EXPLORING INSIGHT: FOCUS ON SHIFTS OF ATTENTION

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There is a famous tale about the schoolboy Gauss, who was able to compute the sum of the first 100 integers with great rapidity. Mathematics teachers and educators frequently use the tale to demonstrate to their students what insight in mathematical problem solving may look like. Hayes (2006) collected and analysed more than a hundred versions of the tale. In all the versions, a pivotal part of the young Gauss's insight is described as noticing the pattern 1 + 100 = 2 + 99 = 3 + 98 and so on. However, there is no agreement on what "the method of Gauss" could be. Hayes (2006) found that the most widespread interpretations of the method are *folding*, *double row* and *average*, which correspond to the formulas

$$\frac{n}{2}(n+1), \quad \frac{n(n+1)}{2} \quad \text{and} \quad n\frac{(n+1)}{2}$$

respectively (see Hayes, 2006, and Tall *et al.*, 2012, for additional approaches to the problem). The three formulas are algebraically equivalent, but the ways of attending to the string of the first n integers leading to each of them are different.

We will never know which method young Gauss used and how he noticed the pattern. In this article, however, we make empirically-based suggestions about how the pattern was found by a 15-year-old student Ron, when he sought, with his classmate Arik, a formula for the sum of the first n integers. Ron and Arik needed to find a formula in order to accomplish a solution to a challenging problem that they had chosen for their research project in mathematics.

In an attempt to account for the three-week long sequence of events that preceded Ron's (seemingly) serendipitous invention of the Gauss formula [1], we use the lenses provided by the Mason's (1989, 2008, 2010) theory of shifts of attention. Accordingly, our goal for this article is to reveal the potential of this theory as an analytical tool that can (at least partially) explain the course of the exploration towards the insight solution.

Insight and shifts of attention

The phenomenon of the sudden and emotionally loaded realization of a solution to a difficult problem, sometimes called an "insightful solution," an "illumination-based solution" or an "aha-experience", has attracted the attention of many researchers. This phenomenon has been described and discussed from different perspectives (*e.g.*, Sternberg & Davidson, 1995). According to Cushen and Wiley (2012), "[M]ost theories of insightful problem solving assert that the solution to 'insight' problems requires a restructuring of the initial problem representation" (p. 1166). Cushen and Wiley further note that the course of the restructuring process remains uncertain, in part because it is difficult to obtain measures of problem representation. A research tradition rooted in works of Wallas (1926) and Hadamard (1945) considers an illumination-based solution as a problem solving stage preceded by the stages of *initiation* and *incubation*. Hadamard (1945) characterized these stages based on reflections of prominent mathematicians of his time. Liljedahl (2009) resurrected Hadamard's study with 25 contemporary mathematicians and elaborated on the roles of suddenness, certainty and significance in the mathematicians' aha-experiences.

As these examples demonstrate, the phenomenon of insight in doing mathematics is often attacked by aggregating across individuals. Siegler (1996) compellingly argues that aggregate analysis may not actually reveal the trajectories of individual students. Cushen and Wiley (2012) add that aggregate analysis may obscure interesting patterns present in individuals. In our search for analytical tools to analyse Ron and Arik's long-term exploration leading towards their main insight, we turn to the theory of shifts of attention proposed by Mason (1989, 2008, 2010).

Mason (2010) defines learning as a transformation of attention that involves both "shifts in the form as well as in the focus of attention" (p. 24). To characterize attention, Mason considers not only *what* is attended to by an individual (*i.e.*, what objects are in one's focus of attention), but also *how* the objects of attention are attended to. To address the "*how*" question, Mason (2008) distinguishes five different *ways of attending* or *structures of attention.* These structures, which are briefly explained below, appeared in the story of Ron and Arik.

According to Mason (2008), *holding wholes* is the structure of attention in which the person is gazing at the whole without focusing on particular details. This course of action precedes the phase of making any distinctions. *Discerning details* is a structure of attention in which one's attention is caught by a particular detail that becomes distinguished from the rest of the elements of the object being attended to. Mason (2008) asserts that "discerning details is neither algorithmic nor logically sequential" (p. 37). *Recognizing relationships* between the discerned elements is a development from discerned details that often occurs automatically; it refers to specific connections between specific elements. For instance, when attending to the string of numbers 6, 2 and 3 one can effortlessly recognize that they are connected by the relationship 6 ÷ 2 = 3. Recognizing the same relationship (between 2, 3 and 6), however, is more effortful when one looks at the string of numbers 1, 2, 3, 4, 5 and 6.

Perceiving properties involves attention structured by seeing relationships as instantiations of properties. To stretch the above example, when one searches the string 1, 2, 3, 4, 5 and 6 for the numbers that fit a division relationship, one can effortlessly discern the numbers 6, 3 and 2. Finally, *reasoning on the basis of perceived properties* is a structure of attention in which selected properties are attended to as the only basis for further reasoning.

Since our story includes multiple shifts of attention on the way to experiencing the insight, we find it useful to consider not only *what* is attended to and *how*, but also *why* the solver's attention shifts. We choose to address this "why" question by inferring from the data obstacles embedded for the solver in attending to a particular object and discerning possible "gains and losses" of the shift to a subsequent object of attention.

In sum, the analysis of the forthcoming story proposes plausible conjectures about what, how and why Ron and Arik attended to on their thorny way to an insight solution. This way of doing our analysis is supported by the long-term tradition of using *abduction* in qualitative research. Following Clements's (2000) interpretation, we refer to abduction as the process of producing an explanatory model that, if it were true, would account for the phenomenon in question. Accordingly, the questions that guided the analysis were as follows. Based on the evidence at hand,

1. *What* might be some of the objects of attention for this pair of middle-school students in the course of re-inventing the Gauss formula in the context of coping, for three weeks, with a problem related to numerical sequences?
2. For each identified object of attention, *how* did the students attend to it? In other words, what might be the structures of attention be when attending to particular objects in the context of the problem?
3. *Why* did the students move from one object of attention to another? Specifically, which obstacles embedded in attending to one object of attention might stipulate the shift to another object and how might the subsequent object help the students to overcome the obstacles?

By pursuing these questions, we attempt to overcome the limitations of aggregate analysis and attain a nuanced understanding of the nature of the students' insight.

The context of the story

The story of Ron and Arik occurred in 2013 in the framework of the "Open-ended mathematical problems" project, which we have conducted in the 9th-grade classes of a school in Israel since 2011. At the beginning of a yearly cycle of the project, a class is exposed to a set of about 10 challenging problems in the context of sequences. Ninth-graders in Israel, as a rule, do not possess any systematic knowledge about sequences; this topic is taught in 10th-grade. Mathematical notation and the meaning of the terms "recursive formula" and "explicit formula" is briefly explained to students but little else.

The students choose a particular problem to pursue and then work on it in teams of two or three. The students work on the problem almost daily during free time at home and during their enrichment classes. Weekly 20-minute meetings of each team with the instructor (Alik Palatnik) take place during the enrichment classes. When the initial problem is solved, the students are encouraged to pose and solve additional, related problems. At the end of the project, the teams present their work at a workshop.

One of the mathematical problems proposed to the students was the Pizza Problem:

> Every straight cut divides a pizza into two separate pieces. What is the largest number of pieces that can be obtained by n straight cuts?
>
> A. Solve for $n = 1, 2, 3, 4, 5, 6$.
>
> B. Find a recursive formula for the case of n.
>
> C. Find an explicit formula.
>
> D. Find and investigate other interesting sequences.

This problem is a variation of the problem of partitioning the plane with n lines (*e.g.*, Steiner, 1826; Pólya, 1954). Ron and Arik chose to pursue it.

We audiotaped and transcribed protocols of weekly meetings with Ron and Arik and collected intermediate written reports and drafts that the students prepared for and updated during the meetings. These data were juxtaposed to produce an initial account. Pencil marks on the student drafts were particularly informative for making inferences about occurrences of shifts of attention. The initial account was shown to Ron, who took the lead in the project, during a follow-up interview. In the interview, Ron provided us with additional information that supported most of our interpretations.

Ron's explanations and clarifications helped us to refine our *account of* the exploration, which is presented in the next section. This section is followed by our *account for* the story. Mason (2002) distinguishes between *account of* and *account for* as follows: *accounting of* happens through addressing a question "what happened?", whereas *accounting for* happens through asking why events occur, and why one has noticed the specific aspect.

Account of the student exploration

At the beginning of their work on the problem, the students produced about 30 drawings of circles representing pizzas which were cut by straight lines (see Figure 1). They counted the number of pieces on the drawings and observed that the maximum number of pieces is obtained if exactly two lines intersect within the circle. The solutions for 1, 2, 3, and 4 cuts were found: 2, 4, 7 and 11 pieces, respectively. It was difficult for the students to find a solution for 5 cuts from the drawings since they became overcrowded.

To overcome this difficulty, Ron created a *GeoGebra* sketch and found that the maximum number of pieces for 5 cuts is 16. The students recorded their results as a horizontal string of numbers. They noticed that the differences between the subsequent numbers in the string form a sequence 2, 3, 4, 5 and used this observation to produce an

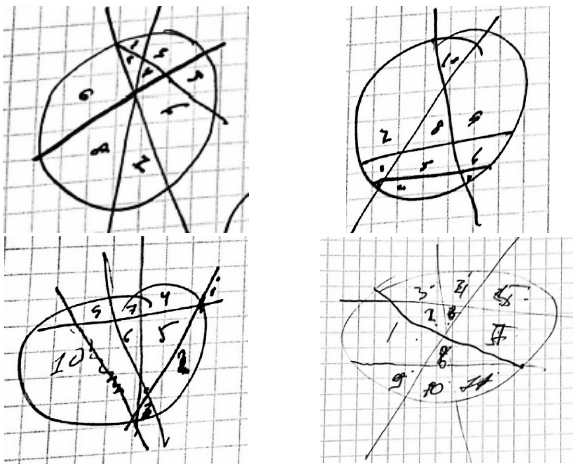


Figure 1. *Some of Ron and Arik's drawings of circles representing pizzas.*

answer for 6 cuts. Their next goal was to find a recursive formula. After several unsuccessful attempts to consider the strings of numbers, they organized their findings vertically and eventually drew a table (see Figure 2).

From this point, the students began exploring the tables. Their exploration strategy can be described as looking for arithmetic relationships between the numbers in the tables and marking them. One of the first relationships that they attended to was a *zigzag pattern* (see Figure 2d). At this stage they introduced some notation: *P* for the number of pieces and *n* for place of *P* in the table. Later they noticed that *n* also represents the number of cuts and introduced the notation P_n. A recursive formula $P_n = P_{n-1} + n$ was written as a symbolic representation of the *zigzag* pattern.

Next, the students began looking for an explicit formula, which would enable them, in the words of Arik, "to find P_{100} without finding P_{99}" [2]. The students tried to find a formula on the Internet and did not succeed. They also attempted to find an explicit formula in Excel since "there are many formulas in Excel." When this plan did not work, they asked the instructor for help. The instructor only helped the students to build a spreadsheet based on their recursive formula and encouraged them to keep looking.

A week later, the students brought to the meeting five tables with marked patterns: a *diagonal pattern* corresponding to the previously obtained formula $P_n = P_{n-1} + n$ (Figure 3b), a *horizontal pattern* summarized by the formula $P_n = (P_{n-1} + n - 1) + 1$ (Figure 3c), a *mixed pattern* accompanied by the (incorrect) formula $P_n = n + P_{n-1} + n - 1$ (Figure 3d), and a *vertical pattern* corresponding to the formula $P = \sum n + 1$ (Figure 3e).

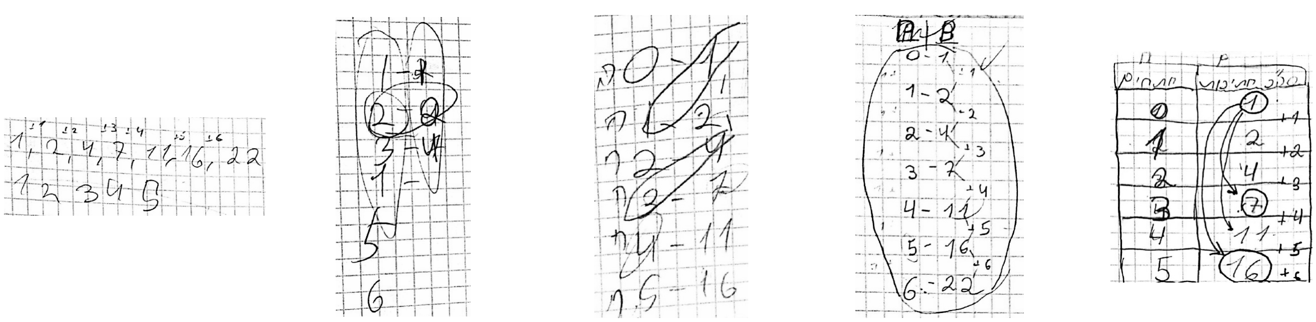
Surprisingly to the instructor, the students presented a *vertical pattern* and formula $P = \sum n + 1$ as just one of their results, and not as an important milestone on the way to the explicit formula. The following exchange happened:

Instructor: [Let's] focus on this way [*the vertical pattern*] […] Tell me, how do I get, for example, 22?

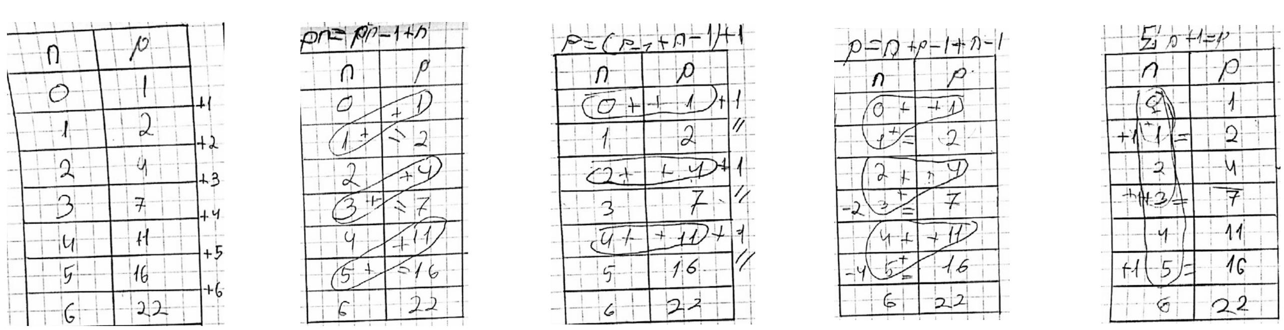
Ron: Twenty-two without 16? It goes ... I make one plus zero and one and two and three and four and five and six.

Instructor: One and two and three and four and five … There is some formula for calculating it.

…



Figures 2a-2e (from left to right). *The drafts produced during the first week.*



Figures 3a-3e (from left to right). *The drafts produced during the second week.*

Arik: So, [you ask] how to calculate it? Without summing the numbers?

Instructor: Yes, without summing the numbers. You know, there is a formula that can give you an answer [immediately]. Do you understand why it is important?

Arik: Because it takes time to calculate [by the formula $P = \sum n + 1$].

At the next meeting, the students introduced the desired formula:

$$P_n = \frac{n}{2}(n+1)+1$$

The instructor was astonished by the students' success and asked them to explain in as much detail as possible how they had obtained the formula. In the words of Ron: "I was stuck in one to six. And I just thought … six divided by two gives three. I just thought there's three here, but I could not find the exact connection [to 22]. I do not know why, but I multiplied it by seven, and voila, I got the result." Based on this account and Ron's clarifications given during the follow-up interview, we infer that in order to say what he said and find what he found, it is highly likely that Ron acted as follows.

Ron focused on the left column of a table similar to the table shown in Figure 3e. He experimented with the vertical string of numbers attempting to somehow, mostly by using the operations of addition and subtraction, create an arithmetic expression that would return a number from the right hand column. He asked his parents and his older sister for help; they tried and did not succeed. Then he explored the table again, and this time he also tried multiplying and dividing. One of these attempts began from the computations 6 ÷ 2 = 3 and 3 × 7 = 21. Ron realized that 7 in the second computation is not just a factor that turns 3 into 21, but also a number following 6 in the vertical pattern. He noticed (not exactly in these words) the following regularity: when a number from the left hand column is divided by 2 and the result of division is multiplied by the number following the initial number, the result differs from the number in the right hand column by one. He observed this regularity when trying to convert 6 into 22, and almost immediately saw that the procedure converts 4 into 11 and 5 into 16. He observed that even when division by 2 returns a fractional result (5:2 = 2.5), the entire procedure still works. According to Ron, the *aha*-experience occurred at this moment.

To verify the regularity, he calculated P_{100} by the discerned procedure and compared the result with the corresponding number in his Excel spreadsheet, which was based on the recursive formula. The last step was to convert the invented procedure into a formula. Finally, from the follow-up interview:

Instructor: How did you convert it [the observed regularity] into the formula?

Ron: It was a difficult part … I did it really in line with the arithmetic operations that I've used. I divided n by 2, and then I like multiplied it by $n + 1$, which is the next n, and then plus one.

Account for: what might be attended to, and how, and why?

The answer to the "what" question stems from the above account. Namely, the students attended, among other things, to the following objects: handmade sketches of a pizza, a GeoGebra sketch, strings of numbers, two-column tables, and the left hand column of a table similar to that shown in Figure 3e [3]. For each of these objects, we now answer the "how" question and the "why" question. The answers for the first four objects are summarized in Table 1.

The last object of attention was the left hand column of the table similar to that in Figure 3e. The structures of attention for this object were particularly diverse and changed kaleidoscopically. Ron discerned sub-sets of the set of numbers 1, 2, 3, 4, 5 and 6, recognized various relationships in the sub-sets, perceived the division property and discerned a sub-set "2, 3, 6" that fitted it. He discerned a subset "3, 7, 22", recognized the relationship 3 × 7 + 1 = 22 and perceived numbers 6 and 7, which have been discerned in the above relationships, as numbers that belonged to the vertical pattern. Ron then perceived the property "3 × 7 + 1 = 22" for additional triples of numbers, namely, (4 ÷ 2) × 5 + 1 = 11 and (5 ÷ 2) × 6 + 1 = 16. (This was his *aha*-experience). The solution to the problem was concluded by means of *symbolic reasoning*, that is, converting "3 × 7 + 1 = 22" into the formula

$$P_n = \frac{n}{2}(n+1)+1$$

Discussion

In the famous tale, young Gauss finds the sum of the first 100 integers almost effortlessly. The problem of finding the sum of the first n integers in the context of the Pizza Problem appeared to be very effortful for Ron and Arik. One can wonder: why did the students miss the "classic" solutions to the problem, that is, the solutions by folding, double row or average? The literature on algebraic reasoning provides us with some initial explanations of the complexity of the students' performance.

In terms used by Duval (2006), coping with the problem required the students to shift representational registers many times, which definitely added to the complexity of the endeavour (recall that the young Gauss was given a numerical exercise, not an algebraic one). In line with Radford (2000), we infer that the problem was difficult because it led the students to make a shift from pattern recognition to algebraic generalization (young Gauss looked for a numerical answer). Furthermore, in line with Zazkis and Liljedahl (2002), we suggest that the problem was difficult for the students because the recursive approach was dominant, and this approach is known to prevent students from seeing more general regularities (young Gauss did not approach the problem recursively).

In spite of the difficulties, the students eventually found the solution, and one of them, Ron, experienced an *aha*-moment. Consequently, it is reasonable to consider the story we have presented in terms of the insight problem solving phases proposed by Wallas (1926) and Hadamard (1945). At the *mental preparation* phase the students confronted the given problematic situation and realized that the looked-for formula should compute any element of the numerical

Objects of attention	Structures of attention: how is the object attended to?	Why did the students move to the next object?
Handmade sketches	*Discerning* the areas bounded by the circles and the cuts in order to count the pieces. *Perceiving* that the maximum number of pieces is obtained if exactly two lines intersect within the circle.	When there are more than four cuts, some areas become small and it is difficult to count them.
A GeoGebra sketch	*Discerning* the areas bounded by a circle and five cuts in order to count the pieces. Counting is supported by the ease of moving the cuts so that small areas can be enlarged.	The drawings, even dynamic, are not convenient for the larger numbers of cuts; results of counting are not ordered.
Strings of numbers	*Discerning* the neighbouring numbers of the string and *recognizing* the relationships between them: the differences of the neighbouring numbers form a sequence 1, 2, 3, 4 …	The number of pieces (P_n) is visible in the string, but the number of cuts (n) is not; realization that producing an explicit formula requires that both n and P_n would be visible.
Two-column tables	*Recognizing* various numerical relationships between the numbers (including diagonal, horizontal, mixed and vertical patterns). *Symbolic reasoning* on the basis of the perceived properties: $P_n = P_{n-1} + n$, $P_n = (P_{n-1} + n - 1) + 1$, $P = \sum n + 1$.	Realization, partially based on the instructor's prompt, that an explicit formula can be produced by looking at the vertical pattern, which is visually situated in the left hand column of the table.

Table 1. Structures of attention for the first four objects.

sequence without lengthy calculations and without the knowledge of the preceding element of the sequence. Thus, the students constructed an adequate conceptual understanding of an *explicit formula* notion. Next, we observed that, after making some initial progress (namely, considering tables instead of drawings), the students were stuck for about a week. However, at this stage the students developed and repeatedly used a particular strategy of looking for the desired regularity. Thus, it is reasonable to suggest that some *incubation* occurred at this stage. We also know that the solution to the problem came suddenly. In terms of the representational theory of insight (*e.g.*, Knoblich, Ohlsson & Raney, 2001), we can say that the insight occurred when a particular representation was put forward among many other representations.

However, consideration of the problem's difficulty due to the students' under-developed algebraic reasoning and explanation of the insight by identification of representational shifts is compatible only with one aspect of the analysis we have presented, the one focused on Mason's "what" question (*i.e.*, what objects were in the focus of attention?). The added value of our analysis is in also putting forward the "how" question (in line with Mason's theory) and the "why" question. We argue that the concurrent focus on these three questions is pivotal for explaining the observed phenomena. Specifically, focusing on the "how" question enabled us to follow at a fine-grained level the interplay of the structures of attention that led Ron to his main insight. Focusing on the "why" question enabled us to suggest some internal logic behind a seemingly serendipitous chain of attempts.

The literature on scientific serendipity [4] can offer a complementary, global, view of the nature of Ron's insight. One of the approaches to making sense of serendipity in science is known as *a prepared mind approach* (Seifert, Meyer, Davidson, Patalano & Yaniv, 1994). The name of this approach is taken from Louis Pasteur's famous aphorism "chance favors the prepared mind." Seifert *et al.* (1994) put forward the importance of the interrelation between indices of failure of the initial solution attempts and opportunistic assimilation of information from subsequent exposures to relevant information. From this viewpoint, Ron's insight was well prepared for by the repeated (and initially unsuccessful) implementation of a particular exploration strategy: consideration of numerical relationships among the numbers of the left hand column of the table (0, 1, 2, 3, 4, 5 and 6) with simultaneous attention to the numbers of the right hand column (1, 2, 4, 7, 11, 16 and 22). The student's success can be attributed to the development of a powerful way of thinking (*i.e.*, the aforementioned exploration strategy) as well as, in Harel's (2008) terms, various opportunities for repeated reasoning during the project.

The insight solution was a local highlight (see Liljedahl, 2013) for Ron, but not the end of the project. After obtaining the solution, the students chose to continue working on the problem. In the words of Ron: "We got it [the formula] by chance, so we do not know what it means." Apparently, the serendipitous nature of the solution became for the students a source of uncertainty and triggered the emergence of a new goal: to make sense of the obtained formula. The students pursued this goal for four weeks. (This part of the project is analysed elsewhere). In brief, the students made sense of the formula when they sensed a geometric mechanism behind the formula, justified it by algebraic means and, in addition, explored several related patterns so that they could coherently insert the formula in question in the cloud of the related formulas.

Our last comment is about possible pedagogical implications of the presented case. We draw the argument from three interrelated sources. Hadamard (1945) wrote: "Between the work of the student who tries to solve a problem in geometry or algebra and a work of invention, one can say that there is only a difference of degree, a difference of level, both works being of a similar nature" (p. 104). Hitt, Barrera-Mora and Camacho-Machín (2010) suggested that student inventiveness can be promoted by research projects based on problems whose solution time is longer than in a regular classroom experience. Liljedahl (2005) argued that *aha*-experiences have a positive impact on students' attitude towards mathematics and raised a question of how to organize learning environments, in which such experiences might occur. We argue that an instructional format of the "Open-ended mathematical problems" project, of which the story of Ron and Arik was part, is an example of such an environment. On one hand, in this environment the students had enough room and time for autonomous problem solving and learning. On the other hand, the chosen instructional format included opportunities for the instructor to focus the students' attention on the most promising idea from the pool of their ideas.

Acknowledgements

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Notes

[1] An abridged version of this paper work was presented at PME38 (Palatnik & Koichu, 2014).

[2] All the excerpts are our translations from Hebrew.

[3] Additional objects (*e.g.*, Excel spreadsheet) were attended to, but turned out to be secondary rather than primary objects of attention in the course of solving the problem.

[4] According to the online World English Dictionary, serendipity "is an aptitude for making desirable discoveries by accident" (see http://dictionary.reference.com/browse/serendipity)

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