TOWARDS A SEMIOTICS OF MATHEMATICAL TEXT (PART 1*)

PAUL ERNEST

In this article and the two following parts, I consider the content and function of mathematical text from a semiotic perspective. My enquiry takes me beyond the written mathematical text for I also consider the spoken text and texts presented multi-modally. In each case, a text is mathematical because it concerns the relationships among a set of mathematical signs that together make up such things as assertions, examples or questions. I explore both the reading/listening and the writing/speaking dimensions of mathematical text in this broader sense. In addition to making the enquiry more extensive, this necessitates the inclusion of a further vital dimension of written mathematical text in use, namely, the social context. For texts do not exist in the abstract, but are always and only present via their utterances and instantiations. Thus, to open up the mathematical text from a semiotic perspective is to explore its social uses and functions, as well as its inner meanings and textures.

Mathematical text is unlike fiction, for it is not merely a doorway to a world of the imagination. It is not just a told tale rendered into written language. This insight, whilst readily available to schoolchildren, is one that philosophers, mathematicians and educational researchers have to struggle to attain. Why is this? How do mathematical texts differ from fiction? This brings me back to the main questions of my enquiry. What do mathematics texts 'say'? What light do the tools of semiotics shed on the content and functions of mathematical text?

In this first of three linked articles, I aim to clear the ground for a tentative semiotic theory of mathematical text applicable to both research mathematics and mathematics education. I do this by addressing two myths. The first concerns the assumption that mathematical texts refer to some external reality. The second is the myth that both research mathematics and learning of mathematics are primarily psychological activities, *i.e.*, 'in the head'. Instead, I assert that the reading and writing of mathematical texts are the central activities of doing and learning mathematics.

These are not simple questions, and they are rendered all the more obscure because mathematical texts are not viewed or read neutrally. Irrespective of the reading subject who is needed to make sense of the text, mathematics is thickly overlaid with ideological presuppositions that prevent or obscure a neutral or free reading. So my first enquiry is into some of these ideological presuppositions that distort a reading of mathematical texts, that is, into what mathematics texts do not say.

What mathematics texts do not say

Rorty (1979) has described the ideology gripping traditional philosophy that sees it, the text, and the human mind as "mirroring nature". In other words, he critiques the traditional assumption that there is a given, fixed, objective reality and that mind, knowledge and text capture and describe it, with greater or lesser exactitude. This traditional philosophy reached its apogee in Wittgenstein's (1922) *Tractatus* with its picture theory of meaning. Wittgenstein's doctrine asserts that every true sentence depicts, in some literal sense, the material arrangements of reality. Language, when used correctly, floats above material reality as a parallel universe and provides an accurate map or picture of it. However, a claim this strong is hard to sustain, and even the logical positivists withdrew from this overly literal position about the relationship between language and reality. They adopted instead the verification principle, which states that the meaning of a sentence is the means of its verification (Ayer, 1946). [1] Without this revised view of meaning, the predictive power and generality of scientific theories is compromised. Ironically, Wittgenstein (1953), in his later philosophy, also rejected this position himself having pushed the picture or mirror view to its limits in the *Tractatus*. [2]

This ideology, applied to mathematics, remains potent in Western culture. Mathematics is seen to describe an objective and timeless superhuman realm of pure ideas, the necessity of which is reflected in the ineluctable patterns and structures observed in our physical environment. The doctrine that mathematics describes a timeless and unchanging realm of pure ideas goes back to Plato, if not earlier, and is usually referred to as Platonism. Many of the greatest philosophers and mathematicians have subscribed to this doctrine in the subsequent two-plus millennia since Plato's time. In the modern era, the view has been endorsed by many thinkers, including Frege (1884, 1892), Gödel (1964), and in some writings by Russell (1912) and Quine (1953). According to Platonism, a correct mathematical text describes the state of affairs that holds in the platonic realm of ideal mathematical objects. Mathematical texts are nothing but descriptions or mirrors of what holds in this inaccessible realm. [3]

Although I shall reject it, quite a lot is gained by this view. First of all, mathematicians and philosophers have a strong belief in the absolute certainty of mathematical truth and in the objective existence of mathematical objects, and a belief in Platonism validates this. It posits a quasi-mystical realm

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into which only the select few – initiates into the arcane practices of mathematics – are permitted to gaze. Secondly, it puts epistemology beyond the reach of humanity's sticky fingers and earth(l)y bodies, locating mathematical knowledge in a safe and inaccessible zone. Because of this strategy of removal, any claims that mathematics is socially constructed are disallowed *tout court* without having their specific merits or weaknesses considered.

But it seems to me that the strategy of saying the text '2 + 2 = 4' is true because 2 + 2 = 4, *i.e.*, there are ideal abstract numbers 2 and 4 and when you combine 2 with 2 by addition the result is equal to 4, is not very enlightening. The real question is what does addition mean, and why, when you add 2 to 2 does the answer happen to be equal to another number and that number is 4 and not something else? I can, of course, answer this particular question, but that is not the point. The point is that the translation from the realm of signs and texts, where '2 + 2 = 4' is located, to the realm of meaning (*i.e.*, the interpretation of the sign in the platonic universe of number where 2 + 2 = 4 holds), is not illuminating, *per se*, in providing the meaning of the expression.

Tarski (1935), in his well-known paper on the semantic interpretation of truth, argues that the sentence 'Snow is white' is true, if, and only if, snow is white. This argument is trivial and uninformative. However, Tarski's project is fundamentally technical. He is concerned with the definability of the concept of truth in formalized languages and, as a consequence of his formal explication of 'truth', provided the foundations of model theory and arrived at one of the important limitative results of modern mathematics – namely, that truth is indefinable in formal languages, on pain of inconsistency. For technical purposes he clarified the notion of syntactic expressions mirroring states of affairs in the semantic realm, with valuable mathematical and foundational results. But this does not lend support to the general ideological 'mirroring' presupposition 'as below, so above' (to invert the fundamental principle of astrology) that I am critiquing.

Philosophers have long been concerned about mathematical ontology: the study of the existence and nature of mathematical objects and abstract entities in general. Balaguer (2004) provides a state of the art survey of *Platonism in metaphysics* and concludes that mathematical objects can be accounted for in three ways: physical (in the world), mental (in the mind), and abstract (in Plato's realm of ideal objects or some equivalent). His account is of course much more subtle and nuanced than this one sentence summary. It is an impressive display of intellectual prowess, although I get the feeling that professional philosophers like Balaguer turn the virtue of subtle reasoning, through excessive application, into a vice. These three possibilities seem limited in that mathematical objects cannot obviously be regarded as purely physical or mental [4], which leaves open only the possibility that mathematical objects are abstract objects, confirming Platonism, or some variant of it. Balaguer himself acknowledges some concerns about his categorization, perhaps as a gambit to stave off the possible criticism that mathematical objects could also be seen as social constructions.

> There are a couple of worries one might have about the exhaustiveness of the physical-mental-abstract taxonomy. First, one might think there is another category that this taxonomy overlooks, in particular, a category of social objects, or perhaps social constructions. It seems, though, that social objects would ultimately have to reduce to either physical, mental, or abstract objects. (Balaguer, 2004, note 3)

Here he makes the unsubstantiated assumption that any other category, such as that of socially constructed objects, can be reduced to one of his three categories. This introduces another ideological presupposition, namely that of reductionism. By reductionism, I mean the positivist doctrine, much beloved of the Logical Empiricists, that that there is an hierarchy of objects, theories or disciplines, and it is possible to translate and replace objects, theories or disciplines further up the hierarchy into the objects, terms, concepts and theories lower down the hierarchy without loss of generality or scientific significance. Thus, according to this scheme, sociology can be translated (reduced) to psychology, psychology to biology, biology to chemistry, chemistry to physics, and finally physics to mathematics. (A further step, the reduction of mathematics to logic, the goal of logicism in the philosophy of mathematics, was shown to be a failure (see, *e.g.*, Ernest, 1991, 1998; Hersh, 1997; Kline,1980). [5]

One of the most notable critics of reductionism is Feyerabend (1975), who argues that there is semantic instability within and across theories or disciplines. He claims (Feyerabend, 1962) that meanings of the same terms and concepts in different theories are not only different but are also incommensurable, and that "theoretical reduction" is not reduction but the replacement of one theory *and its ontology* by another. The same year, Kuhn (1962) also published his seminal work on the structure of science revolutions in which he argues that the concepts of competing theories are incommensurable. Although the claims of strong incommensurability did not stand up to criticism, there is a powerful argument from holism that refutes the reductionist claims.

Consider, for example, the reduction of psychology to biology. We know that human minds are ultimately based in a material organ. Furthermore, specific areas and functions of the organ (the brain) can be correlated with particular mental activities. A range of chemicals have profound and sometimes predictable impacts on thinking. Nevertheless, there is no forseeable possibility that the full complexity of human thinking and behaviour as a whole could be reduced to a biological model. The kinds of mental elements that can be found to correspond to biological processes are so very simple and disconnected that there is no prospect that current biology could explain human thoughts, feelings and behaviour as a whole. The earlier attempt by behaviorism to ignore the mind and to try to explain behaviour scientifically is a well-known failure.

I have suggested that both epistemological and ontological reductionisms are fatally flawed. We cannot simply define away complex objects or bodies of knowledge in terms of simpler ones. In particular, I strongly believe that social constructions cannot be reduced either to physical, mental, or abstract objects, for they combine elements of all three. This tripartite ontology mirrors Popper's (1979) definition of three distinct worlds, each with its own type of knowledge.

> We can call the physical world 'world 1', the world of our conscious experiences 'world 2', and the world of the logical *contents* of books, libraries, computer memories, and suchlike 'world 3'. (Popper, 1979, p. 74)

Ironically, this also mirrors the tripartite division, much beloved of 'New Agers' into Body, Mind and Spirit. I draw this parallel out of more than playfulness. It seems to me that the Platonic realm of abstract entities, World 3, the world of the Spirit, and even Heaven and the Kingdom of God, all require irrational belief or faith. Positing them does not simplify the task of understanding the mathematical text. Rather it defers a key element of that understanding, removing it to what I regard as an inaccessible realm. What is needed instead is what Restivo (1993) has aptly termed *the Promethean task of bringing mathematics to earth*.

One approach, which has some currency if not widespread acceptance, that I have applied elsewhere (Ernest, 1998), is to redefine 'objective' mathematical knowledge (using the term 'objectivity' without subscribing to absolutism in the epistemological realm or idealism in ontology) as social and cultural knowledge that is publicly shared, both within the mathematical community and more widely as well. In mathematics, this includes all that Popper counts as objective knowledge, including mathematical theories, axioms, problems, conjectures and proofs, both formal and informal. However, I also want to include the shared but possibly implicit conventions and rules of language usage and a range of tacit understandings that are acquired through participation in practices. These are shared, in that they are deployed and learned in public for persons to witness. But they may remain implicit if only their instances and uses are made public, and any underlying general rules or principles are rarely or never uttered. According to such an account, explicit objective knowledge is made up of texts, which have been socially constructed, negotiated and accepted by social groups and institutions. Naturally, such texts have meanings and uses both for individuals and for social groups.

What I have briefly indicated is a social theory of objectivity that resembles, at least in part, proposals by, for example, Bloor (1984), Harding (1986) and Fuller (1988). In some variant or another, such a view also underpins much work in the sociology of knowledge and in post-structuralist and post-modernist epistemology. [6] By subscribing to an approach that demystifies 'objectivity', I am suggesting doing away with the ontological category of abstract objects, which often amounts to Plato's world of pure forms. That is, I have applied a principle of ontological reduction. I am in good company here. Quine (1969) has argued for "ontological parsimony", the principle that we should apply Occam's razor and not allow entities and their types to multiply beyond what is necessary. Ryle (1949) has also argued convincingly that the ontological separation of mind and body is also a mistake. Although the mental cannot be simply reduced to the physical, this is an epistemological non-reducibility, not an ontological one. So, if we reject separate ontological categories for the mental (mind) and the physical (body), we end up with a unified realm of being.

In my view, the universe is made up not of three types but one type of 'stuff', namely the material basis of physical reality. Within this unified world there are, among the myriads of things and beings, humans with minds and groups of humans with cultures. Human minds, the seat of the mental, are not a different kind of 'stuff', but are a complex set of functions of self-organising, self-aware, feeling moral beings. Mathematical knowledge, like other semiotic and textual matters, is made up of social objects. These are simultaneously materially represented, given meaning by individuals and created and validated socially.

This discussion may seem like a diversion on the way to opening up the mathematical text, but its function is to show that there is a deeply entrenched ideology, all the way up to the highest intellectual levels, that regards mathematical text as a vehicle that describes superhuman and objective mathematical reality. According to this view, when the text is correct, and it thus counts as expressing mathematical knowledge, it truly describes this reality. So from this perspective, mathematical knowledge texts are mirrors that reflect a true state of affairs in a timeless, objective, superhuman Platonic realm. Furthermore, this widespread image of mathematics has some immediate consequences for the teaching and learning of mathematics. Although Platonism may appeal to some, a number of researchers have reported how many other learners, especially girls and women, are repelled by this image (Buerk, 1982; Ernest, 1995).

Rorty (1979) rejects the views of knowledge or mind as mirrors of nature. He argues that this view is a major stumbling block in philosophy and, in so doing, identifies himself as post-modern, "in the rather narrow sense defined by Lyotard as 'distrust of metanarratives'" (Rorty, 1991, p. 1). He goes on to argue that mathematical knowledge, for example, the Pythagorean Theorem, is accepted as certain because humans are persuaded it is true, rather than because it mirrors states of affairs in 'mathematical reality'.

> If, however, we think of "rational certainty" as a matter of victory in argument rather than of relation to an object known, we shall look toward our interlocutors rather than to our faculties for the explanation of the phenomenon. If we think of our certainty about the Pythagorean Theorem as our confidence, based on experience with arguments on such matters, that nobody will find an objection to the premises from which we infer it, then we shall not seek to explain it by the relation of reason to triangularity. Our certainty will be a matter of conversation between persons, rather than an interaction with nonhuman reality. (Rorty, 1979, pp. 156-157)

Thus Rorty shares the view expressed above that it is social agreement, albeit in a complex and non-whimsical way, that provides the foundation for mathematical knowledge. The truth or otherwise of a mathematical text lies in its social role and acceptance, not its relation to some mysterious realm.

What is mathematical text?

In keeping with modern semiotics, I want to understand a text as a simple or compound sign that can be represented as a selection or combination of spoken words, gestures, objects, inscriptions using paper, chalkboards or computer displays, as well as recorded or moving images. Mathematical texts can vary from, at one extreme, in research mathematics, printed documents that utilize a restricted and formalized symbolic code, to, at the other extreme, multi-media and multi-modal texts, such as are used in kindergarten arithmetic. These texts can include a selection of verbal sounds and spoken words, repetitive bodily movements, arrays of sweets, pebbles, counters, and other objects, including specially designed structural apparatus, sets of marks, icons, pictures, written language numerals and other writing and symbolic numerals.

The received view is that progression in the teaching and learning of mathematics involves a shift in texts from the informal multi-modal to the restrictive, rigorous symbol-rich written text. It is true that, for some, access to the heavily abstracted and coded texts of mathematics grows through the years of education from kindergarten through primary school, secondary school, high school, college, culminating in graduate studies and research mathematics. But it is a myth that informal and multi-modal texts disappear in higher-level mathematics. What happens is that they disappear from the *public* face of mathematics, whether these be in the form of answers and permitted displays of 'workings', or calculations in work handed in to the school mathematics teacher, or the standard accepted answer styles for examinations, or written mathematics papers for publication. As Hersh (1988) has pointed out, mathematics (like the restaurant or theatre) has a front and a back. [7] What is displayed in the front for public viewing is tidied up according to strict norms of acceptability, whereas the back, where the preparatory work is done, is messy and chaotic.

The difference between displayed mathematical texts, at all levels, and private 'workings' is the application of rhetorical norms in mathematics. These norms concern how mathematical texts must be written, styled, structured and presented in order to serve a social function, namely to persuade the intended audience that they represent the knowledge of the writer. Rhetorical norms are social conventions that serve a gatekeeper function. They work as a filter imposed by persons or institutions that have power over the acceptance of texts as mathematical knowledge representations. Rhetorical norms and standards are applied locally, and they usually include idiosyncratic local elements, such as how a particular teacher or an examinations board likes answers laid out, and how a particular journal requires references to other works to be incorporated. Thus, one inescapable feature of the mathematical text is its style, reflecting its purpose and, most notably, its rhetorical function.

Rhetoric is the science or study of persuasion, and its universal presence in mathematical text serves to underscore the fact that mathematical signs or texts always have a human or social context. I interpret signs and texts as utterances in human conversation, that is, within language games embedded in forms of life (Wittgenstein, 1953) or within discursive practices (Foucault, 1972). Texts exist only through their material utterances or representations and, hence, via their specific social locations. [8] The social context of the utterance of a text produces further meanings, positionings and roles for the persons involved. Thus, in any given context, a mathematical text or sign utterance, like any utterance, is indissolubly associated with a penumbra of contextual meanings including its purpose, its intended response and the positioning and power of its speaker/utterer and listener/reader. [10] Such meanings are both created and elicited through the social context and are also a function of the meanings and positions made available through the text itself. Different types of meanings and intentions are intended, but perhaps the most central and critical function (and hence meaning) of mathematical texts in the mathematics classroom is to present mathematics learning tasks to students. A given (*i.e.,* assigned) mathematical learning task [9]:

1. is an activity that is externally imposed or directed by a person or persons in power representing and on behalf of a social institution.
2. is subject to the judgement of the persons in power as to when and whether it is successfully completed (or good enough for now).
3. is a purposeful and directional activity that requires human actions and work in the striving to achieve its goal.
4. requires learner acceptance of the imposed goal, explicitly or tacitly, in order for the learner to consciously work towards achieving it.
5. requires and consists of working with texts: both reading and writing texts in attempting to achieve the task goal. [11]

A more general concept of mathematical task includes self-imposed tasks that are not externally imposed and not driven by direct power relationships. However, even in research mathematicians' work, although tasks may not be individually subject to power relations, particular self-selected and self-imposed tasks may be undertaken within a culture of performativity that requires measurable outputs (Burton, 2004). So, power relations are at play at a level above that of individual tasks. Even where there is no external pressure to perform, the accomplishment of a self-imposed task requires the internationalization of the concept of task, including the roles of assessor and critic, based on the experience of social power-relations, to provide the basis for an individual's own judgement as to when a task is successfully completed.

Mathematical learning tasks are important because they make up the bulk of school activity in the teaching and learning of mathematics. During most of their mathematics learning careers, which in Britain continues from 5 to 16 years and beyond, students mostly work on textually presented tasks. I estimate that an average British child works on 10,000 to 200,000 tasks during the course of their statutory mathematics education. This estimate is based on the not unrealistic assumptions that children each attempt 5 to 50 tasks per day, and that they have a mathematics class every day of their school career.

Some school mathematics tasks concern the rule-based transformation of text. Such tasks consist of a textual starting point, the task statement. These texts can be presented multi-modally, with the inscribed starting point expressed in written language or symbolic form, possibly with accompanying iconic representations or figures, and often accompanied by spoken instructions from the teacher, typically a meta-text. Learners carry out such set tasks by writing a sequence of texts, including figures, literal and symbolic inscriptions, ultimately arriving, if successful, at a terminal text, the required 'answer'. Sometimes this sequence involves a sequence of distinct inscriptions, for example, the addition of two fraction numerals with distinct denominators, or the solution of an equation in linear algebra. Sometimes it involves the elaboration or super-inscription of a single piece of text, such as the carrying out of 3-digit column addition or the construction of a geometric figure. It can also combine both types of inscriptions. In each of these cases, there is a common structure. The learner is set a task, central to which is an initial text, the specification or starting point of the task. The learner is then required to apply a series of transformations to this text and its derived products, thus generating a finite sequence of texts terminating, when successful, in a final text, the 'answer'. This answer text represents the goal state of the task, which the transformation of signs is intended to attain.

Formally, a successfully completed mathematical task is a sequential transformation of, say, n texts or signs ('S_i') written or otherwise inscribed by the learner, with each text implicitly derived by $n - 1$ transformations ('$\Rightarrow_i$'). [12] This can be shown as the sequence: $(S_0 \Rightarrow_0)\ S_1 \Rightarrow_1 S_2 \Rightarrow_2 S_3 \Rightarrow_3 \ldots \Rightarrow_{n-1} S_n$. S_1 is a representation of the task as initially inscribed or recorded by the learner. This may be the text presented in the original task specification. However, the initial given text presenting the task (S_0) may have been curtailed, or may be represented in some other mode than that given, such as a figure, when first inscribed by the learner. In this case an additional initial transformation ($\Rightarrow_0$) is applied to derive the first element (S_1) in the written sequence. S_n is a representation of the final text, intended to satisfy the goal requirements as interpreted by the learner. The rhetorical requirements and other rules at play within the social context determine which sign representations (S_k) and which steps ($S_k \Rightarrow_k S_{k+1}$, $k < n$) are acceptable. Indeed, the rhetorical features of the transformed texts, together with the other rules at play and the final goal representation (S_n), are the major focus for negotiation (or correction) between learner and teacher, both during production and after the completion of the transformational sequence. This focus will be determined according to whether, in the given classroom context, the learner is required only to display the terminal text (the answer) or a sequence of transformed texts representing its derivation.

The final transformational sequence of texts displayed by the learner and the actual transformations derived during the work on the task may not be identical. The former may be a 'tidied up' version of the latter, constructed to meet the rhetorical demands of the context, rather than the working sequence actually used to derive the answer (see, *e.g.*, Ernest, 1993). This distinction is most clearly apparent during the construction of a proof by a research mathematician, as in the distinction between the 'front' and the 'back' of mathematical activity (Hersh 1988). Here, the proof as first sketched and the final version for publication, both transformational sequences of texts, will almost invariably be different. Lakatos (1976) and others have criticized the pedagogical falsification perpetrated by the standard practice of presenting advanced learners with the sanitized outcomes of mathematical enquiry. Typically, advanced mathematics textbooks conceal the processes of knowledge construction by inverting or radically modifying the sequence of transformations used in mathematical invention, for presentational purposes. The outcome may be elegant texts, but they also generate learning obstacles.

Typically, there are strict limitations on the modes of representation employed in research mathematicians' published proofs, although these will vary according to the rhetorical standards of the particular journal and mathematical sub-specialism. Nevertheless, there may well be written linguistic text, mathematical symbolism, arrays of signs, and diagrams, for example, combined in the final text. An even greater number of different modes of representation may be employed singly or together in a school mathematics text, including combinations and selections of symbols, written language, labelled diagrams, tables, sketches, models, and so on (even including arrayed objects and gestures where the text is spoken). In school mathematics, it is common for the transformational sequence of texts produced by students during problem-solution activities to use more modes of representation than the starting text of the task specification, or the final text, 'the answer'. Furthermore, the transformational text sequence produced during task directed activities may be neither monotonic nor single branched. Solution attempts may result in multi-branched sequences with multiple dead ends and only one branch terminating in the 'final' answer text, and this only appears when the activity is completed in a way deemed successful (or good enough for now) by the task setter, *i.e.*, the teacher, in school mathematics.

My claim is that virtually all mathematical activity, throughout schooling, but also in graduate and research mathematics, can be understood in terms of the production of sequences of texts through the application of textual transformations. What distinguishes routine from creative (less routine) school mathematical activity in this description are the types of texts and transformations involved. Routine mathematical activity typically involves relatively simple initial texts and the deployment of restricted (algorithmic) transformation rules in the production of sequences of texts. Less routine or creative mathematical activities, such as problem solving, applications, or investigational work, typically involve more complex task formulations and require some novelty and insight in selecting which transformations to apply and which elements to apply them to, in producing the sequence (Arcavi, 2005).

In this first of three articles, I have addressed two issues, both concerned with the meaning of mathematical text. The first addresses the traditional Platonistic presupposition that a mathematical text refers to some ideal reality beyond the text itself and its surrounding social contexts of use. The second issue is that of the relationship between a learner and the mathematical texts they read/write in the classroom. In each case, I have argued that the texts themselves, viewed from a semiotic perspective that incorporates their functions and uses, should be the primary object of attention, instead of putative (and not directly accessible) meanings in some objective or subjective realm. This semiotic perspective provides an alternative both to Platonism, and to psychological perspectives that focus exclusively on mental structures and functions, or on performances or behaviors. In subsequent articles, I shall present a semiotic theory of sign systems and their personal appropriation by persons within their social contexts of learning and signing which offers ways of conceptualizing mathematical activity in both the research and learning contexts.

Notes

[1] In both the cases of Wittgenstein and the Logical Positivists, mathematics is treated differently from empirical or scientific claims or texts. In the *Tractatus*, Wittgenstein (1922) argues that mathematics is a by-product of the grammar of world description. The logical positivists argue that mathematics is analytic *a priori* knowledge, derived purely by logic and telling us nothing new. Both of these accounts trivialize mathematics and focus on the tasks of accounting for scientific knowledge of the empirical world and the rejection of metaphysics. From the perspective of mathematical philosophy, these accounts are fundamentally unenlightening.

[2] Wittgenstein (1953) went on to develop what might be termed one of the first postmodernist epistemologies, with his doctrines that "meaning is use", and that all text and knowledge emerges from language games embedded in human forms of (social) life, with mathematics made up of a multiplicity of language games.

[3] Otte (1997, p. 50) contrasts Cartesianism, the view that the (mathematical) world is already known in principle, and that in time scientific deduction will fill all the *lacunae*, with Pascal's "subtle intelligence" according to which "the world can be appropriated cognitively only by the constructive incorporation of ever new types of objects into thought".

[4] I put to one side Intuitionism in the philosophy of mathematics, which argues that the concepts and results of mathematics are purely mental constructions. Intuitionism is problematic in that it rejects much of classical mathematics (the non-constructive and infinitary parts) and has never managed to explain how different minds all arrive at identical mathematical results, especially as it holds that mathematics is a pre-linguistic activity. Lakoff and Núñez (2000) also claim that mathematical concepts are mental in origin, but unlike Intuitionism their theory of embodied mind rejects the separation of mind and body.

[5] There has been strong critical and negative reaction among many mathematicians and philosophers to social philosophies of mathematics such as proposed in Hersh (1997) and Ernest (1998). Part of the common critique strategy is to misrepresent such philosophies as saying that mathematics can be reduced to sociology (*i.e.*, that mathematical matters are decided by 'mob rule'). If these critics subscribe to reductionism, which is not uncommon, then the threat is that by 'joining the ends' mathematics is knocked off its pedestal as the foundation of knowledge, and the whole chain of disciplines closes on itself in a vicious circle, knowledge eating itself like the worm Ouroboros. No wonder social constructivist philosophies of mathematics are viewed as a nightmarish threat!

[6] Some thinkers in postmodernity, *e.g.*, Noddings (1990) and Rorty (Ramberg, 2002) have styled themselves as post-epistemological, because they want to reject the absolutist and foundationalist presuppositions of traditional epistemology. However, just as I want to retain but redefine 'objectivity' in a non-absolutist and non-idealistic way, so too do I wish to retain the term 'epistemology' for the theory of knowledge without any similar presuppositions. I do not believe we need to reject important terms because we do not agree with the ways they have been used by others. Terms are all the while growing, changing in meaning, and in mid-renegotiation, as Lakatos (1976) demonstrates for mathematical concepts.

[7] Hersh draws his analogy from Goffman's (1971) work on how persons present themselves in everyday life.

[8] Especially in the age of mechanical reproduction we speak as if different utterances (tokens) represent the same text (type). In such cases, much can be shared between the different utterances, *e.g.*, when the students in a class each has a copy of a set school mathematics textbook, because they are located in a shared social context. Even in this example, there can be significant differences in reader reception of the text utterances. When the tokens are located in different social contexts it makes less sense to refer to them as the 'same' text, even if this is common parlance.

[9] What I am describing are standard, routine tasks that occur in many school mathematics textbooks and are also regularly observed being used in the classroom (*e.g.*, Goodchild, 2001). I am not endorsing this type of practice, nor denying that other, perhaps better, models of mathematical activity take place too, on occasion. Elsewhere, I have argued for the importance of problem-posing and investigational work (Ernest, 1984, 1991). But here I am presenting a model that describes a widespread pattern of practice.

[10] Clearly, these are attenuated if not lost when a text is taken beyond its intended audience or appears accidentally in some social context. Such appearances are different utterances with different social contexts, meanings and intentions. For example, a school mathematical text taken from the shelf of a bookshop does not have the specific associated directives and positionings for the customer as it would have for a reader in a particular classroom (although it might elicit some comparable meanings through memories of schooling). A Babylonian clay tablet would be unlikely to have any of the same meaning for a US army soldier finding it in Iraq as it did for the ancient scribes who created and used it.

[11] Gerofsky (1996) adds that tasks, especially "word problems", also bring with them a set of assumptions about what to attend to and what to ignore among the available meanings.

[12] Normally, learners of school mathematics are not expected to specify the transformations used. Rather, they are implicitly evidenced in the difference between the antecedent and the subsequent text in any adjacent (*i.e.*, transformed) pair of texts in the sequence. In some forms of proof, including some versions of Euclidean geometry not generally included in modern school curricula, a proof requires a double sequence. The first is a standard deductive proof and the second a parallel sequence providing justifications for each step, that is, specifications for each deductive rule application. Only in cases like this are the transformations specified explicitly.

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The world of mathematical [enquiry] [...] is very different from that to be found in the mathematical literature where the results and the techniques are reified and the participatory practices of the mathematicians [...] are expunged. If, however, we think of these mathematicians as learners, their enquiry processes provide an excellent model to use for less sophisticated learners.

(Leone Burton (2004) *Mathematicians as enquirers: Learning about learning mathematics*, Boston, MA, Kluwer Academic Publishers, p. 133.)