Communications

The right to be wrong

I enjoyed the piece by David Fielker entitled "Observation Lessons" in the February 1990 issue of FLM. He dealt sensitively with a number of issues that transcend the topic of observation lessons — an important teacher training strategy inspired by Gattegno in the 1950's I was particularly impressed with his discussion of the need for greater subtlety in handling the concept of student participation, as well as his suggestion that an important ingredient in the education of teachers is sensitizing them to the distinction between fact and conjecture as they observe children in action

Of the many interesting issues he discusses, one in particular strikes me as in need of further exploration, for it has deep implications for the nature of dialogue in mathematics classes. Apparently teachers who reacted to his observation lessons were angered by the fact that, in his role primarily as a moderator rather than judge, he did not correct errors at the time they were made in class.

Fielkner wisely points out that there are many good reasons for his not correcting errors in the classroom in general. They have to do with such issues as the locus of responsibility for learning, the nature of deductive thinking (the tracking of subsequent errors to original ones made), and the establishment of a climate for critical thinking

There are however, important issues derived from his discussion of the handling of errors that are left unspoken He tells us that it is rare that students do not locate errors before the end of the lesson and he suggests that it would be valuable for teachers who observe him to think of the subtle cues he gives off that assist students in discovering their errors.

What does it say about the nature of our curriculum and about the kind of dialogue we pursue in our classes if teachers are *angered* by colleagues who do not sound the alarm whenever errors are made by their students? What does it say about the kind of question teachers and students explore if essentially all errors are eventually detected by the end of a period?

Errors and mistakes become an important category primarily when the concepts of correct, right, proper, and true dominate our curriculum concerns. I fear that our present international pre-occupation with problem solving in the curriculum may have the ironic effect of subtly entrenching some of these values to the exclusion of others The dominance of these categories in mathematics classes suggests that important parts of one's emotional and intellectual education are being neglected.

What are the kinds of issues, problems, situations, questions that might legitimately be explored in the context of a mathematics classroom for which the concept of "being mistaken" or "committing an error" might not apply at all?

One obvious (only so because I have thought about it for 20 years) candidate is that of student problem generation. Given a conjecture, a physical object, a situation, a definition, an hypothesis, a theorem or whatever, one can invite students to pose problems or ask questions about the phenomenon Though the concept of "better" or "more interesting" prevails with regard to problems posed by students, and though one might wish to educate students in creating problem posing strategies, the concepts of "right" and "wrong" do not apply in problem posing as they do in mathematical activity which focuses upon proof, propositions, procedures, and even problem solving

Another candidate is one that focuses on personal and idiosyncratic dimensions of the mathematical experience Take the case of any mathematical concept that has been learned at some point by a student — either something that has been constructed by the student or merely "received" What kinds of devices does the student use to enhance retrieval when it has to be used at a future date? That is, what kinds of associations, visualizations, special cases, metaphors do our students create and hold in mind to enable them to recall important characteristics of the concept? Remembering and retrieving are important intellectual acts, both given short shrift when they are inaccurately associated with mechanically memorizing. It is a very narrow notion of mathematical thinking which precludes such personal yet intellectually challenging acts as remembering and retrieving. Once more, however, these are not categories for which "right" and "wrong" apply in a comfortable way

Another interesting category that may not be mathematical thinking *per se* but that surely impacts powerfully on the mathematical experience is that of the emotions. Look back at Fielker's article In about a half a dozen different places, he mentions that students or teachers were *angry* with him because of some mathematical or pedagogical act he performed in an observation lesson or in his teaching more generally. Though he deftly discusses with both audiences the issues that generated the anger, at no point does he tell us that he acknowledges and discusses the phenomenon of their anger *per se*.

Does it not seem surprising that teachers are "angered" by the fact that Fielker neglects to correct errors committed by students at the point they are made? Why were they "angry" rather than (for example) "surprised" or "intrigued"? Would it not be worthwhile for him to discuss these matters with the teachers? Would it not be worthwhile to discuss explicitly with students the contexts that seem to generate emotional reactions to mathematical ideas? Might it not significantly affect how students do mathematics if they were to be made sensitive to the emotional climate that affects their learning? Are these not once more categories that transcend the bounds of "right" and "wrong"?

We as teachers and teacher educators do not have ready made strategies for dealing with categories of teaching mathematics that defy the concept of error. Even when we are clever enough to pass the mantle to our students (as Fielker does), we tend to reify dialogue that further acknowledges such a perspective on the nature of mathematical thinking.

Before we create strategies for an expanded design, however, we need to be aware of what the terrain is within which these strategies might apply. Borasi's 1987 essay in FLM ("Exploring Math Through an Analysis of Errors") is an interesting start, for she inquires into how we as teachers might think of errors as an invitation to explore rather than to correct. That is surely a valuable part of the story that needs further analysis Here I am raising a different question however: What is there of intellectual and personal worth that we ought to be doing in mathematics classrooms for which the concept of error (regardless of how imaginatively we deal with it) makes little sense? I may be "in error", but I believe that I have suggested a few elements of that terrain I will wager that fellow readers of FLM can do a good job of expanding — if not challenging or modifying — the essential issue I am addressing.

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A mathematics lesson

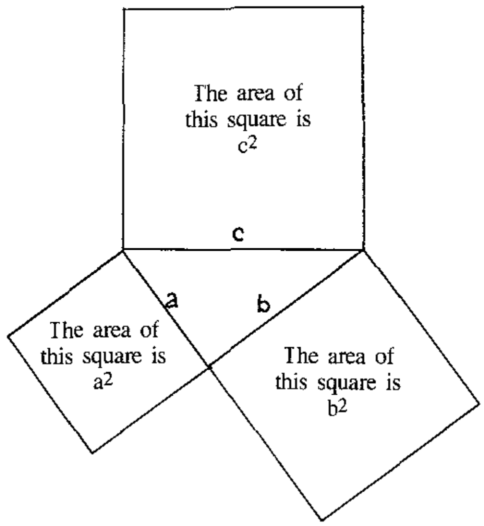
I used to address groups of girls, aged 15 or 16, who were to be persuaded to stay in mathematics Here is a description of a typical session

I tell the silent, uptight group that I am not their teacher, that I am not going to subject them to a test, and that one can't do mathematics unless one feels like a member of a posh club about to be served cognac and cigars A few of the girls titter That is a good sign.

Their teacher told me that they know Pythagoras' theorem. This makes me ask them the following question:

> You are given two carpet squares. Can you *always* find a third carpet square whose area is the sum of the areas of the given squares?

After a while I take a vote. The result is refreshingly democratic Some vote "never", some vote "always", and some vote "sometimes yes and sometimes no" Then I ask the group if they know Pythagoras' theorem The answer is "$a^2+b^2=c^2$". Without commenting on the answer I hint at the fact that each algebraic term can be thought of as the area of a square Also, there are three squares in $a^2+b^2=c^2$ and three squares in my original question. Can they bridge the gap? By this time a few make the great leap from the squares in the equation to the squares in the original question Eureka! *Given two carpet squares there is always a third carpet square whose area is equal to the sum of the areas of the given squares.* It turns out that some things are not settled by a vote. Now some of the girls have discovered the remarkable connection between the algebraic and geometric formulations of Pythagoras' theorem *Now* some of them have discovered what Pythagoras' theorem is "really" about We spell it out



$$a^2+b^2=c^2$$

I ask: is Pythagoras' theorem obvious? They do not know what to say I tell them that it is remarkable and nonobvious and that these conclusions can be reached only by struggling with a proof I tell them that it would be wonderful if they tried to find a proof and that there are some 150 proofs of Pythagoras' theorem.

To give a significant application of Pythagoras' theorem I ask them to help me answer the following question:

> Given two sticks. Can one always find a third stick that fits a whole number of times, say p, in one stick and a whole number of times, say q, in the other stick?

I ask for a gut reaction. By now a few trust me enough to risk an answer These few say "yes" I tell them that my own gut reaction is also "yes" but that together we'll show that *sometimes the answer is "no"*. Can they exemplify the "yes" answer? They can and do. Now we'll exemplify the "no" answer

Take one stick to be the side of a square and the other its diagonal. Suppose that some third stick fits p times in the side and q times in the diagonal. By Pythagoras' theorem, $p^2+p^2=2p^2=q^2$ with p and q whole numbers. We quickly agree that every whole number greater than one can be written in *just one way* as a product of primes (we don't distinguish between, say, 2.3 and 3.2). Now comes a series of questions in "slow motion" Is it clear that the number of 2s in the square of a whole number is even? Yes, it is.* Is the number of 2s in $2p^2$ even or odd? It is odd. Is the number of 2s in q^2 even or odd? It is even. Does this contradict the uniqueness of the representation of a whole number as a product of primes? Yes, it does. Can we then conclude that no stick fits a whole number of times in the side of a square and its diagonal? Yes, we can. We have proved that there exist segments having no common unit of measure.** In this way we have also shown that our gut reaction is false.

Could the length of the side of a square be given by a fraction a/b and the length of its diagonal by a fraction c/d? If this were the case, then we would have $(a/b)^2+(c/d)^2=2(a/b)^2=(c/d)^2$, that is $2a^2/b^2=c^2/d^2$. But then $2a^2d^2=c^2b^2$. This is the same as $2(ad)^2=(cb)^2$. Since ad and cb are whole numbers, we are back to the case disproved earlier.

I tell the girls that *this discovery,* first made by some Greek mathematicians around 450 BC, *persuaded the Greeks that numbers — to them the equivalents of our positive fractions — are not adequate to all tasks of geometry.* Having demoted numbers they relied on geometric proofs even in situations that we now think of as quintessentially algebraic (here I give geometric proof of the identity $(a+b)^2=a^2+2ab+b^2$). What was so bad about this? For one thing, products of more than three factors were declared meaningless and the evolution of algebra, and thus also its use in geometry, was delayed for centuries. So much for the lesson (whose description is far slicker than was the real thing. Now for some "didactic" comments.

The two key elements that "make or break" a lesson are its mathematical content and its "metamathematical" content. The mathematical content must be intellectually respectable and must contain an element of surprise. The metamathematical content is harder to describe.

I am not always the teacher and the students are not always students; sometimes we switch roles. Some students can sometimes teach me mathematics but all students can teach me how I can best teach them. I try to win their trust and form a partnership with them. I try to teach them to be critics. I try to prove to them that they have unsuspected powers of reasoning and insight. I try to help them experience the sense of awe associated with the contemplation of an unpredictable mathematical insight. Last but not least, I share with them the "existential axiom" that mathematical prowess is as nothing compared with a person's "prowess" as a decent human being. Coming from a teacher of mathematics this is a surprise.

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* There are zero twos in the prime factorization of an odd whole number greater than one, and zero is even.

** Such pairs of segments are called incommensurable.

Some thoughts on the use of Instructional Technology in schools and colleges

This paper has been written in an attempt to raise issues for discussion. Particular, yet probably partial, solutions will be found in institutions where such questions are being addressed. In an environment where change is inevitable and ever-accelerating those working solutions will be reviewed at regular intervals. Can we develop strategies for review which depend on a re-interpretation of issues, rather than the preservation of a past working solution?

- Some students will probably be able to use certain software packages and particular machines more confidently than their teacher.

As teacher, how does this make you feel?
As another student, with less experience and confidence, how does this make you feel?

Accepting this situation, what strategies can be developed for an individual teacher to become comfortable with not knowing everything? What strategies for creating a culture where sharing and co-learning are accepted?

There are present concerns here about the individual student whose confidence and speed in working at a keyboard leads them to assume control of the technology and gender issues. Where resources are scarce this is a major concern, but what might the implications be when each student is presented with, say, their own graphical calculator?

The issue is related to the one of control of the mathematics by the teacher. Where a group engages on a question raised by a student, it is likely that some methods and ideas for solution proposed by the individuals in the group will be strange to the teacher and to other students in the group. The security of the teacher can lie in simply telling.

- Students cannot use a pair of compasses unless they know that such an instrument exists. On arrival at my secondary school I was the owner of a geometrical drawing kit as requested by the school. I had explored uses of the compasses but in class we did not do anything exciting with them for some time — e.g. drawing "accurate" lines, compared to my "flowers". I remained mystified by the set-squares.

What does this story make you remember? What would be the similarities and differences between this story and the question of when and how to introduce students to particular software packages which they could use as problem-solving tools?

More or less the same geometrical drawing equipment was the standard issue for decades. Students had access to the equipment before anyone had formally taught them how to use it. Later on log tables and/or slide rules were added to the toolkit, usually with some instruction first. Now it is increasingly likely that no one in the classroom will know that a particular new development, which would be extremely useful for the problem being worked on, actually exists. Any individual in the group might, through parents, relations, the media become aware of its existence and then share.

More importantly, currently there is a range of programs such as spreadsheets, databases, graph-plotters ... which are quite general tools for problem solving. Which package is to be used? When should students be told of their existence if they are not already aware of them? Should every student be able to use such tools in application to a problem? What packages are due on the market that could be equally useful, e.g. algebraic manipulators?

Should each student be working with a "state-of-the-art" hand-held programmable, graphical and algebraic and ? calculator from an early age? from 11? from ??? Is it possible to ensure that all students do not explore buttons which have yet to be discussed?

- Learning to use a word-processor for the first time cost me quite a lot in terms of time, memory and worry. After I had made the transfer to using a number of other word-processors effectively I had begun to know what to look for with a new one. So, initially my attention was placed in particular syntax and key-operations, often learning by rote. Then in transferring to other packages I became aware of similarities and differences, working on what a word-processor actually does for me. Now, on needing to use a word-processor and finding that only one which I have not used before is available, I know what questions to ask, what things I need to know, and make the transfer relatively quickly and painlessly

Many years ago the "drive chart" for a program was considered to be necessary for study before using the program itself. Using *Derive,* an algebraic manipulator on a Nimbus, I was recently struck by how little I needed to know. Certainly if this had been the first program I had ever used the story would have been different, but there has been a lot of work done on the "user-interface" of programs. The manuals, instead of including complex instructions for getting programs to do their task, can now give case studies of the sorts of problems and situations the programs have been found useful for.

A possible model for use by a teacher considering how to introduce a particular tool such as a spreadsheet might be:

1 What context shall I use?

This could be the offer of a problem which the teacher knows is suitable for solution using the tool and, if no student suggests using it, the teacher can share a method of solution. Alternatively, exploring the environment of a package such as a spreadsheet or, say, *Cabri-géometrie,* can lead to questions being formulated by the students and worked on. Alternatively a problem can be introduced through the package ...

2. Particular

Given this particular package how do I get the program to run? If the program uses a "windows" environment and I have used this before then I will feel comfortable, but there will be specific things to learn for everyone — these will just differ in kind dependent on previous experience with computers. If I have used a spreadsheet before, but not this particular one, I may need to learn very little

3. General

What can this program do for me? As experience using the program to solve a range of different problems is built up it might be possible to identify classes of problems for which the program is powerful. Then, on encountering such a need, it is possible that a student could independently reach for the tool.

When operating at the general level (3), given a context (1), using a new particular (2) tool becomes easier since it is known what questions to ask. This seems to give a paradigm for acquiring the flexibility to transfer between different manifestations of the same application software. Just using one package may not allow me to be so clear about what the general class of programs is useful for.

Two possible activities to encourage sharing and co-learning:

- Set aside time at regular intervals — once a week?, once a fortnight? once a half-term? — for the group to share the ways in which a particular piece of software has been used by the different individuals in the group.

This first activity can prepare the ground for the second

- Explicitly invite students to bring to the attention of the group new developments, uses of programs, new programs, new problems as they meet them

The regular times become unnecessary, but can be brought back at any stage if the group so desires, and the teacher can choose to share new tools which seem not to be in the larger culture.

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Papuans are almost always right

A native of Papua New Guinea is reported (see the reference in Christine Keitel's article in FLM Vol. 9, No. 1) to have told how the roughly rectangular gardens of his village are measured by adding length and width. He had learned the white man's method of multiplying when he was at school, but at home he would always add.

What happens when the Papuan compares two rectangular gardens? How often will he rank these in the same order of size as the white man?

Consider a field, a units long and b units wide, and another field, x units long and y units wide, which will compare with the first differently by semi-perimeter than by area. This means that

$$\text{either} \quad x+y > a+b \quad \text{and} \quad xy < ab$$
$$\text{or} \quad x+y < a+b \quad \text{and} \quad xy > ab$$

The values of x, y corresponding to these constraints yield points in the shaded area below. Within an N by N boundary this area is A + B where

$$A = (a-b)^2 + 2 + b(a-b) - \int_b^a \frac{ab}{x}\,dx$$

$$= (a^2 - b^2) + 2 - ab \log a+b$$

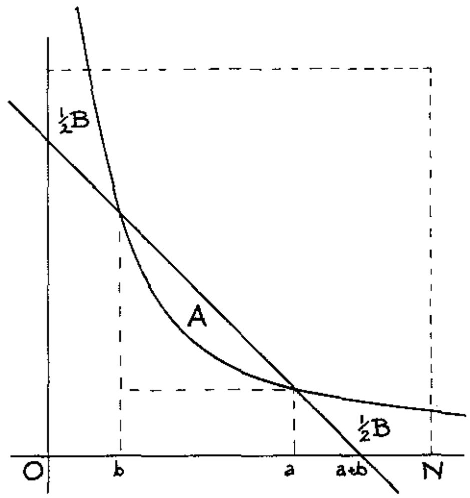
$$B = 2\int_b^a \frac{ab}{x}\,dx - 2b^2 + 2$$

$$= 2ab \log N+a - b^2$$

The probability that Papuan calculation of size by semi-perimeter fails to give the same comparison as calculating by area is

$$(A + B) + N^2 = ab(\log N^2) + N^2 + O(1 + N^2)$$

This tends to zero as N tends to infinity. In other words, the Papuan is "almost always" correct in his use of semi-perimeter rather than area to compare rectangular gardens



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