"-Pat + Pat = 0": Intellectual Play in Elementary Mathematics

HELEN FEATHERSTONE

Some years ago, walking around a third-grade classroom whose students had been invited to "write as many number sentences as you can that start with a negative and equal zero", I spotted the following entry in a math journal:

-Pat + Pat = 0

Above this equation were quite a few others of the sort I had expected when I suggested the assignment: $-3 + 3 = 0$, $-14 + 14 = 0$, etc Daniel seemed to understand that for any whole number a, $-a + a = 0$. It was the playful quality of the last entry that caught me, however. Daniel's imagination had leapt from whole numbers to Pat Cummings (an author and illustrator of children's books who had recently visited his school) and back again If there is a number -2 which, when added to 2, gives a sum of 0, why could there not be a -Pat which, when added to Pat, gives a sum of 0?

My response to Daniel's flight of fancy had two quite different sources. The first was my experience as an elementary school teacher The six-year-olds I had taught years earlier in an inner-city Boston school were eager to read, but they had had little exposure to books or print before they entered first grade. Many struggled mightily with the alphabet and the task of learning to recognize common words. It often took many months of reading very simple books - the publishers called them 'pre-primers' - before they could read a page from a primer.

The glacial pace of their progress terrified me at first: I thought that my students would never learn to read (and that I was a failure as a teacher) Events proved me wrong, however: when children finally got to a point where they were able to read a primer, their skills improved astonishingly rapidly. And it was obvious to me why this had happened: as soon as they could read a real book, even if that book was only *Go, Dog, Go*, they wanted to read constantly Their increased fluency - and the literary riches P. D Eastman and Dr. Seuss had provided for them - led them to read for pleasure whenever they saw the opportunity. When this breakthrough occurred, and often it happened quite suddenly, their reading skills developed meteorically: some children went from reading primers to reading third-grade books in a few weeks time. When reading became play, children learned on their own far faster than I could have taught them.

Looking at Daniel's math journal, I thought: "This is math in the ludic zone Daniel is playing with - and in - the math assignment". In his play, he had connected the math we were studying to an event that had excited him. I wondered whether, if we could help children to see math as an arena for play, their mathematical understanding and skill might not develop in ways that could astonish us

My pleasure in Daniel's fanciful equation had another source besides my experience as a primary grade teacher: it was rooted in my understanding of the discipline of mathematics itself As I considered what Daniel had written, it seemed that he was 'doing mathematics' in a way that eight-year-olds rarely can. He had gone from the regularity he had observed as he counted on the thermometer he had drawn in his math journal to an abstract generalization: having noticed that for any number a, $-a + a = 0$, he proposed (playfully) that for any entity, including a writer named Pat, there would be another entity, which, when added to the first, would yield a sum of zero.

This is an abstract idea indeed - one that was, in fact, a challenge to my own imagination. It goes well beyond observing patterns It reminded me that the spirit of play can figure in the invention of mathematics - for new ideas, in mathematics as elsewhere, often have an element of the preposterous. (The writings of Richard Feynman offer a number of splendid examples of the way in which a playful mind can lead a discipline into new and productive territory)

Daniel had explored the meaning of the minus sign in a way that went beyond what we had done as a group. In his equation, it was not simply a symbol to add to a number that was below the zero mark on the thermometer, but a negation, a symbol that creates an additive inverse: -Pat is that which, when added to Pat, totals 0 Such an entity is close to unimaginable if we ground our thinking in the familiar stuff of school mathematics problems: "Ms Koblinski has 15 stickers . " or "The fourth grade is making cupcakes for the bake sale . .". However, its theoretical existence can be extrapolated from the existence of a number, -3, which, when added to the familiar 3, gives a sum of 0.

What might be the role of play in the learning of mathematics?

These reflections on Daniel's journal entry led me to two questions: what are the connections between doing mathematics and playing? Could it be helpful to think of mathematical invention as a kind of play? Desiring a deeper understanding of play, I sought out Johann Huizinga's 1950 classic [1] *Homo Ludens: a Study of the Play Element in Culture*. At the outset, Huizinga makes a number of assertions about the nature of play: play steps outside of ordinary or 'real' life; it is bounded in time and space; it is orderly and, in consequence, beautiful; it is governed by rules; play involves tension; play creates social groupings; play is voluntary.

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I found it helpful, as I read, to think about some specific examples of play - four-year-olds playing 'school', the fourth graders in the playground near my house kicking a ball around the soccer field during recess But I was also struck by the ways in which many of Huizinga's points about the nature of play connected to mathematics and the doing of mathematics.

Like play, real mathematics certainly steps outside of ordinary or 'real' life Although in recent years, US reformers have urged teachers to emphasize the connections between mathematics and the real world (NCTM, 2000), from the time of the ancient Greek geometers, the most prestigious mathematics has focused on purely theoretical entities (one-dimensional lines without thickness or breadth, etc.). In the world of Euclid and Pythagoras, arithmetic, the real-world analogue of this pristine science, was a convenient tool for keeping household accounts and dealing with the material universe, suitable for use by slaves and women.

Even first graders realize that the correspondence between the mathematics they study and the behavior of objects in the real world is in some sense idealized. You can find the answer to 8 + 9 = p by laying down a row of 8 cubes, then a row of 9 cubes, pushing the rows together, and counting the cubes in the resulting pile; you can use apples, shoelaces or pennies and get the same result However, if you substitute drops of water for pennies, you will get one bigger puddle of water, and that will not mean that in mathematics 8 + 9 = 1.

Play, Huizinga asserts, is bounded: it is 'played out' within certain limits of time and space In other words, it begins at a certain moment and is over at another and it takes place within a spatial frame.

> the arena, the card table [ ] all are temporary worlds within the ordinary world, dedicated to the performance of an act apart. (p 10)

Although the mathematician may scribble out an equation on a café napkin, it is certainly true that the work of mathematical exploration and creation is set apart for the most part from daily life - it is done either alone or in the company of a few initiates. The historian and the landscape architect are far more likely - and more able - than the mathematician to describe the substance of their work to casual acquaintances Mathematics, like play, creates social groupings

Like play, mathematics also creates order:

> Into an imperfect world and into the confusion of life [play] brings a temporary, limited perfection. (p 10)

Huizinga links this orderliness of play to beauty:

> The profound affinity between play and order is perhaps the reason why play, as we noted in passing, seems to lie to such a large extent in the field of aesthetics. Play has a tendency to be beautiful. It may be that this aesthetic factor is identical with the impulse to create orderly form, which animates play in all its aspects. The words we use to denote the elements of play belong for the most part to aesthetics, terms with which we try to describe the effects of beauty: tension, poise, balance, contrast, variation, solution, resolution, etc (p. 10)

Although mathematics does not seem especially beautiful to laymen, aesthetics plays an important role in its appeal to mathematicians. Poincaré (1946) wrote eloquently of:

> the feeling of mathematical beauty, of the harmony of numbers and forms, of geometric elegance. This is the true aesthetic feeling that all real mathematicians know (p 391)

He argues that an appreciation of mathematical beauty is the key to mathematical invention, because in mathematics, "the most useful combinations [of ideas] are precisely the most beautiful", and that it is the mathematician's appreciation of this beauty, his "special aesthetic sensibility", which enables him to invent new mathematics.

Rules figure centrally in play While this is most obvious in relation to competitive games, the Russian psychologist Lev Vygotsky (1933/1976) has shown that imaginative play - children playing house, or hospital or super heroes - also has rules: children playing at being mothers and fathers must behave like mothers and fathers and this obligation constrains their actions in much the same way that the rules of canasta or golf constrain the players of those game. [2]

For the mathematician, 'doing mathematics' means exploring the regularities (or rules) that govern the behavior of mathematical entities. (The rules that govern the behavior of mathematical entities also, of course, influence the behavior of the mathematician and of the child at work on her mathematics assignments) According to Huizinga, rules:

> determine what holds in the temporary world of play [...] Indeed, as soon as the rules are transgressed the whole play-world collapses (p 11)

The same is surely true in mathematics.

Virtually all of the foundational writing on play agrees that the lay view, which sees play as aimless, spontaneous activity, is entirely mistaken (Dewey, 1916/1966; Piaget, 1962; Huizinga, 1955; Groos, 1898). Play, these writers insist, always has an aim.

> The player wants something to 'go', to 'come off'; he wants to 'succeed' by his own expertise (Huizinga, pp 10-11).

> Persons who play are not just doing something (pure physical movement), they are trying to do or effect something. (Dewey, p 203)

Although the goals of play, unlike the goals of work, are always in some sense 'inside' of the play, they are real and shape the activity of the players, whether they are trying to impersonate Batman, serve a tennis ball into the corner of the serving court or avoid tripping on a jump-rope. Because players are trying to accomplish something, tension is a part of their play experience Clearly the same is true in mathematics: the effort to arrive at a new understanding of mathematical entities or of mathematical relationships and the uncertainties about success create a certain tension.

But what of the most central aspect of play, the feature of my first-graders' reading that struck me most forcibly as I appreciated it several decades ago: play is free, voluntary, uncompelled? Poincaré's (1946) account of his own mathematical inventions pointed me towards an interesting way of thinking about the role of freedom in doing mathematics He explained that mathematical creation consists in discovering connections among mathematical facts or entities which have in the past been seen as unrelated but which can, in fact, be connected in ways that reveal hitherto unknown mathematical truths.

> Among the chosen combinations, the most fertile will often be those drawn from domains which are far apart. (p. 386)

The number of possible combinations is astronomical (and most, of course, are "absolutely without interest"), so the process of identifying fruitful ones is a bit mysterious After examining his own experiences of mathematical creation, Poincaré declares that mathematical invention requires not only periods of disciplined work on a problem, but also a period in which the conscious mind is occupied elsewhere and the unconscious mind works on the ideas without this sort of discipline. [3] In the unconscious mind, ideas and entities combine and bounce off one another quite randomly

> In the subliminal self [...] reigns what I call liberty, if we may give this name to the simple absence of discipline and to the disorder born of chance. Only, this disorder itself permits unexpected combinations. (p 394)

Although the vast majority of these momentary connections are useless mathematically, a few are not. The aesthetic sensibility of the mathematician enables his unconscious mind to select the most beautiful combinations for careful conscious analysis (see above)

So the freedom from certain sorts of mental restraint may be central to 'doing mathematics' in ways that lead to new ideas According to Poincaré, in order to discover or invent in mathematics, we need to be freed from strict mental censorship that permits us to see only 'logical' connections. We need a kind of intellectual emancipation which makes room for disorder, for play with apparently unconnected ideas. [4]

Poincaré's construction of the process of mathematical invention and the role of a certain sort of liberty in that process reminds me that there are, in fact, moments of relative freedom even in the doing of school mathematics (and that Daniel's '-Pat + Pat = 0' was the fruit of such moment). Daniel had 'done the assignment': he had written plenty of number sentences in his notebook and identified a pattern. He was now at liberty to read a book or talk to a friend. Furthermore, writing this equation was not really doing the assignment - the equation was not a number sentence and Daniel did not volunteer to put in on the chalkboard during the ensuing discussion.

This activity was, in Huizinga terms, "a voluntary activity", one which, like the free play of ideas that Poincaré identified as a central part of mathematical invention, connecting the previously unconnected - thus suggesting the freedom of the unfettered (and playful) mind. I would argue that this connection carried Daniel beyond the limitations of the thermometer representation to a deeper understanding of the negative sign.

There are, then, some interesting correspondences between the phenomenon of play and the phenomenon of mathematics. Certainly, mathematics and play are not identical, and I am not arguing that teachers ought to try to make a mathematics period into a play period Rather, I want to think about the moments in which doing mathematics becomes playful and about the ways in which 'play' might expose children to aspects of the discipline that may not ordinarily be visible to them

Mathematically imaginative play

The argument for play in the teaching of mathematics usually centers on practice (as in 'drill and practice'): games in which children practice their number 'facts' (8 + 9 = 17; 7 x 6 = 42) are a staple of the elementary curriculum, perhaps especially the computer-based enrichment curriculum. Often in these games children get to make a move (shoot at an alien, get information about the whereabouts of hidden treasure, ...) if they solve a computation problem quickly and correctly Some teachers appreciate the way in which these games create an opportunity for practicing computation that does not feel too burdensome to their students

It is easy to see how a responsible mathematics teacher could feel this way: computational facility matters, and not just for success in traditional classrooms and on standardized tests. Developing facility with multiplication, addition and subtraction can allow a student to explore the world of number in ways that are nearly impossible if s/he has to work out each computation by counting. Yet many good teachers hesitate to use mathematics periods for the computational worksheets that teachers have traditionally assigned in order to help students to develop this sort of automaticity. Games that help children to learn their facts seem like a perfect solution.

However, this kind of mathematics 'play' has serious drawbacks. First, as Ainley (1988) points out, when the mathematics is extrinsic to the 'play' - as it is when solving a computation problem allows you to take an extra shot at an alien - children learn to see mathematics as a chore and mathematical thinking as something that they should be rewarded for doing. Such games, Ainley writes, encourage the idea that there is a dichotomy between 'playing' and 'doing' mathematics. (She worries when she hears students commenting enthusiastically, "That was great! We didn't do maths. We just played games".)

Second, the emphasis on speed of response encourages the notion that knowing mathematics is about remembering rather than about reasoning And third, the games are usually competitive and to the extent that they actually do reward the skill they are attempting to teach (or 'reinforce'), they provide a pleasanter experience for those who find computation easy than for those who find it difficult and are likely to reinforce or create a dislike of mathematics on the part of students who are struggling with their 'facts'

As I explored the connection between play and mathematics, I began to see something interesting: it seemed that perhaps play in school mathematics, instead of being an activity centered in very concrete sorts of activity - work with Cuisenaire rods or unit blocks, for example - might be seen more often in the area of abstraction. Perhaps the ideas that seem not to have concrete referents are the ones that attract children most powerfully and ignite the playful impulse.

Deborah Ball (1995) reports that in her third-grade mathematics class, 'pure mathematics' engaged children in a way that problems grounded in real life often did not

> Rarely do I see students become as engrossed as when they are debating whether zero is an even or an odd number, what a sensible answer might be to 6 + (−6), or from which bag [one containing 2 green chips and 2 yellow ones or one containing 3 green chips and 4 yellow ones] one would be more likely to pull a green chip (p. 673)

Of a discussion about whether zero was odd or even, she remarks:

> Neither useful or practical, it was nonetheless engrossing. And so engrossed, [the students'] activity was as much play as intellectual pursuit Zero appeals to children's fancy, not unlike the appeals of magical characters in fiction they read and write. Exploring what zero might be capable of being and doing is an activity of imagination for eight- and nine-year-olds. (p 675)

What I am arguing for is a mathematics curriculum in which children are tempted to play in and with mathematical ideas, as Daniel did. The mathematical play that interests me parallels the sort of play these children did when they were somewhat younger and explored ideas about human work and relationships by pretending.

After analyzing audio-tapes of the imaginative play of the kindergarten and pre-school children in her classroom, Vivian Paley (1988) reports:

> The children were actors on a moving stage, carrying on philosophical debates while borrowing fragments of floating dialogue. Themes from fairy tales and television cartoons combined with social commentary and private fantasy to form a tangible script that was not random and erratic
>
> A relentless connection-making was going on, the children inventing and explaining their rules and traditions every time they talked and played. "Let's pretend" was a Socratic dialogue, and the need to make friends, assuage jealousy, and gain the sense of one's own destiny provided reasons for agreement on goals and procedures. An astonishing marketplace of ideas flourished in the kindergarten classroom. (p. 12)

For the four-year-old, Paley argues, imaginative play serves much the same purpose that a conversation might serve for an adult: it provides a setting and a language for thinking out ideas and examining them with others.

The vignette in the next section provides a glimpse of some mathematically imaginative play that follows the introduction of negative numbers in a third-grade class. The encounter with these new and comparatively abstract numbers seems to prompt our protagonist, Cherise, to play with *two* sorts on unimaginable numbers: numbers which, because they are below zero, have no concrete quantitative referents within the experience of eight-year-olds and numbers which are so big that they represent quantities outside of the experience of a third grader

In the third grade

A third-grade class (in which I am present, at the teacher's request, to help with any difficulties she may encounter as she tries to teach unfamiliar mathematics) has been introduced to numbers below zero on a Fahrenheit thermometer. After together writing some equations representing changes in temperature ("The temperature in Anchorage Alaska was 4° in the morning; by night it had gone down 6° . ..."), the children work in their mathematics journals on the following assignment: "Write all the ways you can think of to get below zero". After the group reconvenes and several children have presented lists of equations, Cherise comes to the front of the room and copies some examples from her notebook onto the chalkboard.

1000	7000	2000
−<u>2000</u>	−<u>10000</u>	−<u>3000</u>
−1000	−3000	−1000

She announces:

> C: Now these problems here, they may look pretty big, but they are easy. Because, the way I look at it, it's the same way as if it doesn't have any zeros

She inserts degree marks after the 2, 3, and 1 in the third example, emphasizing the irrelevance of the zeros.

2°000
−<u>3°000</u>
−1°000

> C: So it's just like 2 take away 3, and it will equal up to negative 1. So I say that is very easy. Now problems like 14 take away 20 are harder, even though they have less numbers.

She counts up from 14, writing each number as she says it − 15, 16, 17, 18, 19, 20 − then counts aloud the six numbers she has written and concludes:

> C: And then all you have to do is decide if it's below zero or not. That's all you have to do And it's my conviction that it is Now, my idea is that numbers like 1000, 2000, 3000, they have lots of numbers, but they still aren't hard: it's just like 2 take away 3 So I say that is easy and it will equal up to 1 I could write a huge big number with forty zeros. But what I am saying is that, even though this is a huge big number, you don't even have to worry about the zeros.

Making her point visually, Cherise crosses out all nine zeros.

"Why would you cross out the zeros?", Nathaniel asks.

"You don't have to worry about them", Kevin explains.

"Well, look at it: the zeros are zeros," Cherise agrees. "What you can say is, '1 take away 2 is negative 1 and you don't even need to worry about the zeros You just put the zeros on' "

"Cherise", I interject, "could you say that as a conjecture?"

The third graders have been listening quietly to Cherise's explanation for the last few minutes, but as soon as I make my request a dozen voices, all breaking with emotion, protest: "How can you cross out the zeros?", "I don't get it!"

C: Now the reason I crossed out the zeros is that people will look at this and say, 'Oh, no, I don't know this. This is too big a number.' But it is really just 1 take away 2

Feeling, as she explains to me later, the same pressure I am feeling to bring the mathematics lesson to a close, the teacher urges Cherise to state her conjecture.

C: I need to say the second thing Now 14 - 20, is harder. You can't just say 1 take away 2, you have to think about it.

Cherise writes all the numbers from 14 to -6 vertically on the board, explaining, "You can't just cross out the 4, because 4 is a number".

At this point I intervene

Cherise, can I try to say your conjecture in a short sentence that we could write on the board, and you can tell me whether I am right?

When Cherise grins and nods, I tell the rest of the class:

I'm going to say Cherise's conjecture and other people can listen and see if they think I got it right. I think Cherise is saying, 'A great big number with a lot of zeros can be easier than a smaller number'.

I turn to Cherise and ask: "Is that right?" Cherise nods cheerfully and asks if there are any questions. The teacher, observing that we have gone well beyond 10:30, says that we will have questions tomorrow and sends the children back to their seats to put away their notebooks

That afternoon I write in my journal:

Cherise was the last one to come up, I think, and she announced that she had a conjecture Then she wrote 2000 - 3000 on the board and explained that although this looked hard, it was really easy, because you could ignore the zeros - she proceeded to x them out. She then wrote 14 - 20 and said that this was really much harder because you had to count up the difference. I'm not sure those were her exact words, but she wrote the numbers 15, 16, 17, 18, 19, 20, counted them up (6), and wrote -6 as the answer.

The first time around I thought that she was proposing that you can get the answer by counting up in this way - that she had noticed that the below-zero answer is the negative of the difference between the two numbers, and that this was her conjecture But when someone said that they did not get it she went through the whole spiel again, complete with the business about the thousands So I asked her if her conjecture was that sometimes a problem with great big numbers could be easier than a problem with little numbers, and she said, "Yes!" She talked much louder than usual.

As my journal makes clear, I was confused by Cherise's presentation at the time, unsure of her point. I was certain that she planned to make a conjecture Indeed, I claim that when she came to the board "she announced that she had a conjecture", although in fact she does not appear to have done anything of the kind.

Looking now at Cherise's presentation, however, through the lenses provided by Vygotsky, Dewey, and Huizinga, I see it as an intriguingly suggestive example of mathematically imaginative play. If we keep our eyes trained on Cherise and our ears tuned to the long conversation about play, a number of points command our attention.

First, goals shape Cherise's activity, but these are goals that are 'inside' of the play itself. Working alone in her mathematics journal, Cherise challenges herself to write equations using very big numbers whose solutions are paradoxically easy and to create equations involving familiar numbers whose solution is paradoxically challenging: later, during her presentation to the class, she tries to persuade her classmates that they could do what she had done

Second, the challenge Cherise has created for herself goes well beyond the teacher's assignment, although it is clearly responsive to it. Instead of simply creating subtraction problems that can be solved by counting on the thermometer - problems involving numbers between 40 and -40 - she has set herself the challenge of writing problems involving numbers greater than 1000 that can nevertheless be solved with first-grade number facts The fact that she has created this challenge without external suggestion or compulsion makes her enterprise to some degree voluntary, to use Huizinga's terms, even though it is done in the context of a school mathematics assignment

Cherise invites her classmates into a realm that is outside of the familiar world of school mathematics, one that seems to appeal to her imagination. She is trafficking in numbers that are doubly 'outside of real life': some are so large as to be almost inconceivable to an eight-year-old, while some are unimaginably far below zero (2000 - 3000 = -1000). When we remember that Cherise does this work in the context of a thermometer representation (and when she attaches the degree marks to the non-zero numbers in 2°000 - 3°000 = -1°000, she refers us back to that representation), the use of the four-digit numbers seems particularly striking - and particularly effective as an appeal to the imagination. [5]

Cherise is offering 'rules' that can help children to navigate in this territory Her move helps us to see a less-obvious function of rules in play: they help novices to operate in a new realm Actually, the rule Cherise gives her classmates as a guide to making the new territory less intimidating oversimplifies what she knows: she tells them that they can ignore all zeros – "you don't even need to worry about the zeros" – although in fact, she has, in her second example, 7000 – 10000 = –3000, ignored *only* the last three zeros in each number. Had she followed her own rule literally, she would have treated 10000 as 1000 and ended up with an incorrect answer.

Cherise's enterprise also illuminates the social aspect of play: her purpose seems to be to entice others into joining her in a mathematical territory of which she thinks they may be frightened. Her vocabulary is that of invitation and reassurance: foreseeing "that people will look at this and say, 'Oh, no, I don't know this. This is too big a number'", she reassures: "But it is really just 1 take away 2"; "you don't even need to worry about the zeros." Repeating phrases like 'people would worry' and 'they will say "this is too hard"', she offers understanding of their fears and a helping hand across the intimidating territory.

Her enterprise here seems deliciously playful: both her words and her demeanor indicate that she is having a good time and that she is eager to have others join her. Huizinga argues that participating together in the play world creates such strong social bonds that communities originating in shared play often endure long after the game is over. Watching Cherise, I realize that recruiting playmates is a fundamental part of all play.

Finally, and this point must remain a conjecture, it seems to me that Cherise was trying to communicate the aesthetic qualities of her discovery, the way in which there was order in these two alternative realms and that this order both surprised and delighted.

Cherise's presentation suggests two further points about mathematically imaginative play. First, adults may have difficulty both in recognizing it and in seeing its value. At the time it was happening, Cherise's presentation confused me. I had an image of how contributions to the mathematical discourse would go: I expected children either to raise a question about an earlier presentation, present their own work on the assigned task – another child had put eleven problems on the board – or offer a conjecture. I expected the linearity of 'work' [6]. Instead, Cherise produced an invitation into an unfamiliar intellectual playground. My confusion provides, I think, a clue that Cherise was stepping outside the regular discourse. Her broad smiles suggested that she was stepping into the realm of play.

Second, a close examination of the behavior of Cherise and her classmates during this sequence of events suggests that, like other play worlds, the world of mathematically imaginative play is fragile, vulnerable to the unthinking intrusions of well-meaning adults – like me. When I ask Cherise to say her idea as a conjecture, the intellectual and emotional climate of the discourse instantly changes: children call out anxiously, "How can you cross out the zeros?" and "I don't get it!" Having been calm, though occasionally restless, the children suddenly seem anxious and upset.

The *content* of students' protests suggests that at least some of the children suddenly felt a greatly intensified pressure to understand Cherise's idea. It is as though they have been interpreting Cherise's contribution as an invitation to intellectual play and are responding in that spirit, not yet worried about whether they completely understand her idea (or whether it is true) – until my talk of a 'conjecture' convinces them that it is instead a serious business for which they will in some way be held 'accountable'.

Revisiting Vygotsky: why do we care whether children play in mathematics?

But even if we agree that there are some interesting correspondences between the realms of play and mathematics, what can we accomplish by encouraging play in mathematics? The first reason for encouraging such mathematical play is implicit, I think, in my observations of my first-grade students' progress in reading: just as children who enjoy reading and do it for fun develop fluency and skill more quickly than children who read only when told to do so, children who play with number, probability puzzles or geometric entities will probably develop a fluency that comes with familiarity in the areas in which they play.

Fluency with number facts is a part of mathematics, even though it is only one part (NCTM, 2000). Learning to navigate in the territory of very large and very small numbers, to make sense of fractional relationships or to think sensibly about the probability of two events occurring simultaneously is partly a matter of experience and familiarity; the child who plays in and with mathematics develops familiarity through exposure and exploration.

I want to propose, however, a second, less obvious, argument for a playful approach to mathematics, one grounded in the ideas of Vygotsky. In addition to deepening our understanding of the relationship between rules and imaginary situations in play, Vygotsky helps us to see how imaginative play contributes to children's cognitive development, arguing that it is through play that pre-school children free themselves from the dictatorship of concrete objects and develop the capacity to behave in accordance with meaning.

Before they reach the age of three, Vygotsky explains, objects elicit children's actions: seeing a door, they seek to open it; finding a cup, they prepare to drink. In symbolic play, however, pre-schoolers give objects a meaning different from the common one – a shell may become a cup, a small pillow becomes a baby – and then act according to the meaning they have invented, drinking from the shell and feeding the pillow.

> It is here [in play] that the child learns to act in a cognitive, rather than an externally visible, realm, relying on internal tendencies and motives and not on incentives supplied by external things. (p. 544)

According to Vygotsky, play creates the intellectual space – the zone of proximal development – for development in the pre-schooler, for it is here that s/he learns to sever thought from the concrete objects in her or his environment.

> Thought is separated from objects because a piece of wood begins to be a doll and a stick becomes a horse. Action according to rules begins to be determined by ideas and not by objects themselves. This is such a reversal of the child's relationship to the real, immediate, concrete situation that it is hard to evaluate its full significance. The child does not do this all at once. It

is terribly difficult for a child to sever thought (the meaning of a word) from object [... In play,] a stick - i.e. an object - becomes a pivot for severing the meaning of horse from a real horse; one of the basic psychological structures determining the child's relationship to reality is radically altered. (p 546). [7]

During the school years, play may be able to serve a parallel function in the child's learning of mathematics. In the early years of schooling, when we are introducing children to work with numbers and shapes, we try to situate mathematics in the concrete objects that are familiar to the child. Doing this helps children to find their own ways into mathematics, to use their experiences of counting, etc , to make sense of the symbols they find in primary mathematics

The importance of grounding early mathematical work in children's experience with concrete objects cannot be overestimated: until and unless children make these connections they must rely on adults to *tell* them what holds in the world of mathematics; once they begin to see ways in which mathematical symbols connect to and represent objects and actions that are already familiar, they are able to figure much out on their own, to make new connections within mathematics

To the adult mathematician, however, mathematics is not primarily a way of representing relationships in the real world. Rather, it is a system of ideas about idealized mathematical entities. Some of what holds in mathematics does so because mathematics is an internally consistent system. Thus, a negative multiplied by a negative is a positive not because this is the most illuminating way of describing what happens when a bank subtracts money from a debt, but because if it did not much else in the alternative world of mathematics would not work the way we wanted either.

Ball (1995) has shown that this aspect of mathematics can and should be made accessible to elementary-age children. I want to conjecture that just as play helps pre-schoolers to free meaning from the concrete objects that dominated their thinking and behavior as toddlers, play can provide the school-age child with a venue for approaching the more abstract side of school mathematics. Following the language and logic of Vygotsky, I suggest that mathematical play might be a place for freeing mathematical meaning from dependence on cupcakes, thermometers and other familiar objects

We see this in Cherise's presentation. She and her classmates have begun their exploration of integers with a concrete representation of this new mathematical territory: a thermometer which extends to -16°. This representation works well for the children as long as they confine their explorations to subtracting one positive integer from another; it may also enable them to subtract a positive integer from a negative integer. It will serve less well should they try to use a negative integer as the subtrahend and will probably prove entirely useless for explorations of multiplication.

Cherise's play with the numbers carries her some distance from the thermometer representation into a realm of numbers which seem almost magical in their power and in their remoteness from a third-grader's daily experience.

Features of Play (Huizinga, 1955)	**Exploring integers in elementary school**
Play is voluntary, *free, "never a task"*	Although school mathematics is rarely completely voluntary, there is actually, for elementary students and their teachers, a kind of freedom in this territory because few 'high stakes' achievement tests include questions about negative numbers. Because teachers rarely feel pressure to be sure that their students have mastered a particular set of facts and skills related to the integers, some say they feel freer to let students explore and to enjoy their discoveries than they do when they are teaching subtraction, fractions or even geometry.
Play creates a separate, alternative world outside of the real, a *"stepping out of 'real' life"* outside of the immediate satisfaction of wants and needs.	The territory below zero is a separate world for elementary students. It is outside the 'real' world of natural numbers - numbers that are in daily use both inside and outside of school. The fact that this territory is literally a mirror image of the territory above zero may make it particularly and intriguingly separate. In the territory below zero, like Narnia or Middle Earth, some rules hold because they must if the whole - the real number system and the operations defined in that system - is to be coherent and consistent.
Play "creates order, is *order*. In consequence, it is beautiful"	The orderliness of integers is especially visible because the numerical territory below zero mirrors the more familiar world of natural numbers. The correspondences between the familiar and unfamiliar parts of this world can seem beautiful - and exciting
There is an element of *tension* in play that is created by the fact that the player is trying to accomplish something.	Trying to make sense of and operate within this new territory does create a kind of tension. The intriguing combination of the familiar and the strange may make it particularly possible (and attractive) for children to challenge themselves.
All play has *rules* and "[t]hey determine what holds in the temporary world of play ... Indeed, as soon as the rules are transgressed the whole play-world collapses" (p 11).	The 'rules' for operating in the territory below zero and in the larger territory of the integers do indeed create the 'play world'. Indeed, history suggests that negative numbers could be made to 'work' only when the effort to see them as a metaphor for or representation of some sort of reality was abandoned and they were reconceived as purely formal entities

Integers as a territory for mathematically imaginative play

During her turn at the chalkboard, Cherise shows her classmates that they can use what they are familiar and comfortable with – subtraction involving single-digit numbers – to operate with numbers that are outside of their experience She uses the familiar vocabulary – "thousand" is a common *word* in the third grade, even though children this age have little experience with *quantities* of this size – as a bridge to exciting mathematical territory

To use Vygotsky's very apt term, Cherise invites her classmates to use familiar number facts as a 'pivot' for operating with numbers that are well beyond their experience. She invites them to move from the concrete world of third-grade arithmetic to more abstract mathematical possibilities.

Consciously to create occasions for mathematically imaginative play will require us to think about *what* mathematics is taught as well as how. We will need to introduce children to areas of mathematics that invite particular sorts of mathematical play. The study of integers and the introduction of negative numbers may create a particularly fertile ground for the cultivation of mathematical play of the sort I would like to encourage.

The territory of negative numbers

The idea of numbers below zero did not easily gain acceptance among Western mathematicians (Smith, 1923/1958; Barrow, 1992; Hefendehl-Hebeker, 1991): they noticed the *possibility* of negative numbers long before they accepted the legitimacy of such numbers and used them in their work

The idea of extending the number line in such a way as to include numbers below zero arose as early as the third century BC, when Greek mathematicians encountered problems with potentially negative solutions. The idea did not, however, take hold: around 275 BC, Diophantus, in a work on arithmetic, wrote about an equivalent of the equation $4x + 20 = 4$ that it was 'absurd', because in order to satisfy the terms of the equation, x would have to equal -4 (Smith, 1923/1958, vol. 2, p. 258).

For the next eighteen centuries, Western mathematicians appear to have shared Diophantus' skepticism. And even after negative numbers were 'rediscovered' around 1500, they played to mixed reviews: the sixteenth-century mathematicians who did write about numbers below zero referred to them by a variety of disparaging names Some stuck with Diophantus, labeling the new numbers *numeros absurdos* and the numbers above zero *numeros veros* – "true numbers" Others called them 'defective' or 'deprived' numbers. [8], [9] Although skepticism dwindled as the decades passed, it did not disappear.

In 1759, the English mathematician Baron Francis Masères wrote of negative numbers as solutions to equations:

> they serve only, as far as I am able to judge, to puzzle the whole doctrine of equations, and to render obscure and mysterious things that are in their own nature plain and simple (cited in Kline, 1972, p 593)

And even in the mid-nineteenth century Augustus de Morgan, a distinguished mathematical logician and algebraist, asserted:

> The imaginary expression $\sqrt{-a}$ and the negative expression $-b$ have this resemblance, that either of them occurring as the solution of a problem indicates some inconsistency or absurdity. As far as real meaning is concerned, both are equally imaginary, since $0 - a$ is as inconceivable as $\sqrt{-a}$. (cited in Kline, 1972, pp. 592-593)

De Morgan argued that a person only got a negative number as the solution to an equation if he (or she) had set the equation up incorrectly. He illustrated this point with the following problem: if a father is 56 years old and his son is 29, when will the father be twice as old as the son? Defining x as the number of years that must pass before the son's age is half the father's, he sets up the following equation:

$$56 + x = 2(29 + x)$$

He solves the equation, and gets $x = -2$ as the answer. [10]

A modern high school student who was comfortable with negative numbers might explain that this means that the father had been twice as old as the son two years earlier: de Morgan, however, declares this solution "absurd". He observes that the difficulty could have been avoided by phrasing the original question differently, asking *how long ago* the father was twice as old as the son, letting x equal the time elapsed since this was true, and writing the equation as:

$$56 - x = 2(29 - x)$$

Having made this change, he obtains the more satisfactory answer $x = 2$ (Kline, 1972, p 593)

Particularly troubling was the notion that a negative multiplied by a negative yields a positive product. The novelist Stendahl (1783-1843), who as a young man took a great interest in mathematics, devotes considerable space in his autobiography to the difficulty that this notion gave him and the frustration he felt, at age 14, when his teachers failed to provide any satisfactory answers to his repeated questions about how this could be so. He finally concluded:

> It *must* be that minus times minus is plus After all, this rule is used in computing all the time and apparently leads to *true and unassailable* outcomes (cited in Hefendehl-Hebeker, 1991, p 27)

Negative numbers simply do not 'work' very well as a metaphor for everyday activities. They do not help second graders to describe transactions with stickers, sandwiches or even money (although many of us have tried to create a convincing representation of adding a positive number to a negative by talking about the addition of money to a debt). Most children would agree with Baron Masères that the invocation of negative numbers in such a context:

> serve[s] only [...] to render obscure and mysterious things that are in their own nature plain and simple.

Streefland (1996) has shown some of the conceptual difficulties that arise from the "false concretization" [11] of various European curricular representations of negative numbers – witches stirring pots containing hot (positive) and cold (negative) cubes, small trains heading in opposite (positive and negative) directions and capable of running both backwards and forwards (a negative times a negative, etc.).

We can perhaps hear echoes of Streefland's concerns in Cherise's efforts to direct her classmates' attention away from a representation of integers that emphasizes their ordinariness As she invites them to invent problems like 2000 - 3000 = -1000, she seems to be suggesting that the imagination might be more helpful to an effort to understand negative integers than a thermometer could be.

It was only after Hermann Hankel in 1867 extended the number system to include complex and negative numbers, doing away with the idea of magnitude and completing the transition from a concrete to a formal number system, that the idea of numbers less than zero was fully accepted. Instead of making reference to concrete objects, Hankel derived fractions mathematically from natural numbers and the operations of multiplication and division He built up the negative numbers in a similar way, arguing that:

> the condition for the construction of a general arithmetic is that it be a purely intellectual mathematics detached from all intuition, a pure science of forms in which what are combined are not quanta or their number images but intellectual objects to which actual objects, or relations of actual objects, *may* but *need not*, correspond. (cited in Hefendehl-Hebeker, 1991, p 30)

In Hankel's formulation, the numbers so created were not discovered but invented (here we hear echoes of Cardan's *numeri ficti*). And although his formulation removed the need for the new numbers to correspond to or serve as a metaphor for some real-world phenomenon, Hankel insisted that the new formal system must avoid internal contradictions and also not conflict with formulas that hold in the world of natural numbers The negative integers were derived from the natural numbers, using the equation $x + n = 0$: for each natural number n, there is a negative number x such that $x + n = 0$ (Thus $x + 3 = 0$ gives us $x = -3$.)

Daniel's construction of '-Pat' parallels Hankel's construction of the negative integers. Just as Hankel derives the integer -3 from the equation $3 + n = 0$, Daniel derives the mathematical entity -Pat from the equation -Pat + Pat = 0, creating an 'intellectual object' without any obvious counterpart in the visible world. Like Hankel, Daniel apparently felt a need to push past the concrete and actual towards a more abstract formulation of the meaning of the negative sign

Conclusion

I have argued that mathematically imaginative play can contribute greatly to the learning of mathematics in elementary school, that mathematics educators can easily overlook the meaning and value of playful contributions to classroom discourse and need, therefore, to attune themselves to the faces and value of play I also argued that it would be worthwhile for mathematics educators to put effort into learning more about what it might take to foster more of this sort of intellectual activity in elementary classrooms

John Dewey (1916/1966) reminds us that play differs from work not in being aimless - players have goals - but in the *type* of goal players have: in work, the goal of action is "the production of a specific change in things"; in play, the goal is "a subsequent action" (p 203). It is for this reason, perhaps, that play is often the precursor to invention: activity pursued for its own sake, or for the sake of creating a venue for further explorations, permits more exploratory moves. "Play", writes Dewey, "is free, plastic" (p 203)

Cherise's actions open up more possibilities for future action than do those of the children who simply do the assignment and the mathematical actions she points her classmates towards are playful ones. Play in the territory she opens up puts the third graders in a position to invent mathematics their teachers have not thought to lead them towards.

Because play is free and plastic, mathematically imaginative play can create multiple points of access: it can provide a way for a student who finds an assignment relatively easy to create a new challenge or investigate - to play with - a mathematical idea that is not formally up for discussion - as it did both for Cherise and for Daniel. It may also, as I have argued above, help elementary students to experience the more abstract side of mathematics, to experiment with mathematical ideas that are not tied firmly and directly to familiar operations involving everyday objects. And because players need playmates, it may mobilize children's pedagogical impulses in a way that ordinary mathematics assignments often do not

Cherise's effort to persuade her classmates that they too can write subtraction problems involving very large whole numbers is fundamentally an educative one; it is education undertaken for the purpose of sharing play. We may have many reasons, it seems, to value and cultivate mathematically imaginative play It will do us relatively little good, however, to cultivate mathematically imaginative play *unless* we also make a point of studying the faces of this play: as we saw from my journal description of Cherise's time at the board, an adult who is accustomed to a relatively straightforward sort of discourse can miss the value of mathematically playful ones

We do not, at present, know very much about the circumstances that invite mathematically imaginative play. However, I have suggested that the mathematical territory of integers may be particularly hospitable to this sort of activity precisely because it does *not*, in fact, map particularly well onto familiar activities with everyday objects The very features that created difficulties for earlier generations of mathematicians can ignite children's imaginations The looking-glass world of below-zero numbers, I have asserted, shares many of the defining features of play that Huizinga identified more than half a century ago.

Ball (1995) has compared the appeal of zero to her third-grade students' imaginations to:

> the appeals of magical characters in fiction they read and write. (p. 675)

Many teachers see something similar in their student's investigations of the mathematical territory to the left of zero on the number line. Putting students in a position to enjoy the pleasures of this sort of mathematical play could be a way to invite them into the heart of the discipline of mathematics.

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Notes

[1] The edition I have used for all the citations is the 1955 Beacon paperback edition

[2] To illustrate this point, Vygotsky analyzes the behavior of two little girls who have agreed to play "that we are sisters" What follows clearly involves an imaginary situation (even though the girls really *are* sisters), but it is also, Vygotsky explains, rule-governed: the behavior of the little girls is now governed by the rule that, for the duration of this play episode, they can do only what sisters would do. Indeed, as we eavesdrop on their conversation, it seems that they do only what emphasizes the peculiarities of the sisterly bond Thus, they hold hands; the older admonishes the younger, "that is theirs, not ours". In play of this sort (and in the sort described in Vivian Paley's accounts of her pre-school and kindergarten classrooms), the rules emerge from the imaginary situation. Similarly, Vygotsky argues that all play involves an imaginary situation; he offers chess as an example. Here the rules - that the rook travels in straight lines, the knight can jump over other pieces, etc - create an imaginary battleground in which each piece *always* follows particular strict rules In this imaginary world, the combatants never break the rules. As Huizinga said: "Into an imperfect world and into the confusion of life, [play] brings a temporary, limited perfection" (p. 10)

[3] See also Jacques Hadamard's (1945) classic *The Psychology of Invention in the Mathematical Field*

[4] Poincaré's delighted invocation of 'disorder' here, coupled as it is with his references to the central role of the mathematician's aesthetic sense in selecting fruitful combinations, makes me revisit and reconsider Huizinga's claim that order and beauty are inevitably linked and that the beauty of play is an outgrowth of its orderliness We might take Poincaré's description of his own moments of invention as being about harvesting order from naturally-occurring disorder

[5] Egan (1992) writes persuasively about the ways in which extremes of all sorts capture the imagination of children of this age. The size of Cherise's numbers seem clearly to do this, and because the class has talked earlier in the morning about how very cold -20° is, adding the degree marks in the third computation intensifies the effect.

[6] Karl Groos (1898), in the first of two early and important books on play, quotes the German philosopher Grosse who argues that play occupies a middle ground between art and work Grosse suggests that it may be helpful to imagine art as a circle, play as a spiral, and work as a straight line.

[7] Vivian Paley pursues this line of thought in her accounts of play in her kindergarten and nursery school classes She argues that play is the place where children examine large subjects together - like jealousy, fairness and little sisters - that for pre-schoolers play can function in the same way that conversation functions for their elders: it creates a place and a means for pooling and examining ideas on a subject of common interest In play, moreover, the implicit rules of interaction require the participants to step back and examine the subject 'at one remove': Paley (1986) describes a scene in the doll corner in which three-year-old Molly, pretending to be a baby, speaks of her little sister Leslie - and is politely ignored by her four-year-old playmates.

> Molly doesn't know that real babies, the ones at home, are not mentioned in the doll corner The whole point of pretending to be a baby is to review the subject one step removed, in the abstract. (p 35)

Paley provides persuasive evidence that imaginative or 'make-believe' play is the place where the kindergartner is an intellectual. "In play", she quotes Vygotsky as saying, "a child is above his average age, above his average behavior; in play it is as though he were a head taller than himself" (p xiv). Her books both support and develop Vygotsky's formulation that play is the leading edge of development We might say, perhaps, that play is the place where young children are accustomed to examining that which is not concrete and actual In play, children *construct* a 'what if' or an 'as-if' world Like the mathematical world, the world of play is tied to the observable world in multiple ways (Vygotsky says that in the beginning symbolic play is almost a pure imitation/repetition of reality; imagination comes later); children playing mother and baby take cues and material from the interactions of mothers and babies that they have observed. But in play, children dig more deeply into a subject, beyond what they have actually seen They 'make believe' in the sense that they construct the situation and the world in which it is embedded. They reinvest this world with meaning

[8] One of the few exceptions was Fibonacci who, in 1225, interpreted a negative answer in a financial problem to mean a debt - rather than a mistake or absurdity.

[9] See Smith (1923/1958). Stifel, writing in 1544, spoke of zero as placed between *numeros veros* and *numeros absurdos*. Cardan, in 1545, adopted a different and intriguingly modern terminology, calling the new numbers *numeri ficti*. *Ficti*, the past participle of *fingere* (and root of the English word *fiction*), means "molded" or "created"; this terminology emphasizes the degree to which these numbers are a human construct - and, by implication, the extent to which mathematics is invented rather than discovered.

[10] Toom (1999, p. 38) writes about this problem in the context of arithmetic word problems.

[11] He borrows this rather helpful descriptor from Freudenthal (1991)

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