject an unit in the ſecond left hand figure of the index of the ſum of the ſecond and third terms. But when the radius occurs in the ſecond or third term, you muſt ſuppoſe an unit to be added to the ſecond left hand figure of the index of the other term, and ſubtract therefrom the logarithm of the firſt term.

RIGHT-ANGLED TRIGONOMETRY.

Solution of the ſix caſes in right-angled trigonometry.

CASE I.

The angles and hypotenuſe given, to find the legs.

Given the hypotenuſe AC 250 leagues, and the angle C oppoſite to the ſide AB=35° 30′, to find the baſe CB and perpendicular AB.

By PROJECTION.

Draw the baſe CB of any length; on C as a centre, deſcribe the arch DE; from E to D lay off 35° 30′; through C and D draw the line AC, which make equal to 250; from A let fall the perpendicular AB, to cut CB in B, and it is done; for CB will be 203,5, and AB=145,2.

By LOGARITHMS.

By making the hypotenuſe CA radius, it will be,

To find the baſe BC.		To find the perpendicular AB.	
As radius	10.00000	As radius	10.00000
Is to the hypot. AC 250	2.39794	Is to the hypot. AC 250	2.39794
So is ſine ang. A 54° 30′	9.91069	So is ſine ang. C 35° 30′	9.76395
To the baſe BC 203,5	2.30863	To the per. AB 145,2	2.16189

By GUNTER’s SCALE.

In all proportions wrought by gunter’s ſcale, when the firſt and ſecond terms are of the ſame kind, the extent from the firſt term to the ſecond, will reach from the third to the fourth;

Or when the firſt and third terms are of the ſame kind,

The extent from the firſt term to the third will reach from the ſecond to the fourth; that is, ſet one point of the compaſſes on the diviſion expreſſ.

I

Figure 2. The first trigonometry problem in Bowditch (1802) (photo courtesy of Smithsonian Libraries).

As shown in Figure 4, the publication of Bowditch’s book in 1799 appears to have had an influence on the content of cyphering books within Salem. Given that Nathaniel Bowditch was a native of Salem, it is not surprising that his text would be widely used by Salem students. The manuscripts from Harvard, in contrast, do not appear to favor one source over the others. I conjecture that this difference is likely due to the makeup of the two collections and the manner in which they were formed. The collection of cyphering books at Phillips Library in Salem was acquired over a number of years as small area libraries were consolidated and thus represents a local focus. The collection at Houghton Library, in contrast, appears to have been gathered over the

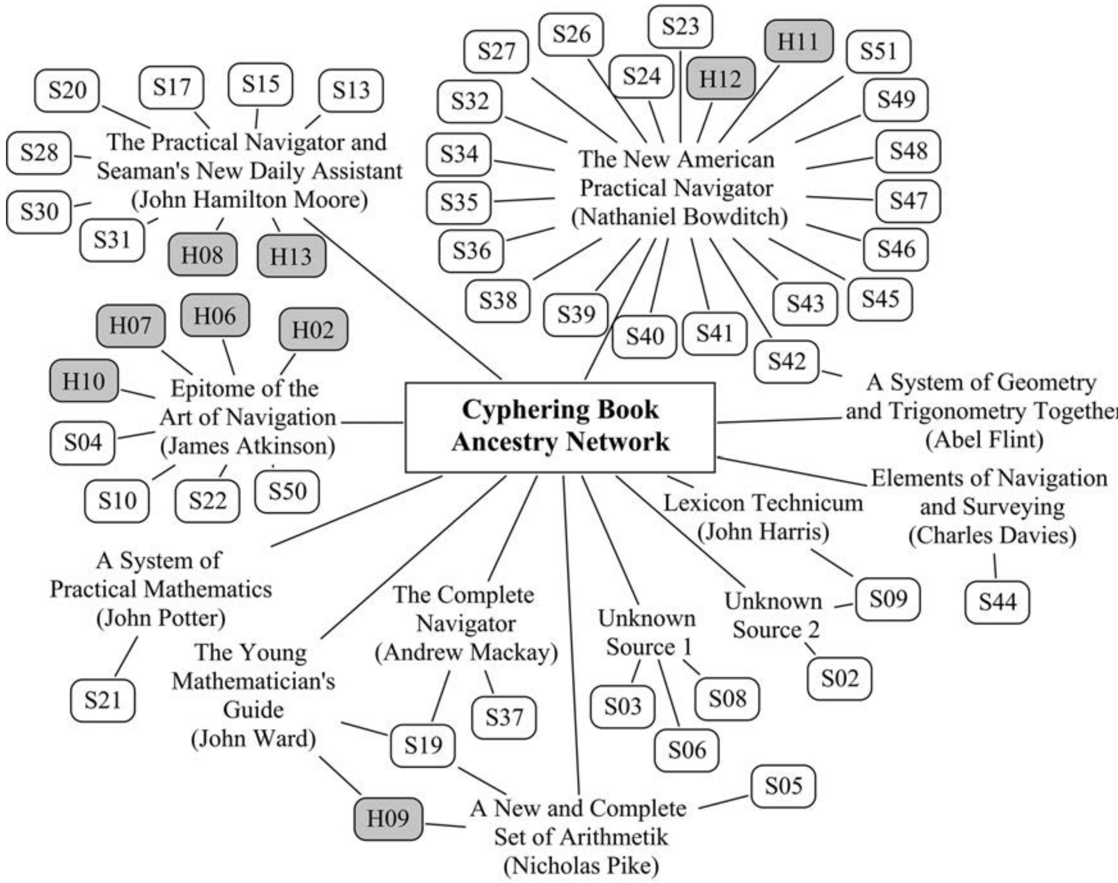


Figure 3. The cyphering book ancestry network for navigation cyphering books.

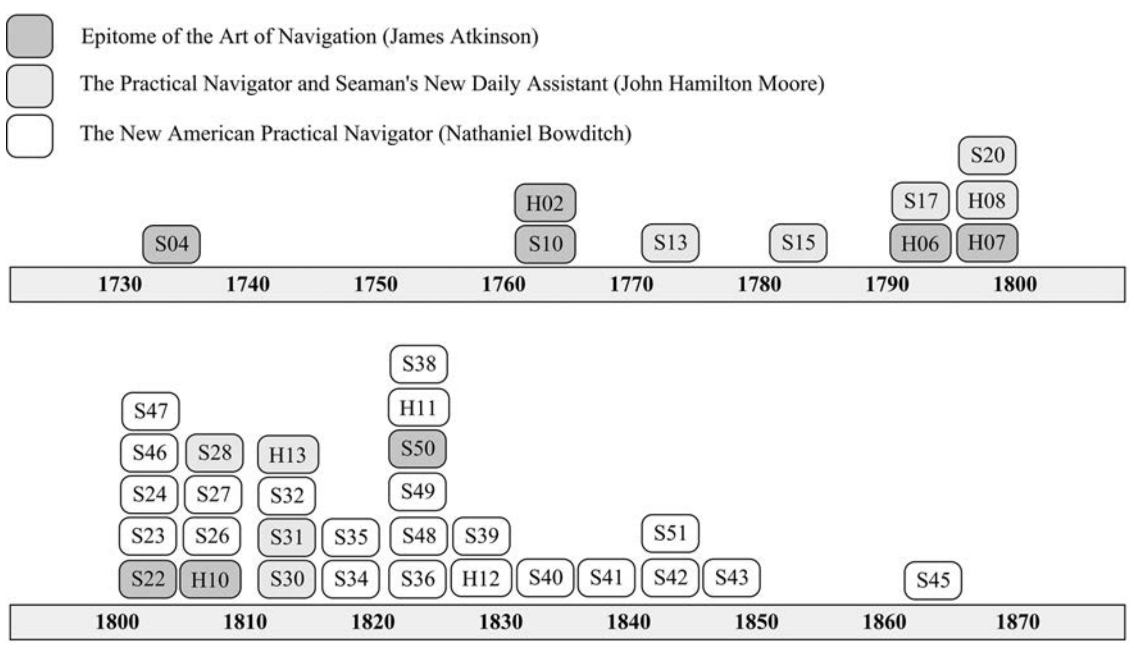


Figure 4. A timeline of seeds linking to prominent texts in the cyphering book ancestry network.

years through donations and represents a more regional focus that is less influenced by a single text.

Uncovering the implemented curriculum

The commonalities in structure across the cyphering books provide an initial outline of the implemented curriculum. Using this outline, I developed the following characterization of a navigation cyphering book:

> A cyphering book prepared as part of training in navigation contains sections on plane sailing, Mercator's sailing, traverse sailing, middle latitude sailing, and right-angled trigonometry. Also likely to be present, but to a lesser extent, are sections on geometry, oblique trigonometry, and parallel sailing. Content objectives focus on navigation first, trigonometry second, and geometry third.

This characterization provides a guide for determining whether or not a particular manuscript is a navigation cyphering book. To use an analogy, characterization of a modern-day algebra classroom would include discussion of linear equations at some point during the year followed by a study of quadratics, *etc.* Similarly, in a "classroom" preparing navigation cyphering books during the seventeenth or eighteenth century, one could expect to observe students working on plane sailing problems, trigonometry problems, and several other topics noted previously.

The evidence shows that this content coverage was stable over the 100-year period of the books and that instructors were consistent in their selection of content. Furthermore, when manuscripts were linked to texts via seeds, I found consistent evidence of omission of text content. For example, solution methods using Gunter's scale, which were in nearly every one of the various period texts, were uncommon in the cyphering books. Moreover, there were several cases where I found seeds linking to content before and after a solution method using Gunter's scale, but the method itself had been omitted. Therefore, it appears that solution methods using Gunter's scale, which were part of the intended curriculum within printed texts, were excluded from the implemented curriculum. Noteworthy about this finding is that there is no evidence instructors were working collaboratively or were formally trained on what to teach, or that there were efforts to standardize the curriculum itself. In contrast, within the cyphering tradition in England there were focused efforts to standardize the curriculum, particularly at the Royal Mathematical School at Christ's Hospital in London (see Ellerton & Clements, in press). Thus, the evidence shows that the cyphering tradition within New England developed its own accepted practices that were passed on from individual to individual.

My characterization of navigation cyphering books also suggests that they were only one component of a student's training in navigation. Most compelling is the fact that the majority of books did not contain preliminary sections on numbers, basic operations, *etc.* It is unlikely that students could have successfully completed a navigation cyphering book without this knowledge, but where or how they learned it is unclear. Perhaps, as is documented in other cases, such as George Washington's cyphering books (Crackel, Rickey & Silverberg, 2013), students had created an earlier book focusing on introductory mathematics; however, evidence of this practice as a widespread phenomenon is lacking. Similarly, although the calculation method was the most prevalent, none of the books contained a reference table, which is an essential tool in using the method. This absence makes sense if we consider that the books were intended for illustrative purposes. However, it may also suggest that such tables could be obtained elsewhere when needed. Likewise, the practice of keeping an accurate log of a journey, which is a critical task for navigation, was present in only 20 (31%) of the manuscripts. This omission of log keeping suggests that students were not expected to learn the practice by means of cyphering and that it would be incorporated into their training at some other point.

Reflecting on the modern curriculum

Although there are substantial differences between the implemented curriculum within navigation cyphering books and the modern intended mathematics curriculum, there is also overlap. The cyphering curriculum shows the importance of trigonometry, geometric constructions, and logarithms for practical tasks. All three of these topics are still a part of the modern curriculum to a degree. For example, geometric construction and applied trigonometry problems are recommended in various US Standards documents.

The connection between cyphering and modern curricula also raises a number of questions. For example, to what extent have the rich practical applications of topics, which made them valuable to the cyphering curriculum, survived to the modern classroom? Can we assume, as those teachers who worked within the cyphering tradition did, that the examples presented are useful to students in a practical sense? Or is our inclusion of "practical" examples merely a ruse to justify that there are instances where the mathematics can be directly applied? To what extent can we argue that generic examples (*e.g.,* finding the height of a lighthouse) illustrate the usefulness of the mathematics for students the same way that they did hundreds of years ago? Furthermore, the implemented curriculum of navigation cyphering books shows that, far from being a modern development, the focus on mathematical procedures without understanding has been present for well over a hundred years. How can this enduring challenge be addressed? Far from being rhetorical, these questions are central issues that many modern mathematics teachers routinely face in the classroom.

The notion of students preparing a mathematical guidebook for their life stands in stark contrast to the way that mathematics education is approached in many modern classrooms. Imagine, if you will, a modern mathematics educator tasked with helping a student prepare a cyphering book. What content would be included for this student? What content would be excluded? What things would we wish to showcase about a student's understanding of mathematics? How would we anticipate the student using their cyphering book later on in life? These and related issues can serve as useful starting points for curriculum conversations.

Final thoughts

Given the evidence of a common curriculum, a lingering question remains: how was this curriculum transferred? Was it an exact copy of a specific work or seminal piece? Or, perhaps, comprised of different pieces gathered from a variety of sources? The shared cyphering ancestry (Figure 3) is an initial picture of the complex relationship among navigation cyphering books and printed texts. This ancestry suggests that New England navigation cyphering books primarily drew from a single source rather than multiple sources. The nature of this single source, whether a teacher's cyphering book or a printed text, is debatable. Given the cultural value of cyphering books, high regard given to the manuscripts, and the scarcity of printed texts, it is likely that some manuscripts were prepared by referencing material in other cyphering books. However, answering this question requires a more robust analysis of navigation cyphering book content. Moreover, work is needed that looks more deeply into the shared ancestry of cyphering books in order to begin building an understanding of the lineage of the manuscripts.

Regardless of the source used for construction of manuscripts, the evidence shows that there was an implemented mathematics curriculum that was different from the intended curriculum of textbooks. Based upon my work with these books, I envision the implemented curriculum akin to the border around a patchwork quilt. This border ties together various patches of concepts and ideas. Although individual patches may differ in color or design, they are bound together by the same thread. It is my sincere hope that this research will help to refute notions that cyphering books were mere "copybooks" and promote a more accurate view of these manuscripts as forms of individualized instruction drawn from a common implemented curriculum.

Notes

[1] I use the words "cyphering book," "manuscript," and "book" interchangeably to describe handwritten manuscripts prepared during the eighteenth and early nineteenth centuries. The word "text" will be used to refer to printed manuscripts from the same time period.

[2] Note that there is some disagreement among scholars about how the abbaco tradition was introduced to Europe. For example, Høyrup (2014) claims that the tradition was already in France before Leonardo of Pisa brought it to Italy.

[3] The distinction between intended curriculum, implemented curriculum, and received curriculum, was made by Westbury (1980) and has been used by Ellerton and Clements (2012, 2014) in their examination of cyphering books. In brief, the intended curriculum is that which can be seen in textbooks and surviving institutional documents (*e.g.,* syllabi, curriculum guides). The implemented curriculum is that which was taught by teachers and is represented by cyphering books. The received (or realized) curriculum is that which students learn from the implemented curriculum.

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