Which Operation? Certainly not Division!

DAVID FIELKER

DF: What do you get if you double 3?
Children: Six.
DF: And if you double it again?
Children: Nine.

No, it does not always happen. But it has happened many times with groups of 10-year-olds.

The trouble with doubling is that children do it in their heads, and are perhaps not aware of which operation they are using. Doubling is a fairly primitive idea with a long history, and features in problems like the grains of rice on the squares of a chessboard demanded by the emperor of China, and in algorithms like 'Russian peasant' multiplication However, it is not clear to all young children, as the following anecdote shows [1].

The book said, "John scored double four at darts." Peter, age seven, was puzzled.
His teacher explained what 'scored' meant.
Then she asked, "What's double four?"
Peter was still puzzled.
"What are two fours?"
"Eight," he said.
His teacher went away

It is interesting that the teacher chose to explain "scored" explicitly, but only explained "double" by implication And the question of which operation is used is not necessarily answered, because we do not know whether, to Peter, "two fours" meant 2×4 or $4 + 4$.

Now watch what happens when a calculator is used. I asked a class of 9-year-olds to enter 7 on their machines, and then double it. I had a brief word with those who obtained 77! Everyone else had 14, but they all arrived at it by *adding on* another 7.

I asked if there was any other way of turning 7 into 14, and several suggested "timesing by 2". We actually tried both methods to double various numbers, counting keystrokes to see which was quicker, but I am not sure whether this made them feel that doubling had anything to do with multiplication.

I asked them to enter 10, and then to halve it. They all *subtracted 5*!

I asked them to halve 11, and they decided it could not be done!

I gave up this idea, and instead asked them what they had to double to get 12. They knew it was 6.

I asked what they had to double to get 56 This took longer. Some began to try various numbers, doubling them to see if they got 56. One pair of boys doubled 25 to get 50, decided they needed another 3 to get the extra 6, and so doubled 28. Two girls (I suspect, unfortunately, prompted by their teacher) calculated $56 \div 2$.

I gave them other numbers to get by doubling, and various methods were used, like getting 324 by doubling the sum of 150, 10 and 2. When I slipped in 729 another pair were trying 369, 368, ... finding that 365 was too much and 364 was not enough, but not knowing what to do about it.

The two girls divided 729 by 2 to get 364.5. They did not know what it meant, but they knew that if they doubled it they got 729!

We can see that children of 9 or 10 have a decided feeling that doubling is more to do with addition than with multiplication. Even if we are not sure about this when children are working in their heads, then doing it on the calculator forces them into making an explicit choice about which operation key to press

This is not, of course, to say that there is anything wrong with seeing doubling as addition, and it is certainly an advantage to appreciate the relation between addition and multiplication. We can recognise this ability to some extent even in those all-too-numerous children who calculate 6×7 by marking 7 dots 6 times and then proceeding to count them, though one wonders if the relation is still there when each is reduced to counting.

The calculator can raise more interesting problems about the relation

DF: If you have a calculator, what's the best way of working out your four times table?
Helen: Four and press add
Grant: Press one times four equals four
Nicola: Press four times one. and then go to the two and then the three.
Helen: You press four then you press add then you press equals and then you press the equals till it gives you the four times table, like that.

Here the children were being invited to create a calculator algorithm for producing multiples of 4 We discussed the two methods, and their respective efficiency It was agreed that if you wanted, say, 7×4 it was quicker to multiply, but if you wanted all the multiples of 4 in order then it was quicker to add. It seems a fairly sensible situation in which to create algorithms that were appropriate, and it shows a clear understanding of addition and multiplication.

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It was not quite so sensible, perhaps, to say on two other occasions that the multiplication key on my calculator had broken, and to ask how I could calculate 72×13, especially when Grant retorted, "I'd get another calculator!" But the children each time immediately resorted to adding, either way round, and realised that adding 72 thirteen times was quicker. Then the problem of 72×99 implicitly carried a big hint to subtract 72 from 7200, and this suggested quicker ways of dealing generally with such questions by using multiples of 10 calculated mentally. Some of them transferred written methods to the calculator, dealing with tens and units separately, but there were several interesting ideas for particular problems.

One led us back briefly to a discussion of doubling, when I followed 37×6 with 37×12.

Alan: You know what 37×6 is, so you add the same answer again.
Grant: If you know 37×6 is 222, and 12 is twice as much as 6, then you know it must be more than 6×37.
Helen: I know what the answer is. 'Cos it's 37×6, and if you ... 6 times table, and 2 sixes go into 12, so it must be double the answer, it must be 444.
DF: O.K., and you double that in your head? How do you double that on the calculator?
Helen: Press the "equals".
Nicola: By timesing.
DF: Ah, but if we can't press the "times" button, how do we double?
Martin: You get 37×6, right? You get the answer, and you add it together two times.

It was perhaps strange that this time multiplying was a first choice, rather than addition.

In spite of errors (like Karen assuming that 59×99 was the same as 58×100) there seemed to be a general flexibility about operations and algorithms, varying the method to suit the particular question. This is the sort of strategy children (and adults) frequently use when calculating mentally, yet rarely do when they are expected to work things out on paper. Maybe the absence of paper in the sessions I conducted contributed to an appropriate blend of mental work and the pressing of keys, and a consequent freedom to vary methods, but I am aware there are other factors as well.

However, most of this flexibility is to do with the relation between addition and multiplication. The main difficulty about doubling occurred when it was transformed into the inverse relation, halving. And this difficulty stemmed not so much from the necessity to see doubling in terms of multiplying rather than adding, as from the inability to transform multiplication into division.

Although subtraction came more readily as a means of halving, generally children seem to find the inverse operations more difficult. Both they and shopkeepers prefer to treat subtraction as complementary addition. Yet I used to find it curious when visiting the U.S.A. some years ago that there were so many teachers' courses called "The Missing Addend", which was obviously something that children found extremely troublesome.

Division in a sense is traditionally done by multiplication, as we can clearly see by vocalising mental or written algorithms for it. One says "How many twos in eight?" or "Two into eight" rather than "eight divided by two", and computation is based on the multiplication tables rather than a memory of division bonds. Indeed, the reversed verbalisation of "two into eight" often leads to $2 \div 8$ on the calculator rather than $8 \div 2$.

What the calculator does, though, is to make the division explicit, and unreliant on multiplication. At least, it would do, if children thought about division problems in terms of division. But children, as the next few examples show, still prefer to use multiplication when algorithmically division would be easier.

The following extract has already been quoted in [2] as a classic example of how a teacher can blindly pursue his own idea, ignoring all the useful suggestions the children are making. But notice how the idea which seems to be inaccessible to them is division, and how they persist in replacing it by strategies mainly based on addition or multiplication.

John: A prime number is ... a number which only one and the number itself goes into.
DF: Aha. So, is 52 prime?
All: No.
DF: Why not?

(Pause. They are not sure.)

Jonathan: 2 goes into it.
Cheryl: There's another number as well.
Several: 4 ... 5 ... 52.
DF: If 2 goes into it, what else do you know goes into it?
Paula: One.
DF: Cheryl said something about another number as well. Does the calculator help?
Cheryl: Yes.
DF: How does it help?
Cheryl: You can use it to times all different numbers to see if you can get 52.
DF: If you *know* that 2 goes into 52, *how* do you know that 2 goes into 52?
Cheryl: 'Cos it ends in 2.
Others: Or 4. Or 6. Or 8. Or nought.

We discussed even numbers, and the idea that 2 divided into them.

DF: So, if 2 goes into 52, what else will go into it?
Edward: 26.
DF: How do you know?
Edward: Because if you add 26 and 26 you get 52.

A suggestion about using the calculator only resulted in some apparently fruitless button pushing.

DF: Well, if 2 goes into 52 exactly, how many times does it go in?
Steven: 26.
DF: How did you know that?
Steven: 'Cos two twenty fives are 50, and another one makes 52, and if you share out the other 2 you get 26.
DF: Yes. Is there any other way you could work out how many twos in 52?
Cheryl: Use a calculator.
DF: Yes. How?
Jonathan: You would times the thing you thought by 2.

DF:	Is that what you did?
Jonathan:	Yes.
DF:	What did you start off with?
Jonathan:	26

The session of which the above was part took place as a result of my asking the class what they had recently been doing. On another occasion a different class of 10-year-olds told me they had been calculating areas of rectangles. The "inverse" strategy for me seemed an obvious one, and I asked them to calculate the height, given the area and the width. With simple whole numbers they could do this easily in their heads. So I then gave them a rectangle with area 10 and width 3.

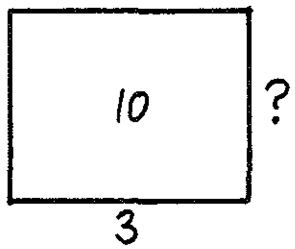

Figure 1

Most busied themselves with their calculators, multiplying different numbers by 3 to see if they could get 10. The problem has many features in common with that of finding what to square to get 10 (see [3] for descriptions of children's work on this problem), with one important difference: usually 10-year-olds do not know about square roots and do not recognise the sign on the appropriate key; but they do all know about division. Yet only one bright girl thought of using the calculator to divide 10 by 3

A similar activity was to multiply two whole numbers on my calculator, tell a class I had obtained 119, and pretend I had forgotten what the numbers were, except that neither of them was 1 The ideas these 10-year-olds used were interesting. Discussion amongst them established that the two numbers were odd, but not multiples of 5. One common strategy was then to try 3, 7, 9, 11, etc., but each of these numbers in turn was *multiplied* by other numbers to see if 119 was possible. Another strategy was to start with 59 × 2, which was almost 119, and then move up or down from 59, but again multiplying to try to get 119 No one ever thought of dividing.

When they asked for harder ones I asked how I could make them harder! (It is not often children ask for harder examples; it was the first time I had actually thought of discussing with them what makes examples harder.) They suggested I chose larger numbers, or multiplied three numbers together. After 289 and 177 I gave them 646, which I said was the product of three numbers. They realised one of the numbers was 2. and could calculate mentally that that "left" 323, but they continued to multiply 2 by two other numbers to try to get 646, rather than ignore the 2 and work in 323. I feel that this more efficient strategy would have been more accessible had they been able to think of division rather than multiplication.

One of the uses of division as an efficient algorithm is for changing common fractions to decimal fractions, a use that has become far more important now that children are expected to use calculators. Yet this, like other division algorithms, does not come naturally to children, nor is it very easy to understand.

Furthermore, traditional methods very often hamper the way to a division algorithm by concentrating on the place value aspect of decimals, for example by beginning only with tenths and establishing the equivalence of those This is not to say that traditional methods are wrong. They make use of practical materials like Cuisenaire or multibase blocks which provide useful visual models for an understanding of tenths, hundredths, etc., and we can see these ideas in the following extract.

DF:	How would you find out what a third was as a decimal?
All:	A decimal (Pause.)
DF:	How would you find out what half was as a decimal?
Jamie:	A half is ...
Simon:	0.5
DF:	Yes, but how do you find out?
Jamie:	Because a half goes into ten two times No!
Simon:	Because a half is five tenths
DF:	We can do it that way. What about a third?
Jamie:	Ninths ... (Pause)
DF:	How many tenths in a third?
Simon:	Three
DF:	Three tenths is a third?
Jamie:	No. Three and a half. No Three and a third.
(Agreement.)	
DF:	So, you can put down 3 in the tenths column. Then you've got a third of a tenth. What are you going to do with a third of a tenth?
Simon:	(quietly) Throw it away!

There is something nicely intuitive about Jamie's "ninths", and his "three and a third" tenths is a useful idea which I was trying hard to develop into something like the normal written algorithm, but the idea of dividing was still missing.

Another group of ten-year-olds began to develop an oral facility for transformation between tenths and fifths, seeing a fifth as two tenths, a tenth as half a fifth, and a quarter as 2½ tenths I drew a number line to support all this awareness,

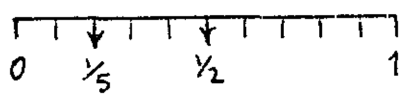

Figure 2

and this and further discussion inspired an intuitive "3 1/3 tenths" as equivalent to a third, but again the number line model does little to aid the more general algorithm.

With this group I resorted to asking if they could think of a calculation involving whole numbers for which the answer would be one half, so that we could check that it gave 0 5 on the calculator. This sparked off a long discussion, with a lot of different ideas before we settled on an equivalence class of divisions like 8 ÷ 16, 4 ÷ 8, 3 ÷ 6, of which the simplest was obviously 1 ÷ 2. The generalisation to other fractions was inductive rather than deductive, and as the class settled down with their calculators to record the decimal equivalents of some of the simpler fractions with unit numerator I do not feel that I had done much better than teach them the algorithm by rote!

With yet another group of ten-year-olds I tried the idea the other way round and asked what one divided by three was.

"One third."
"What does the calculator give you?"
They calculated 0.3333333.
"Is that one third?"
"No," said Sean, "because when you multiply by 3 you get 0 9999999." He explained something about the threes continuing, and said, "You'll never get one."
"What is a half?"
"0.5," they said
"How do you get it on the calculator?"
"One divided by two," said Sean
They did that on their calculators
"How do you get one seventh?"
"One divided by seven," said Sean. "Yes, that's right, because it's one divided into seven parts."
After a little discussion there was some agreement on this, and they calculated, and wrote down:
1/7 = 0 1428571.
"How do you get two-sevenths?"
Several suggested doubling one seventh. They did that and wrote:
2/7 = 0 2857142.
"Three sevenths?"
"Just add that on to there," said Lorna, indicating the two lines
I asked them to work out other numbers of sevenths
For seven sevenths Lorna had 0 9999997 I asked what they would expect for seven sevenths.
"That," said Lorna, indicating her result.
"One," said Sean.
"Is that one?"
"No," he said.
"How much out is it?"
"Three ten-millionths," he said!

Sean was in fact a bright nine-year-old, and normally even ten-year-olds do not reach this level of sophistication.

The teaching strategy here seems to work a little more successfully, mainly thanks to Sean's perception, but note that having calculated one seventh by dividing they then calculate other numbers of sevenths by adding. Even multiplying one seventh by 2, 3, 4, ... in turn would not of course obviate the problem of the compound inaccuracies produced by the rounding off of the calculator, and Sean seemed to have more insight into this right from the beginning.

The problem baffled the other ten-year-olds, the group which included Simon and Jamie whose comments were quoted above. They and five friends were working on a problem posed by Tristan who, after a session on the "squaring" produced by pressing "times-equals" on a calculator, wondered what "divide-equals" would do, tried it on 3, and produced 0.3333333 (This method of squaring works if the calculator has an automatic constant on multiplication, and if it has an automatic constant on division then "divide-equals" gives the reciprocal. A fuller account of the group's work is given in [3].) This extract preceded the one already quoted.

DF: There was another problem with 3, wasn't there? If you put in 3 what do you get out?
All: 0.3333333 (Laughter.)
Jamie: Nought point seven threes.
Simon: It's a third. (General agreement.)
Tristan: No, it can't be because three thirds equals one whole, and that, if you times it by 3, it'll come out as 0 9999999
Jamie: Just a minute In class—right?—Mr B— said, if you get nought point seven threes, it is a third
Tristan: Yes, but he's not an expert!

Tristan's idea caught on They tried multiplying 0.3333334 by 3, but that gave 1.0000002. They tried desperately to find a number between the two, with—not surprisingly—little success!

I suggested they tried one eighth. Annette now remembered the algorithm Mr. B— had presumably taught them, and divided one by eight, but the others looked for a number which gave 1 when multiplied by 8 When I suggested one seventh, Susan and Elizabeth now followed Annette's lead and divided 1 by 7 to get 0 1428571. Susan multiplied it by 7, and got 0.0000007, then multiplied 0.1428572 by 7 and got 1 0000004 It was time to discuss rounding errors.

However, apart from all the other complications, it is again evident that even these children, who were quite bright, felt a certain comfort in multiplication as both an algorithm in itself and as a means of checking division. That is, if they *were* thinking of the multiplication as a check, because I did have a strong feeling that they were dividing merely to find a good approximation on which to use the multiplying algorithm!

There seems to be enough evidence here to suggest very strongly that children around the age of ten tend to avoid doing division if they can find other ways

Perhaps it needs to be said that this evidence arises because of two contextual factors

First, most of the work has been done using calculators. (A much fuller account of these and other activities appears in [3], together with some personal suggestions about the implications for the curriculum for children up to the age of about eleven.) As has been stated already, one important feature of the calculator is that the operations used are made explicit by virtue of the keys that are pressed. Mental methods, or those parts of written methods that are mental, can be fairly woolly as far as the operation is concerned When faced with 2 and 8 and having to associate with those two numbers an "answer" of 4, it is difficult to say whether we or the children are consciously thinking of multiplication rather than division, or vice versa. The forms of words we use generally favour multiplication Yet when a calculator is used multiplication becomes a clumsy algorithm and division is almost essentially an explicit operation.

One should add that the actual layout on paper of a written algorithm for division also encourages children to think in terms of multiplication, and the placing of the divisor and dividend adds further to the difficulty of getting the numbers in the correct order into the calculator. (This is discussed at length in [3].)

The second context factor is to do with the teaching approach used in the activities quoted. Children were being asked to solve problems. They were not given methods of solution. Many of the problems involved inventing suitable calculator algorithms. The children were not taught algorithms or told which to use. Furthermore, they were generally not helped very much by the teacher, who listened to their discussions, perhaps challenged or asked for justification, but on the whole left them very much to their own devices. This approach, by the way, stems from a general philosophy about teaching, so it is one I have used as a regular teacher, and is not adopted merely for the purpose of research.

What the approach does in the activities described is to leave the children free to make choices, in particular choices about methods and operations. It is this freedom which has highlighted the concerns about division which are the subject of this article.

It is difficult to suggest what we can do about it. For one thing, I am not sure if we want or need to do anything about it. There is obviously a flexibility about choice of operations, based on the relations between them, that we wish to preserve. And these children are ten years old or younger, so they have plenty of time to develop further understanding and appreciation, not only of the operations themselves, but of the sophistication of efficient algorithms.

All I can do, then, is to raise a few questions about the teaching of division which may affect the issues raised here, and hope that some of them can provide a relevant basis for discussion.

1. We usually teach the operations in the order: addition, subtraction, multiplication, division. It may be useful to consider changing the order, even if this consideration only provides us with stronger reasons for keeping to tradition. But it depends what we mean by "teach" here. If we are interested in the concepts involved then we tend to base these on physical operations with materials, structured or otherwise. In which case it is as difficult sometimes to distinguish individual operations as it is when one works mentally; for example, putting 5 red Cuisenaire rods against 1 orange rod could represent addition, multiplication or division, and at a pinch subtraction!

Figure 3

2. So, should we provide a variety of such concrete experiences for young children, from which there will gradually emerge four equivalence classes which can be identified as the four arithmetic operations, and ...

3. ... what part in this is played by language (maybe equivalence classes of descriptions), and ...

4. ... symbols (perhaps a labelling of the equivalence classes)?

5. Is action → verbalisation → symbolisation a reasonable order?

6. There is an equivalence class we call division, which can be partitioned into two subclasses we call quotition and partition. Is it as easy as that? And are there similar partitions for the other three operations? (I was about to apologise for using the word "partition" with two different meanings, but the meaning is the same!)

7. If children are to use calculators for any calculation they cannot do it in their heads (this is now a serious suggestion at least in the U.K.) then can we dispense with all written algorithms? In particular can we abolish written algorithms for division, and so always write "8 divided by 2" as $8 \div 2$? Will this also render obsolete all forms of words which put the 2 before the 8? Does it matter?

8. Is it possible to reconcile the multiplication tables with division in such a way that division "facts" or "bonds" or "tables" are seen as division, in their own right, without resorting to the equivalent multiplications? Do we want to do this anyway?

9. How do I calculate $748 \div 37$ on my calculator if the division key does not work?

10. I divide one whole number by another on my calculator and get 0.6099426. What were the two numbers?

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