Mind, Matter, and Mathematics

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There are those who hold that there is nothing else but Mind and there are those who think there is nothing else but Matter Those who hold these views often tend to be very certain that they are right and that the others are wrong. In this case right and wrong often become moral as well as logical judgements Faced with such conflicting certainties it is tempting to call a plague on both houses No matter! Never mind![1] But it does matter and I, for one, do mind So rather than sit on the fence between these cartesian dualities, it may be worth seeking a third mediating term

The same issue is often found in the case of Mathematics The problematic dichotomies re-appear as soon as we ask questions about the nature of mathematical reality There are those who hold that mathematics is constructed in the inner world of the mind and there are those who think that it is found in an outer world of matter Those who hold these views often tend to be very certain that they are right and that the others are wrong In this case right and wrong often become moral as well as logical judgements Faced with such conflicting certainties it is tempting to call a plague on both houses Neither constructivism! Nor empiricism! But these points of view are both in some way indispensable, they cannot be arbitrarily discarded and many people find it difficult to choose one at the expense of the other So rather than sit on the fence between these cartesian dualities, it may be worth seeking a third mediating term.

A mediating third is often found in an independent world of ideas It is not surprising, then, that many — perhaps most — professional mathematicians see themselves as platonists (at any rate, some of them say, on weekdays) This point of view is rejected (often quite intemperately) by those mathematics educationists who are currently taken by some form of constructivism It would also be rejected by those who take an empirical point of view, for example by those neuroscientists who have in recent years become very interested in how and where mathematical activity occurs in the brain Their approach tends to link mind and matter in that they seek to find a physical description of mental activity in terms of electrical and chemical activity in neurons and synapses

My current interest in these issues — and indeed the title I chose for this review article — derives from a remarkable book. *Conversations on mind, matter, and mathematics* [2], which translates some conversations between Jean-Pierre Changeux, a neuroscientist author of an important book on the biology of the mind, and Alain Connes, a mathematician and Fields medallist The background question is whether a machine could reproduce the activity of the brain This would involve knowing more about the way the brain works Since mathematical activity occurs in the brain, neuroscientists are interested in the nature of mathematical activity Hence this interdisciplinary dialogue which ranges very widely and provides some exciting cut and thrust of argument, with at times the feel of a genuine conversation, a meeting of minds

I cannot here cover all the fascinating issues they raise. In general, the speakers keep to one or two central themes The opening third of the book addresses the nature of mathematical objects, about which they take opposing sides The central third discusses a neurobiological point of view The final third consists of a chapter on "thinking machines", another chapter that was added on for the English edition in order to tie up some loose ends, and a final digressive epilogue — almost entirely a monologue by Changeux — on some ethical questions There are throughout various other digressions into technical matters (which are sometimes too condensed and difficult to follow despite the provided glossaries of biological and mathematical terms) but the conversation always returns to the way in which mathematics might be held to mediate between mind and matter

In this article, I give a brief account of some highly selected passages of a book which I commend to anyone also interested in its main themes, and I end with some further comments of my own

An evolutionary point of view

To start in the middle — namely the central and possibly most important part of the book which introduces some recent brain research that has focussed on mathematical activity The fourth chapter (called "The Neuronal Mathematician") starts with a discussion of Jacques Hadamard's four phases of mathematical activity: preparation, incubation, illumination, and verification Connes confirms that these make sense for him He adds a relatively rare revelation by a creative mathematician of some of the emotions involved

> [Co] The final phase, verification, can be very painful: one's terribly *afraid* of being wrong Of the four phases it involves the most anxiety, for one never knows if one's intuition is right — a bit as in dreams, where intuition very often proves mistaken . The moment illumination occurs, it engages the emotions in such a way that it's impossible to remain passive or indifferent On those rare occasions when I've actually experienced it, I couldn't keep tears from coming to my eyes [p 76]

Changeux helpfully asks some probing questions and recasts some answers in biological terms Connes describes the mulling over a problem as "having a framework for thinking, a neighbouring field for exploring the problem indirectly" For Changeux this enlarging of context permits "variability" — a perfect metaphor, he says of Darwinian evolution Connes mentions the process generalisation and Changeux sees this in terms of creating a larger framework through a "diversity generator" that produces a range of mental variations called "pre-representations" Connes thinks the Darwinian model is appropriate for a computer playing chess

The program for this will involve some selection function that measures the advantage of any sequence of moves. But it is not clear to him that there is an analogue in the human brain.

The conversation continues: the following extract illustrates how they begin to pick up and use each other's ideas (though there are plenty of other occasions where they remain in dogged disagreement); it also introduces some themes that become quite important later

> [Co] To establish the existence of a Darwinian mechanism in the brain, it would be necessary to understand what type of evaluation function is at work during the incubation period of selecting the solution to the problem. One could then very roughly say that the first stage — preparation — consists in consciously constructing an evaluation function connected with affectivity, which could be crudely expressed by the formula: "That's the problem I want to solve" The Darwinian mechanism would correspond then to the second stage — incubation — with illumination occurring only when the value of the evaluation function is large enough to trigger the affective reaction
>
> [Ch] A kind of "pleasure alarm" goes off, in other words, rather than a danger alarm signalling —
>
> [Co] That what's been found works, is coherent, and, one might say, aesthetically pleasing. But the word "Darwinian" seems to suggest that there is something hidden that controls the selection function that determines the quantity to be optimised
>
> [Ch] Of course But nothing's hidden. Selection is built into the mechanism [pp 82-2]

Changeux seeks a neurological account of mathematical activity and distinguishes his point of view from those in the field of artificial intelligence who look for an algorithmic description of thought processes. He points out the need to define a hierarchy of levels of organisation — atoms, molecules, neurons, circuits, assemblies, assemblies of assemblies — to describe increasingly complex interactions. These are related to the three epistemological levels described by Kant: Sensibility, Understanding, and Reason. [3]

> [Co] It would be simple enough to more or less precisely define three levels of mathematical activity, but I'm not sure how these would relate or not relate to Kant's three levels I'd rather we use a different terminology
>
> [Ch] Be my guest! And then I'll see if I can make your three levels square with the neurological evidence. [p 86]

Levels of thought

Connes suggests a first level of algorithmic calculation. He emphasises its richness, saying that it covers everything one does in a pre-university mathematics course ("the most idiotic kind of math!", adds Changeux). A second level involves checking and the possibility of changing strategy; for Connes this involves feelings and so cannot be achieved by computers. The third level is then that of discovery — Connes is a platonist and speaks of "unveiling a still unexplored area of mathematical geography".

Changeux then sets out to relate these to levels of organisation in the brain, which may be described in terms of neurons (there are about a hundred billion of these basic nerve cells, linked by synapses, about ten thousand of these for each neuron!). A first level of organisation includes networks of neurons that handle specific automated actions such as walking, speaking, looking, and so on (whether these are thought of as innate or acquired), as well as some simple mathematical operations. Part of the current neuroscientific interest in mathematical thinking derives from the various experiments with people suffering from brain damage in some way. Some subjects can read letters but not numbers and this is associated with damage to part of the left hemisphere; on the other hand ability to read and order numbers simultaneously is dependent on controlling eye movements, and this is located in the right hemisphere. (It is, incidentally, a critical feature of such accounts that they always refer to "numbers" and it is not clear what sort of number — for example, ordinal or cardinal — the experimenter, or indeed the subject, may be invoking.)

Changeux suggests that at another level of complexity assemblies of neurons define what he calls "mental objects" or "representations". These are located in the frontal lobe and Changeux cites experiments that suggest damage to the frontal lobe interferes with the ability to form simple hypotheses. He agrees that this is a second level function, but thinks that — at a third level — assemblies of assemblies forming "chains of representations" will yield more complex thought and that this will also be located in the frontal lobe. But Connes thinks that the tests given to patients are at the level of organisation, not imagination. Perhaps, says Changeux, the issue might be resolved by looking at a Darwinian model for the transition between levels. He explains this in some detail But Connes remains unconvinced that the mental representations, which he agrees exhibit the coherence he particularly prizes, constitute mathematical reality. "We've got back to our original point of disagreement", he says, "I think it's time we got beyond this".

Darwinian ideas are developed in the following chapter. But Connes wants to introduce some topological ideas because these, he thinks, provide the right framework for the way in which the brain is characterised by "a large degree of diversity", but also by a certain invariance across individuals". He associates neurons with vertices of a simplicial complex, links between neurons with simple connections of vertices, and assemblies of neurons with multiple connections. He proposes a model of his own in terms of a "hyperbolic simplicial complex" (The jargon here must be as opaque to non-mathematicians as some of the biological expositions were to me.) The introduction of topological ideas is surely a promising source of further developments in this field as in so many others — though Changeux points out that such theoretical models need to be converted into feasible laboratory experiments

They continue to try to find some agreed way to characterise the third level How one might capture judgement seems to be the main issue. For Connes, the third level is characterised by the illumination when the "harmony and power of a new object" is grasped instantly, independently of any specific problem. The resulting exhilaration, he supposes, is accompanied by excitation of the limbic system (those parts of the brain associated with emotional behaviour). His description reminds Changeux of the mystical ecstasy of St Teresa; Connes imagines this would also occur in the same region of the brain, but for other reasons. Illumination, in mathematics, involves a sense of coherence with other objects, some near in the sense that the brain is already familiar with them, and others quite far. And it does so instantaneously. This reminds Changeux of face recognition: the brain can instantly distinguish between faces it knows and those never seen before. Changeux sees this as a selection mechanism, but Connes wants to say that selection is but the means by which coherence is manifested. The brain perceives coherence between familiar objects and new ones previously unknown. Whether the latter previously existed in some way brings them back to their fundamental point of disagreement

Thinking machines

They do, however, find considerable agreement, in one of the later chapters, when they discuss artificial intelligence and the more recent developments in cognitive science with which Changeux has been involved. Inevitably, the prevailing metaphor is still that of the digital computer. In the extreme version (which they both reject) the mind is seen as a computer program and it therefore does not matter what the brain is made of — what matters are the algorithms which the brain is deemed to be carrying out. So the discussion begins with Connes being prompted to discuss Gödel's theorem. He emphasises that the incompleteness theorem cannot be interpreted as limiting our understanding. Questions which are undecideable in one system create bifurcations into different systems. We can choose whether there are to be no cardinals between those of the rationals and the reals, or whether there are to be several. "We mustn't accept the static picture of a world in which there exists a fixed, finite number of axioms supplying an answer for everything."

Changeux then recalls the question with which Turing began his famous 1936 paper: "I propose to consider the question: can machines think?" Connes sees a parallel in attempts in quantum theory to explain what happens when particles collide: Heisenberg's theory analyzes the property of a matrix, it does not describe the mechanism involved in the collision. But for Connes all this is to stay at first or second levels. They both reject a functionalist point of view that identifies the brain and its functions as a Turing machine. Changeux observes that it is satisfying — "for once, at least" — that their views converge. The human brain is an *evolutionary* machine and this is its main difference from a computer. In particular, they consider intentionality: although a machine can be built with an evaluation function determined by some simple intention (for example, winning a game of chess), it cannot decide whether a particular evaluation function is the best for a given intention. Connes seeks an evaluation function for evaluation functions — and, indeed, a hierarchy of such functions. This is why, for him, computers are still at his first level; apart from anything else they lack the affectivity that plays such a large part in adaptation and which is the mechanism that provides access to the second level

The philosopher, John Searle, has pointed out that because we know so little about the working of the brain we are always invoking some current technology to provide a model. In his childhood it was the telephone switchboard or a telegraph system. Freud had invoked hydraulic and electromagnetic systems; for Leibniz it was a mill, and apparently for some ancient Greeks it was a catapult. Work on artificial intelligence has invoked the computer. Current cognitive research still does so, but is not so ready to ascribe thoughts and feelings to machines. And this is where Connes would agree with Searle that the computer metaphor is ultimately inadequate. One might envisage a machine that could adapt its strategy to some given intention, but in third-level creativity the goal itself is unknown. Connes finds both "a peculiar novelty and harmony" in the creative outcome. And it is this harmony — and the feeling it arouses — which is an important feature for him. He agrees that this could be given a physiological description, but still feels that harmony is apprehended as an external reality.

Nature of mathematics

It seems that the conversation keeps on returning to initial disagreements about the nature of mathematics. These are discussed at length in the first three chapters where familiar arguments (familiar, at least, to mathematical educationists) are put forward about such issues as: whether mathematics is invented or discovered, whether and in what way mathematics is a language, how far it is independent of culture, race, or gender, and so on. Changeux tends to take a constructivist approach but does so in physical terms: "mathematical objects exist in the neurons and synapses of the mathematician who produces them". Connes tends to take what he calls a realist approach: "the working mathematician can be likened to an explorer setting out to discover the world". For him, this world also has "a material reality", but this is neither in the brain nor in the external physical world.

> [Co] You say that nothing proves the reality of these objects outside our brain. Let's compare mathematical reality with the material world that surrounds us. What proves the reality of the material world apart from our brain's perception of it? Chiefly the coherence of our perceptions, and their permanence — more precisely, the coherence of touch and sight that characterizes the perceptions of a single person, and the coherence that characterizes the perception of several persons. And so it is with mathematical reality: a calculation carried out in several different ways gives the same result, whether it is done by one person or several. [p. 22]

As already noted above, the notion of coherence plays an important part in the later discussions. For Connes, "coherence guarantees that if one works correctly, one will always detect mistakes". For Changeux "coherence of perception" is a cerebral property, but at a lower level of abstraction than that of mathematical objects. "The fact that this coherence hasn't yet been explained doesn't prove that it's unexplainable — still less, as you claim, that it's independent of our system of reasoning". Connes describes how mathematicians recognise the internal coherence and generative character of certain concepts. "Investigating these, one truly has the impression of exploring a world step by step — and of connecting up the steps so well, so coherently, that one knows it has been entirely explored. How could one not feel that such a world has an independent existence?"

At this point Changeux sharply intervenes to question the use of the world "feel". He asks whether Connes considers mathematics to be more a matter of feeling than reflection. "I fear that the "feeling" you have of "discovering" this wholly platonic "reality" amounts to nothing more than a purely introspective — therefore subjective — analysis of the problem". Connes re-iterates his view that reality is defined by the coherence and invariance of perceptions.

The discussion moves forward a little (but soon gets bogged down in verbal niceties) when Connes distinguishes mathematical objects and the thought tools developed to investigate them. He grants that these tools have a cerebral basis. But "the progressive elaboration of concepts and methods of investigation doesn't alter the reality.. in the slightest." Changeux demurs: it feels like a stand-off. There seems at this stage to be a lack of awareness or of interest in the other's views. The comments tend to be on the argument itself rather than on what is being argued about.

They battle on and cover some further issues such as the distinctions between formalists and constructivist philosophies of mathematics, the "unreasonable effectiveness" of mathematics in describing the world, and the way theoretical physicists now have to look to mathematics for further insights into natural phenomena. Connes quotes the change in Einstein's views of mathematics. In 1921 he was still emphasising that his theories were based on the need to fit observed facts. By 1933 he was convinced that the key concepts for the understanding of natural phenomena would be found in the traditionally understood way — "constructing theoretical models which are then elaborated on the basis of experimental data". Connes is hopeful about the generative role mathematics — particular topology — might eventually play in biology as well. Changeux is curious about the notion of indeterminacy in quantum mechanics, but finds it difficult to accept the notion that there may be physical phenomena that are irreproducible.

> [Ch] The thought of accepting ignorance as a law of nature troubles me
>
> [Co] What our discussion makes clear is that you still don't understand the problem of indeterminacy. [p 70]

At which point the reader might well give up — which is why I chose to start my review with the middle part. But the fact is that the stage is now well set for the more collaborative later discussion.

The third area

The *triality* of the title is echoed in various ways throughout the book. Changeux and Connes both describe *trivalent* levels of thought, roughly follow Kant's *trinary* epistemological distinctions.[4] In their preface they refer to the reader of the book as a third party in their conversation, so I am emboldened to start a *trialogue*.

The problem is that the authors only offer me a dichotomy — "to agree or disagree with either of us, or both" — and I have already indicated that I would myself prefer some excluded third. Inevitably while you follow such a debate, you take sides — though these may alternate. My own initial preconceptions were with Changeux, and I also liked his readiness to elicit, and be interested in, the other's point of view. But I gradually warmed to Connes' confidence in his own subjective experience and I became intrigued by his account of it. Each of them is clearly very committed to his own view — and this commitment seemed naturally linked with their particular fields of study. But I did not feel I had to choose between them. I was glad that they disagreed, because in this case I found it stimulated my own thought. I was left with two useful issues to work on that I would like to mention briefly here.

The first stays with dichotomy. Philosophical discussions in general, and those on the ontology of mathematics in particular, often present stark either/or (but not both or neither) choices. I commend a Japanese "pillow education" in which you mediate on a statement by patting the four sides of a pillow in turn while entertaining the corresponding four thoughts: that the statement is wrong, right, both wrong and right, neither wrong nor right.[5] Meanwhile, I am often amazed by the emotional conviction with which proponents assert one particular view and anathematise an opposite one. I am tempted to avoid asking what is the truth in such matters, but rather ask why is this particular person expressing this point of view so vehemently? This is, of course, dangerous ground. It is difficult to tread it in general terms and not always appropriate, let alone possible, to focus on the particular. So I merely raise the question here.

My other issue comes back to *triplicity*, and in particular the notion of a mediating third. What is it about *three*?

> Heaven's dearest number, whose enclosed centre
> doth equally from both extremes extend,
> the first that hath a beginning, midst, and end.[6]

Well, you can take your pick from various accounts. My choice tends to be psychoanalytic and I am prepared to see echoes from early childhood in any triangle. In this case, not so much the oedipal one, but the one whose third vertex is the "potential space" between inner and outer worlds, described by the psychoanalyst, Donald Winnicott.

> From the beginning the baby has maximally intense experiences in the potential space between the subjective object and the object objectively perceived, between me-extensions and the not-me. This potential space is the interplay between there being nothing but me and there being objects and phenomena outside omnipotent control.[7]

Winnicott has drawn attention to a baby's use of "transitional objects" — such as comforters, teddy bears, and the like — which may be said to mediate between Mummy and Not-Mummy. Such objects are symbols that unite what are to become two separate things. The "third area" is where we experience our first use of a symbol and our first experience of play. And all this, according to Winnicott, starts at the breast: "the baby creates the object, but the object was there waiting to be created."[8] It is such mediation between fantasy and reality that seems to be invoked in the experience of creativity in any field — and in the emotional charge which, as noted above, was so important for Connes. Another mathematician, Philip Maher, has emphasised the implications of Winnicott's account

> the motion of transitional object — like that of potential space — carries over into adult life. If we accept the view that one's mathematical reality is an instantiation of one's potential space that occurs when one is doing mathematics then the object in this psychological space — the mathematical objects one plays with (tellingly, "play" is a verb mathematicians often use to describe their activity) — are — or, more accurately, *function as* — transitional objects. From this perspective there is little psychological distinction between, say, a teddy bear and a

a self-adjoint operator . . both *function as* transitional objects at the appropriate stage of one's psychological development [9]

I find that these notions cut across traditional discussions of the nature of mathematics to offer a much more helpful approach to the problem that concerns me more than the epistemological issue — namely, that of trying to answer Poincaré's question about why people fail at mathematics. This is a complex question and I do not imagine there are any simple answers. But I conjecture that some may be found in what happened to people in their initial experience of the mediating third area.

A final parade

It seemed to me that in their conversations Changeux and Connes began with fixed pre-conceived positions, but that they began to *play* with each other's ideas and, though reminding themselves regularly of their differences, they did find some common ground. In one of the later chapters, Connes invokes the notion of an archaic mathematical reality. It turns out this is tied up with ideas about the creation of the universe, about archaic time. Changeux wants to describe the evolution of order in the universe in Darwinian terms of selection and so on. Connes points out that the idea of an emergent order depends on the notion of time. "The four-dimensional universe of space-time is completely given: its entire evolution over time is fixed Selection occurs among universes instead."

In the beginning there was no time. What sort of reality can there be in the absence of time? For Connes, this can only be understood in terms of mathematical constructions In effect, external physical reality is for him part of archaic mathematical reality.

> [Ch] If you insist on this we may find it harder to reach some sort of final agreement
>
> [Co] Well, then, let's say that these two sorts of reality are on the same level —
>
> [Ch] With the human brain evolving afterwards to constitute a new internal physical reality?
>
> [Co] Exactly . [p 206]

And they can also finally agree that it is mathematical reality (wherever it is located) that defines the regularities, the intrinsic order, that we find in the external world.

I end with a final extract, a story recounted by Connes to emphasise his belief that, though physicists may use mathematics as a language, what they are investigating is something different. The intuitive development of tools like the Feynman integral which cannot be described with mathematical rigour provides yet another indication of the distinction between a conceptual framework that everyone can agree is a human construction, and which therefore can take different forms, and some sort of independent reality.

> [Co] A physicist goes off to a conference. After a week his suit's gotten soiled and rumpled, so he goes out to look for a dry cleaner Walking down the main street of town, he comes upon a store with a lot of signs out front. One of them says "Dry Cleaning". So he goes in with his dirty suit and asks when he can come back to pick it up.
> The mathematician who owns the shop replies, "I'm terribly sorry, but we don't do dry cleaning."
> "What?", exclaims the puzzled physicist, "the sign outside says "Dry Cleaning"."
> "We don't clean anything here," replies the mathematician, "we only sell signs." [p. 7]

Where dry cleaning actually occurs remains an open question.

Notes

1 "What is matter? Never mind What is mind? No matter." [T. H Keys, in *Punch,* July 14th, 1855] Cf. Byron, *Don Juan,* XI.1: "When Bishop Berkeley said, "there was no matter" / And proved it — 'twas no matter what he said"

2 J P Changeux and A Connes, *Conversations on mind matter and mathematics,* Princeton University Press, 1995, 260 pp; translated by M B DeBevoise from the original publication, *Matière à pensée* Editions Odile Jacob, 1989

3 In this context, Understanding, (*Verstand*) is associated with concepts ultimately derived from experience gained through the senses, and so only provides knowledge of the "appearance" of things: it is phenomenal, whereas Reason (*Vernuft*), being concerned with things-in-themselves, is noumenal — and so quite different from the reason of eighteenth century rationalism

4 This trinity had an influence — via Coleridge, and possibly Carlyle — on Hamilton's search for triads and triples. See T L Hankin, Triplets and triads, *Isis,* vol 68 (1977) pp 175-94

5 Quoted in A Wilden, *System and structure,* Tavistock publications, 1980 p 307

6 From a sixteenth century French poem quoted in A Schimmel, *The mystery of number* Oxford University Press, 1994, p 58

7 D W Winnicott, *Playing and reality.* Tavistock, 1971, p. 100 Winnicott's "third area" roughly corresponds to the Symbolic order described by another analyst, Jacques Lacan: see A Brown *et al.*, Mathematics on Lacan's couch, *For the Learning of Mathematics,* 13 1 (1993) 11-14.

8 David Wheeler has brought to my attention the picture of "objects springing into being in response to our probing" that Michael Dummett once proposed in order to break what he called the false dichotomy between platonist and constructivist accounts of mathematics. See "Wittgenstein's philosophy of mathematics", *Philosophical Review,* LXVIII (1959) pp 324-48; reprinted in P. Benacerraf & H Putnam, *Philosophy of mathematics: Selected readings.* Blackwell, 1964.

9 P. Maher, Potential space and mathematical reality. In P Ernest (ed), *Constructing mathematical knowledge.* Falmer Press, 1994, p. 136-7