Curriculum and Epistemology

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It may not be clear a priori that every curriculum proposal stands on an explicit or, more generally, an implicit epistemology. Ways of teaching, too. But these and curricula affect each other and go hand-in-hand with an unconscious system of social values which determine what to include in, as well as what to omit from, a proposal for a curriculum.

In this paper we shall try to restrict ourselves to the terms of the title, if that is at all possible.

Most of the readers of this journal have had many years of experience of curricula, those they covered as students at school and college and those they have taught or perhaps offered as an alternative. But perhaps not as many have attempted to struggle with what is covered by the word epistemology.

A dictionary definition of epistemology can be: "The theory or science of the method or grounds of knowledge" (O.E.D.). Whether this helps or generates further doubts in understanding what it is will probably depend on who its reader is. In my case, it can only serve to polarize my mind towards knowledge, and that does not help, for knowledge may be things remembered but not understood, as well as profound insights which open up new fields.

For Piaget's school of genetic epistemology, the adjective "genetic" signifies that they concentrate on the successive stages of learning concepts from their non-existence in the mind to that level of mastery available to the researcher. Piaget himself was less preoccupied with the content of curricula and more involved in trying to sort out the overall frameworks within which the essential notions of modern science find their place. Thus he wanted to reach first a grasp of the fundamental structures which would permit him to say that he "knows" what number, space, or time, or probability, and so on are, and only then move on to the young people who do not have these notions and watch how they acquire them. Occasionally he suggested that if teaching were guided by his findings, students would display more satisfactory evidence of their learning of those notions. Few people have worked out in detail a curriculum which follows Piaget's suggestions step-by-step, although a number of people use the word "piagetian" to qualify their proposals.

It is possible to be inspired by Piaget and to try to find how to introduce notions like number, space, or probability, to students and thereby suggest what can be called a curriculum, but there is still no guarantee that there will be a superior assimilation of arithmetic or geometry than before. In fact, this has not yet happened during the 60 years since Piaget produced his first monograph on what was to become his genetic epistemology.

The reason is to be found in an insufficient study of the field (epistemology) and of its applications (one of which can be curriculum design). To progress substantially on this challenge may require that the researchers ask questions, not asked or insufficiently stressed, about human learning, both at the level of working scientists and at that of newcomers: infants or students of any age.

In the field of mathematics, for example, the scientists who are busy producing theorems and theories do not feel equipped to tell us much about the actual activity which leads them to their perception of new mathematical facts. They count on other people to make that clear and, when they are encouraged to say, just the same, what they went through, they either lose their readers in the technicality of their statements or choose trivial material as examples.

As a result, we are not much better off for letting them enlighten us. Even as talented an expositor of mathematical thinking as the skilled and influential mathematician Georg Polya, cannot say that he has made a dent in the field of mathematics teaching. The breath of fresh air his contribution represented remains just that. We are today facing odds at least as great as fifty years ago since just as much work remains to be done.

Where *will* new ideas come from?

Could we, for example, instead of being concerned with existing knowledge, consider knowing and ways of knowing?

Could we, for example, notice that most of us as babies manage alone to teach ourselves matters much wider than anything we find in any elementary school curriculum? and, therefore, can we learn about learning (one meaning of epistemology) by studying children's spontaneous learning?

Could we stop being impressed and intimidated by the names of great men who, taking advantage of their celebrity, make statements about matters in other fields than their own as if their work entitled them to? Could we instead learn to put fruitful questions which keep us in close contact with the challenges we face? Piaget used to say to me more than 30 years ago: "Tell me what *you* think and I'll find it in children!" As he did when Einstein asked him: "Do you know how children think of time?"

Could we, for example, ask all in one breath: "What is

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mathematical in language and linguistic in mathematics?" Would this, when it is known, affect our teaching of both mathematics and languages?

Could we, for example, distinguish between algebra and geometry in terms of what the mind does in one and in the other?

Could we, for example, concern ourselves with awarenesses and how it is that we become aware of a host of different components of our inner and outer world? and ask ourselves whether thinking of awareness and of becoming aware might not be a more appropriate way of studying a field than being concerned only with the end product, called knowledge?

Could we, for example, recognize that the way our mind always works—by stressing something and ignoring the rest—is equivalent to what is called abstraction, as a mental activity? and, from there, see that it is possible to constitute a cascade (or hierarchy) of abstractions by stressing attributes or properties and ignoring others in already-stressed items?

Could we, for example, recognize that our perceptions and our actions, like our feelings, are real energy transactions which convey to us the reality of the world as well as nourish our sense of truth? generating at the same time the feelings of conviction and of certainty?

Could we, for example, see our images as substitutes for reality? but also endowed with dynamic properties which allow us to entertain virtual actions as preferable to actual actions in a number of transactions? Once virtual actions are grasped as the substance of the mind, and the intellect as entertaining their dynamics, do we find ourselves equipped to understand why it is so easy for babies to learn to speak so early? Could we at the same time realize how all pervasive algebras are present in the workings of the mind?

There are many more such questions which have not been asked by students of learning and teaching. Because they are new we can expect that they will reveal new and valuable results provided they are applied carefully and responsibly. Whether the new crop is of significance to many people involved in education, and mathematical education in particular, we shall see. For this writer they have altered altogether the appearance and the reality of the field.

To illustrate this profound alteration of perspective, I would like to summarize the contents of some courseware for microcomputers which at the same time objectifies my new epistemology, proposes a new curriculum, and represents a new foundation for mathematics. In addition, it will exemplify the difference between a broad and sweeping theory and a set of pinpointed treatments of successive items which together yield the detail of the field and lead to its mastery by newcomers. If there is a broad theory I shall not refer to it. In fact, at this moment, I do not know if it exists and don't care much whether it does or not. I think of practitioners (the teachers) and of the mass of learners of math the world over and I care only to propose exercises which are justified epistemologically, psychologically, pedagogically, and that end up representing a mathematization of a corner of the intellectual universe.

Numbers will not make their appearance until the middle of the twelve-disk course when a number of structures merge to produce them

The first four disks are about numerals, which are items originating in the language and are studied first as the signs and names that the English-speaking society has adopted for its own reasons in the past. Disk 1 surveys the set from 1 to 999; Disk 2 extends that to the thousands up to 999,999; Disk 2a takes that to millions and billions (in the USA meaning). It is only in Disk 3 that we find the "ordering" of numerals and its availability for the operation of "counting." This complex operation requires adding a property to the numerals—which can already be named, written and read—serving to attach a label to every finite set of objects, called its "cardinal." Cardinals are recognized as being invariant with respect to the choice of the order of the counted items: they are a "property" of a set.

It has been possible also, in passing, to cover two things which are considered as having to belong to an elementary arithmetic curriculum: (1) place value, (2) bases of numeration (although the second has mainly come about on the back of the "modern math" wave of the early '60's).

The first group of four disks therefore treats numeration first as a language problem, totally unrelated to counting which is seen here as a much more complex mental activity, met at the end when all the notions which have appeared separately can be integrated into an action (called counting).

Starting with the spoken language that children take to school when they start at 5 or 6 years of age, we let them recognize the three forms which go with each numeral: the spoken, the notational and the written. They do this first on the set 1,2,3,.. 9 whose order is *not* stressed and which must be known individually in the three forms just mentioned. This is easy with the computer with its random function. Let's call them the "units."

Following that we meet the name "hundred" vocalized after each of the nine previous numerals; in the corresponding notation 00 is placed next to each unit and on its right, so that with the one new name, nine new numerals (100, 200 ...900) are integrated. Immediately after that we meet the 81 numerals which start with any of "the hundreds" followed at random by any of "the units." The computer shows that these numerals are written in notation by substituting the 0 on the right in the hundreds by the digit names, e.g. 700 and 5 becomes 705. Thus ten memorized names allow the figuring of 99 numerals.

The next name is "ty", introduced as the name of a 0 placed on the right of any one of the five digits, 4, 6, 7, 8, 9. These five new numerals 40, 60, 70, 80, 90 have regular verbal forms in English and they can be compounded with the 9 digits or the 9 hundreds to produce $5 \times 9 + 5 \times 9$ new names; the computer shows how these compositions are written and named. For example, (60, 7) is written 67; (100, 40) is written 140 and the notation is mastered as we respond to the randomly-called pairs of numerals—starting either with the "hundred" followed by the "ty", or with the "ty" followed by the "units".

But we can do more: we can form "triplets" starting with any one of the "hundreds", followed by any one of the "ty's", and finishing with any one of the "units", in this order. This produces five hundred new three-digit numerals without a zero in them

The English language has irregular names for 20, 30, 50, which are introduced and integrated separately.

"Ten" offers four new demands: *one*, its name is a single word as if it were a unit, but its notation has two digits, one being a zero as in the "ty's"; *two*, when linked to the "units", ten changes into "teen"; *three*, the writing of these compound numbers (19, 18 .) differs from the ordering of their names. In writing, as with all the "ty's", one first puts down the digit of the second group ("ty's") and then the unit. But when reading, usage requires that the unit is said first and then "teen" tacked on to it; *four*, two of the compounds, 11 and 12, have a different structure for their names, while 13 and 15 change the name of the unit—though there is an echo here of the irregular formations of 30 and 50.

Let us note that at the same time as we present these 999 numerals and give plenty of practice in making their names, their appearances, known, we have given "place value" its conventional meaning and made sure through plenty of practice that the ambiguous cases (e g. 206 and 260) can no longer be seen as ambiguous. The conventions of reading and writing numerals are traditionally tested under the title of place value.

Disk 2 is actually devoted on the one hand to the function of zero in numeration and, while doing that, to extending the reading and writing of numerals up to 999,999—i e. up to the introduction of "one thousand" as the "name" of the comma. Although some schools of teaching suggest that the comma be dropped as unnecessary, we maintain it for pedagogical reasons. Students can be forced to attach a name to a comma when it appears, while to name a space or a pause between the triplet of digits on the left and the triplet on the right seems a more confusing suggestion.

The microcomputer with its visual display on the monitor seems ideal for the rigorous presentation of a challenge which takes a long time to be sorted out in traditional classroom lessons, leaving many students uncertain for a long time

In the previous treatment of the first 999 numerals zero was not conceived as a place holder, it came as a distinct sign (belonging to the hundreds as 00 after the unit and to the "ty's" as 0—except in ten—after the unit). When we move to the thousands, zero must be used to hold the place of unspoken hundreds and ty's as in, for example, seven thousand nine (7,009).

The students already know from work on Disk 1 how to name and write one-digit, two-digit and three-digit numerals. The way they learned to name, say, two hundred and six was by seeing 200 and then 6, then seeing the unit displace the 0 on the right. Now they are made aware that the writings 06 and 006 are only there to trigger the name six. This, of course, takes place only on the right of the comma and needs to become second nature by practice

The pedagogy for this learning, ending in a know-how, is dictated by the new epistemology which makes us take a thorough and pinpointed look at how we make sense of conventions created by ancestors who only asked the questions that came to them rather than those which need to be asked

Disk 2a extends this know-how to millions and billions In these larger numbers commas change names according to whether there are two or more of them in the string of digits organized in periods of three. It is clear that the only thing students have to learn is that the expansion from one period of three digits (as done in Disk 1) to two can be taken further to three by simply accepting to call the first comma of two—when reading from left to right—"million" while keeping the second as it was when there was only one, giving it the same name "thousand". When there are three commas the first met is called "billion", the other two remaining as they were

At this minimal cost students can now read numerals of up to twelve digits as easily as they could those of three. The inventors of the notation for numeration must have been guided by the enormous economy the above procedure presented. These three disks present exactly what it costs to master numeration and offer the exercises which secure this mastery No one can escape learning it

Let us add that in this study we can state categorically what are the difficulties which need to be overcome and in which order. Only the new epistemology makes that study possible. Only it suggests that we show young students who have managed to learn to speak the language of their environment how to handle numeration extended to 999 999,999,999 before even having counted up to ten

The above represents a good example of how epistemology affects curriculum production

Disk 3 takes two steps before contemplating the question of counting. One is to inject a new structure onto the set of numerals, produced in the way described above, and call that structure "order". From now on it is not the regularity of the language, and the yield in numbers of numerals produced, which makes us bring them in, but the fact that the units can be read *in order* from 1 to 9, and that this order can be repeated when each of these digits in turn occupies the next place to its left, producing *in order* 10, 11, 12, . ..19, followed by 20, 21,. ..29; 30,31, ..39; and so on up to 90, 91, 92,.. 99.

This "sequence" can be extended to include the hundreds, so that from 100 to 999 the same ordering is found.

From now on the numerals we generated in Disk 1 and 2 can not only be read, named and written, they can also be *recited* in a uniform way, forming "the sequence of numerals". Sometimes such recitation is called "counting", although we have here only numerals and a procedure on them. The ambiguity which that word brings in should be avoided for as long as the students may be confused by it; but after that we shall not say "recite in order" but "count".

The second step is a more sophisticated one. Going over the processes used in Disks 1 and 2 we can see that there is no compelling reason why (a) we considered all units from

1 to 9, and (b) we stopped at 9.

If we stop at any unit other than 9 from 1 to 9 and repeat the creation of the "ty's" and the hundreds up to that unit, we find that we can do all that was possible when we stopped at 9. This stresses an item of numeration which was there but ignored so far, and that is that we can select *which unit precedes ten* and keep all the others before it in the recitation of the sequence and form a new system of numeration as valid as the one we have studied. To distinguish the systems, we call "base of numeration" the first unit *not* included in the sequence. Thus base 4 does not allow the appearance of 4 nor its name in the sets of numerals formed with the same structuration as "the ordinary system".

To go beyond 9 requires *new* one-digit units, and letters of the alphabet present themselves. If *X* and *Y* are used after 9, the new set produced is the one sometimes called "duodecimal". In computer science the six letters A, B, C, D, E, F, are used as units and the system of numeration is called "hexadecimal". The "binary system" stops immediately before 2, and only has 1's and 0's in the written numerals. The "octal system" stops immediately before 8, and only 0's, 1's, . 7's appear in its written numerals.

The new epistemology tells us that systems of numeration are the outcome of an awareness of a property that belongs to sets of numerals and nothing else. Hence they can be placed before "counting" is tackled in a curriculum concerned with a hierarchy of ideas.

The rest of Disk 3 is taken up by "counting".

If a set of objects is made to appear on the monitor, students can perceive them as separate objects united within their field of vision.

An *action* can now be carefully introduced composed of

i. a recitation of the numerals in order from one on, and
ii. a focusing (by touch or otherwise) on each of the objects of the set so that:
    a. one object is considered every time a numeral of the sequence is uttered,
    b. no object is considered more than once,
    c. no object is left out of consideration,
    d. the recited sequence starts with one.

Such an action we shall call "counting" proper, i.e. "counting the set". The numeral of the sequence which coincides with the consideration of the last object of the set is called the "count" of the objects or the "cardinal" of the set.

While the recitation must follow the order of the sequence of the numerals and starts with one, there is nothing in counting which tells which must be the first object, or which must be the last, or which element in the set must go with any numeral between 1 and the cardinal of the set. This freedom of association of the recited numerals and the items of the set tells us that the cardinal of a set is a unique numeral whichever order is chosen to go through the objects of the set.

Each finite set of objects can be counted and to it will correspond only one numeral, its cardinal. All sets which yield the same cardinal are said to have "as many" objects, or to be "equivalent" to each other, or to be equally acceptable sets with respect to this singular attribute: their cardinal.

So far, we have covered the following ground of the foundation of mathematics:

1. *Numerals are linguistic items which can be acquired as language is, easily, early, for good.*
2. *The set of numerals can be turned into an ordered sequence.*
3. *It can be used when ordered to define the cardinals of all finite sets. This is what is called counting or answering the question: "how many" elements are there in a set.*

By the way, counting puts into evidence that the "name of the cardinal" of a set is a function of the basis of numeration, i.e. there are a number of names for cardinals or numerals according to which numeral in the sequence precedes ten.

Experience shows that all this appears straightforward to ordinary first graders who can speak English.

The new epistemology has also made plain that if numerals are read and written from the left, algorithms for addition and subtraction must exist which also work from the left rather than the traditional algorithms which work from the right, using transformations like "carrying" in the case of addition and "borrowing" in the case of subtraction.

The new algorithms are based on a more profound understanding of the foundations of mathematics. In fact, addition and subtraction are two readings of a more primitive entity, a partition of a set into two "complementary subsets". Looking at a set and a pair of its complementary subsets we can see

1. *that the two subsets together make the original set.*
2. *that the original set and one of its subsets defines the other subset.*

(1) can also be seen as the merger of two subsets to produce their "sum", the basis of addition; (2) can also be seen as defining the "difference" between the whole set and one of its given subset, the basis of subtraction.

Therefore "seeing" addition and *subtraction in the same situation of a partition of one set into two complementary subsets is a matter of mental optics, easily obtainable from students by asking them to focus on this or on that, and to note that the situation has not changed, only the viewpoint.*

Disks 4 and 5 are concerned with that part of the curriculum we call complementarity and its consequences.

We can do the whole work using the fingers of one's hands.

The set of fingers can be partitioned into two sets by the simple "operation" (which is an action) of "folding" some of them; or "unfolding" them if they were all folded. Because of the choices available we can see that there are ten ways of showing one finger; forty-five of showing two. As a consequence there are also ten ways of folding one finger only and forty-five of folding two out of ten. At the same time we find that there are 10 ways of showing 9 fingers or of folding one; forty-five ways of showing 8 fingers and forty-five of folding two. We mention here the count of how many ways, but there is no need to actually demonstrate them; for our present purpose, it is sufficient that students feel only that there are a number of ways, not just one

What we can force awareness of is that 0 goes with 10 and 10 with 0; 1 goes with 9 and 9 with 1; 2 with 8 and 8 with 2; 3 with 7 and 7 with 3; 4 with 6 and 6 with 4; and finally 5 with 5. This is summarized in ordered pairs of numerals (or cardinals) as (0,10); (1,9); (2,8); (3,7); (4,6); (5,5); (6,4); (7,3); (8,2); (9,1); (10,0).

It would have been possible to move straight from these findings to the translation of a pair into a dual notation—that of addition or subtraction—as is done on Disk 6. But since it is possible to cash in on an easy expansion of the above pairs to the case of complements in one hundred and in one thousand by simply placing one zero or two zeros after the components, this can be done as on Disk 5.

The treatment in this section is of interest to people working on epistemology for it shows how to first separate mental dynamics and language and later how to merge these separated items to produce awareness of greater mental power and of wider know-hows. These also translate into what was considered valuable knowledge to previous generations of educators.

Counting does not require that the objects of the set be identical. All fingers are different, but we can count them from 1 to 10. By giving each the attribute of triggering at the same time as the one-digit numeral the immediately following "hundred", say, we can see that all we did on Disk 4 can be extended to the set of ten hundreds. If "hundred" is replaced by "ty" or "thousand" or "million", etc., the relationships singled out in the modules of Disk 4 will also obtain in all these cases. We can say that "the algebra met in those modules extends to the new language introduced by these changes of words and notations."

But we can do more. Instead of having one pair of hands as on Disk 4 we can use a device by which one of the fingers of one pair is exchanged for a pair of hands. We must be careful when we make these operations since we are stating that "one ty" can be replaced by ten units, "one hundred" by "ten-ty", and so on. As a result, if we have more than one pair of hands, only one pair displays ten fingers and the other pairs nine. It is always the pair of hands allocated to the units which displays ten fingers and the other pairs, when they exist, which display nine each.

The enormous advantage of this presentation is that at once we notice:

a. the whole set and two subsets of fingers determined on it by folding (or unfolding) selected fingers;
b. that "place value" becomes a reality since the notation of the numerals displayed now refers to fingers on specific hands;
c. that we can read uniformly, and in a direction we select to be from left to right, the names of the complementary numerals;
d. that we have no doubts that in the writing of the complementary numerals all the pairs of digits in the same place add up to 9 except the last two which add up to ten.

For example, if we started with five pairs of hands, side-by-side, and opened up the pair on the left only, we would begin with ten ten-thousands. By exchanging one finger of that pair and removing it from the count, replacing it by the open pair of hands next to it, displaying now ten thousands, we have the equivalent displays: $10 \times 10{,}000 \sim 9 \times 10{,}000$ *and* $10 \times 1{,}000$.

By exchanging one finger of this open hand for the next pair of open hands we replace $10 \times 1{,}000$ by $9 \times 1{,}000$ *and* 10×100. This display of $10 \times 10{,}000$ (or 100,000) as:

$9 \times 10{,}000$ *and* 9×1000 *and* 10×100 can lead to
$9 \times 10{,}000$ *and* $9 \times 1{,}000$ *and* 9×100 *and* 10×10 and finally to:
$9 \times 10{,}000$ *and* $9 \times 1{,}000$ *and* 9×100 *and* 9×10 *and* 10

At each stage of the production of these displays, once some fingers have been folded, we can read the two pairs of complementary numerals into which 100,000 has been partitioned. Soon we shall see that this operation can be cast in the form of a subtraction in which one can read at once the difference between a number of the form 10^m and any number "smaller than" it (i.e. preceding it in the ordered sequence of numerals).

It is one of the remarkable consequences of this presentation based on the new epistemology that subtractions which are traditionally very challenging now become "child's play", or a simple reading of an answer constructed from left to right, by naming (or writing) the complements to 9 of every digit from the left, ending with the complement to 10 of the last digit.

Getting from complementarity to the algorithm for subtraction requires some digging by the new epistemology, as is shown on Disks 6, 7 and 8. Nevertheless, what has been achieved so far is not trivial. Indeed, we have seen that a change of stress on what is being contemplated is a method of producing awarenesses of new facts, of generating knowledge. We have realized that there are things in what we contemplate which impose a presence that has to be respected and maintained, while there are other things which can easily be replaced, yielding new possibilities and awarenesses.

As examples of this working of the mind we can see

a. that there are so many objects in a set, but that to say how many depends on the basis of numeration used for counting;
b. that objects can be counted whether they are identical or not;
c. that we can name the same set in different ways;
d. that one set can be partitioned in many ways, etc.

One new important awareness leads to the concept of "commutativity" and its true nature, which is an awareness of a certain "indifference" with respect to a certain operation. ***Addition is commutative because we are indifferent about which of two sets****—which are to be merged to become a new one—****is merged with the other.***

In the case of the complement of a subset, or the difference between a set and one of its subsets, since we need to hold in mind both the set and the subset, such an indifference is not experienced and hence subtraction cannot be commutative.

For the mathematics education of our young generation we have to make them aware of the ways the mind uses in the generation of the universe of mathematics. These ways may already be present in the manner challenges are met in everyday life and only need to be stressed in one's awareness to gain their reality.

For example, if we concern ourselves with "transformations", or with "equivalences", or with both together, we can find them deeply ingrained in our use of language, from the start and all the time. It is usually not possible to say the same words as those we hear because speakers use a self-centered reference system in which "I", "my", "one", are prevalent and must not be repeated if one refers to them in certain situations: "my nose" has to become "your nose", "I am on your right" has to become "you are on my left", "my mother and yours" becomes "your mother and mine", and so on.

No doubt transformations are neither met first in mathematics nor exclusively there. Equivalence is a much more flexible relationship than equality because it refers to relationships which link items so that one can replace another in some circumstances without them needing to be the same; e.g. while one is "equal" to one, "one and one" is only equivalent to two, and conversely: the reality of "one *and* one" does not eliminate the oneness of the ones, and two is seen only as another way of saying "one and one".

Every equivalence assumes a transformation and operating on numbers involves both, changing what we start with into what we end with so that the beginning and the end are equivalent.

A curriculum which is true to our ways of knowing (practiced and tested to the point that they become second nature and, therefore, are available for further studies and development) will have to base itself explicitly upon equivalence and transformations. Disks 6, 7, 8 and 9 are used for that purpose while they also yield much material considered essential in all mathematics curricula.

While in terms of addition "integers" form finite classes of equivalence (those of their partitions), in terms of subtraction they form infinite classes.

Both awarenesses are fundamental and essential, and if we can ever make students have them early in their school careers, we shall serve their cause as well as that of education. Each integer becomes "a mathematical entity" when, besides being a word and a concept *per se*, it is thought of as equivalent to all its partitions (in finite numbers, even if very numerous) and also equivalent to the infinite number of differences which can be formed for it. For example:

$$5 \sim 4 + 1 \text{ or } 3 + 2 \sim 2 + 3 \sim 1 + 4$$
$$5 \sim 6 - 1 \sim 7 - 2 \sim 8 - 3 \sim \ldots$$

There will be other classes of equivalence for an integer as soon as other operations, besides addition and subtraction, are introduced.

In this courseware, a new algorithm for addition and subtraction could be made available by exploiting equivalence and transformation. The pedagogy leading to it, while providing a sequence of activities, provides the curriculum which corresponds to this epistemology.

The final step can be said to be "reading the answer within the problem given to solve" by using a string of transformations that replace each given by an "easier" one and that can be telescoped into a lightning operation. In getting the students to this level a number of skills are presented and practiced to the point of becoming attributes of one's sight. Here we may call it "a mathematical sight!"

In the case of additions the "shifting" of units from one of the two numbers to be added to the other can be done very swiftly because of the study of complementarity, to produce 0 in every column. Since there are 0's and non zeros in each column the answer can be read, and read with certainty.

e.g.

637	by shifting 31	640	by shifting 401	600
+254	becomes	+251	becomes	+291

and the answer 891 can be read from the left

All cases are studied so that no addition can escape being transformed by these shiftings, thus generating, at the end, merely a reading pattern.

In the case of subtraction, both equi-addition and equi-subtraction can be used to transform a problem into one where the answer can be read and written immediately. The final challenge, equivalent to the first, has the property of letting the answer be read in it.

e.g.

631	by sub 1	630	by sub 30	600	by add 400	1000
−254	becomes	−253	becomes	−223	becomes	−623

giving the answer 377 by complementarity

Clearly, once the transformations are defined, the number of digits in the given numbers does not add a new difficulty, only more of the same kind of steps to be taken. Likewise, there is no special significance in the base of numeration for the procedures (or algebra) are the same whatever the base.

Disk 10 can give any number of exercises to practice precisely those invariants and provide tests—to feedback to the students—that mastery has been attained. The new algorithm for subtraction takes care of any subtraction in any base and of any length (i.e. number of digits). It blends addition and subtraction, complementation and transformation in a mathematically elegant fashion, giving this chapter its ultimate form. All there was to understand about this matter in order to achieve the required mastery has now been made available to children of eight or perhaps even younger.

Mastery is demonstrated by speed, accuracy and confidence. Students and mathematicians alike demonstrate it.

This is a proper example of what is meant by a scientific curriculum based on the science of education of which the new epistemology is an integral part.