STAGES IN CONSTRUCTING AND COORDINATING UNITS ADDITIVELY AND MULTIPLICATIVELY (PART 1)

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Aiden can consistently count up to 20 blocks, but he cannot count 9 blocks when 3 are hidden. Several months later, he can do both, but he cannot determine how many blocks were added when he starts with 6 blocks and ends up with 9 blocks. Emma easily finds one fourth of a cake, and then easily divides that fourth into a third to get her piece. However, she is not sure how many of her pieces make up a whole cake. A year later, she knows it takes 12 pieces, but now she is not sure how the size of the original cake would compare to a cake that is 15 times as big as her piece. Why are students frustrated for months or even years by tasks that seem similar to ones they have already mastered? What changes in their thinking allow them to suddenly solve the new task (and a host of others, no doubt)?

In this two-part article [1], I lay out a framework of how students develop their ability to construct and coordinate arithmetical units, which explains precisely why Aiden and Emma have these difficulties and what changed in order for them to resolve them. One can undoubtedly come up with some hypotheses about Aiden's ways of operating with numbers that would explain why these tasks are difficult for him and, indeed, the framework is built on Steffe and colleagues' early research into the development of counting (*e.g.,* Steffe & Cobb, 1988). As a researcher into student learning in the middle grades (ages 11–14), I have been surprised by how often seemingly unrelated difficulties dealing with fractions, negative numbers, algebraic notation, and combinatorial reasoning can all be related back to the students' counting schemes with whole numbers [2].

The framework I present in this article is a synthesis of decades of research by numerous researchers into children's construction of number and fractions. Over time, theory has been refined, and new constructs, such as levels of units and multiplicative concepts [3], have been introduced. The aim of Part 1 of this article is to clarify the current theory behind the first two stages and disambiguate several constructs and terms that have been used in a variety of ways either within or outside this line of research. In Part 2, I characterize the final two stages of constructing and coordinating units and describe the insights they provide into additive and multiplicative reasoning.

Overview of the stages of coordinating units

The theory of unit constructions and coordinations is focused primarily on characterizing the quantitative complexity a student can work with. Before delving into my characterizations of the unit types and how students can coordinate them, I will summarize the theory behind how students increase the amount of quantitative complexity they can work with. I will then disambiguate the use of several key terms in this article from related uses by other researchers.

The development of new mathematical complexity

At first, students often need some sort of *figurative* material to help them keep track of an added complexity in their mathematical constructions. For example, when a student first starts attempting to track how many times he has counted by 4, the student might need to use his fingers. Next, students are able to make these mathematical constructions without resorting to figurative material. For example, the student might be able to mentally keep track of the number of times he has counted by four, thinking, "1, 2, 3, 4 (1); 5, 6, 7, 8 (2); 9, 10, 11, 12 (3); ..." During this phase, the student is said to work with the added complexity *in activity.* After students gain experience working with the given level of complexity in activity, three different, intertwined processes begin:

1. Students shortcut some of their mental operations so that they no longer mentally run through all of the mental activity they previously did. For example, the student might now skip some of the counting activity in keeping track of counting by four: "4 (1); ...8 (2); ...12 (3); ..." (see "condensation" in Sfard, 1991).
2. Students are able to reflect on their mental operations during or after activity. For example, the student might be able to mentally replay counting by four and say things like, "When I got to 16, I had counted by four 4 times. I had to count by four once more to get to 20."
3. Students reflectively abstract a link between their mental operations and the result of the operations (activity-effect relationships; see Simon, Tzur, Heinz & Kinzel, 2004), allowing them to anticipate the results of their mental operations to varying extents. For example, the student might think, "If I count by four 3s instead of four 4s, then I'll end up with a number that is four less because I'd have counted one less each time."

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The result of these processes is that the new level of quantitative complexity becomes *assimilatory*. This means that students can make sense of that kind of quantitative complexity before operating in a new situation.

For an example of how assimilatory constructs affect our mathematical reality, consider your own assimilation of the following situation (adapted from Hackenberg, 2013):

> In a classroom, there are four rows of desks with six desks in each row, and then three more rows are added.

We immediately comprehend that there is some resultant total number of rows that can be partitioned into original rows and added rows, and, simultaneously, that there are related numbers of seats in any of these collections of rows. In contrast, a student who can assimilate with very little quantitative complexity might think only about a multiplicity of seats or a multiplicity of rows, but not both at the same time. That student would need to use figurative material or mental re-presentations to actually draw or imagine seats one at a time in order to form rows and make sense of the situation. A student who can assimilate with more quantitative complexity might understand the situation in terms of some number of seats that are laid out in some number of rows and might know that he or she could figure out the number of seats from the number of rows and vice versa. However, this student might need to make sense of adding additional rows by carrying out mental addition or drawing on paper. Thus, characterizing the amount of quantitative complexity students can assimilate is useful for understanding how they experience the world mathematically. In the stages of unit construction and coordination, quantitative complexity is characterized in terms of the types of units students work with and the relationships students construct between the different types of units.

Units and unitizing

Later I will give a technical answer to the question, "What is the nature of a unit?" Here, let us consider more generally what I mean by the term "unit".

We often think of units in the context of measurement—for example, a unit of measurement. In the earlier stages of measurement, students might think of a unit of measurement as telling them what they are counting by, in order to enumerate the size of something. For example, an inch is one thing we can use to enumerate length. A meter is a different unit we can use to enumerate length. A second is a unit we can use to enumerate elapsed time. However, in order to conceive of being able to use a number to describe the magnitude of one of these quantities—length, elapsed time, mass, *etc.*—a child will use their counting numbers to find how many items are in a collection. For smaller collections or well-known spatial patterns, we might be able to find out how many items are in a collection by subitizing (see, for example, Glasersfeld, 1982; Sarama & Clements, 2009), but for most collections, we do it through counting. In this sense, the first unit that a student constructs is the one that allows them to measure the number of items in a collection. It turns out that children can learn to go through the motions of counting before they have a real sense that counting can be used to answer the question "how much". Therefore, by addressing what I mean by a unit in the next section, I will be specifying when I think that students do have a sense of their counting unit of 1 item. Before this stage, Steffe (2010) calls students pre-numerical [4].

In order to discuss the development of units, I will discuss a generalized and generative process of abstracting out the "one"-ness from some aspect of experience. Glasersfeld (1981) dubbed this the *unitizing* operation. In later stages, this is the operation a student uses to chunk together 5 items to form a unit of 5. It is a unit because the student can now count by 5s and figure out how many times one counts by 5 to get to some multiple of 5. However, in its earlier forms, it would consist of a baby chunking together part of their sensory information to form an object, like a cup. As such, Glaserseld is using the term *unitizing* in a much more general way than many mathematics educators (*e.g.,* Lamon, 2007, 2012; Richardson, 2012) in which *unitizing* refers only to grouping or regrouping (or chunking and rechunking) large numbers of items into equal-sized groups (or chunks). That use of the term *unitizing* is a special case of Glasersfeld's use.

Type, level, and stage

There are several hierarchies of mathematical constructions at play in the stages of unit constructions and coordinations. In order to keep some of these separate, I will vary the terms I use to refer to elements of the different hierarchies. The first is a hierarchy of units students construct (see Glasersfeld, 1981). I will refer to these as *types* of unit, although each of the later types of units is in a one-to-one correspondence with a stage of coordinating units (see Table 1 on p. 6). The next two hierarchies—number sequences (see Steffe, 2010) and multiplicative concepts (see Hackenberg & Tillema, 2009)—correspond to (subsets of) the *stages* of coordinating units in Table 1. These number sequences and multiplicative concepts are referred to as stages in order to indicate that they represent developmentally significant milestones that have far-reaching consequences in the numerical activity of the student (see the discussion of *stage* in Steffe & Cobb, 1988, pp. 7–8). The fourth related hierarchy is the number of layers of embeddedness in the composite structures that a student is working with. I use *levels* to refer to these layers, so that *three levels of units* implies that a student is embedding units of 1 (the first level) in composite units (the second level), which are in turn embedded in units of units of units (the third level). The ability to work with a certain number of levels of units is associated with each of the stages of coordinating units, but the term *levels* is meant to focus on the different types of units involved and their relationships, as opposed to a stage in the student's development. In the past, researchers have used the phrase *levels of units* to signify the number of previously distinct composite units that have been multiplicatively coordinated (Hackenberg & Tillema, 2009), which is a subtly different usage.

Construction of interiorized units

What is the nature of a unit? A unit is fundamentally a tool of measurement. A unit in this context is measuring cardinality (see Thompson, Carlson, Byerley & Hatfield, 2014). While babies have been found to have some sense of *more* or *less* (Starkey, Spelke & Gelman, 1990), a baby has not constructed what I call *units*. However, Glasersfeld (1981) theorizes that the same general abstracting operation, which he named the *unitizing operation,* underlies early object formation in the first year of life and the formation of arithmetical units years later. In a 1981 paper in the *Journal for Research in Mathematics Education*, he lays out a theory for how its recursive use leads to number concepts. Here I will give a briefer outline of that development, making links to related constructs where appropriate.

The general function of the unitizing operation is to abstract out a sense of unitary cohesion from experience. For example, a baby would initially experience visual sensory data as a relatively undifferentiated mass. Eventually, the baby would isolate specific experientially bounded segments of the visual field, caused by attending to changes in color, for example, and would start to form recognition templates for specific visual objects: a mother's face, a toy, *etc*. However, using the unitizing operation is not the same as creating arithmetical units. In particular, the child will at first apply the unitizing operation only to perceptual material. Also, the results of early unitizing are not arithmetical in that they are used to produce ordinary items of experience, such as toys, cookies, or chairs, as opposed to determine the cardinality of a set. The stage at which I say that a child can construct (*one level of*) *units* is when the child has constructed *arithmetical units,* which results from students' reflective abstraction of their counting activity (Glasersfeld, 1981). I call this the first level of units because, once a child has constructed an arithmetical unit, the child can use it to count anything, not just that which is perceptually salient, and, most importantly, the counting acts have themselves become items of reflection, opening the possibility that the child can monitor the cardinality of his own counting acts.

Both Steffe (2010) and Glasersfeld (1981) described a type of unit that the child would have constructed prior to the arithmetical unit, namely, the *perceptual unit.* Children who are limited to the construction of *perceptual units* when counting need to physically experience items, such as a handful of marbles, the chimes of a clock, or the beats of their hearts, in order to count them. More simply, they require perceptual aids in order to create units. The next type of unit both discuss is *figurative units.* A figurative unit item is a mentally re-presented item—often paired with a perceptual item, such as a finger—that a child has constructed to stand in for perceptually unavailable items when counting. These re-presentations are representative of the specific perceptual items, hence the term *figurative* (see Table 1). A child at this stage is still pre-numerical: a number word does not yet refer to a specific cardinality before the child starts counting. Instead, a number word serves as a signal to carry out a counting procedure that ends with the utterance of that same number word:

> Because the child is aware of figurative plurality, a child at this level can give meaning to a number word by producing visualizing [*sic*] images of hidden items. This extensive meaning of the number word remains indefinite, and to make it definite, the child has to actually count. (Steffe, 2010, p. 34)

Therefore, I do not consider the construction of figurative units to reflect the construction of (*one level of*) *units.*

In order for the child to construct arithmetical units, they apply the unitizing operation to a record or to a re-presentation of a counting act associated with counting figurative unit items, and so the countable nature of the counting act itself is abstracted from the countable nature of the figurative unit item. Note that this goes beyond mentally re-presenting and reflecting on an activity, which is called *internalization* by Steffe and Cobb (1988) and *interiorization* by Sfard (1991). The re-unitizing of the re-presentation of counting acts leads to what, following Steffe and Olive (2010), I will call the *interiorization* of records of counting. Interiorization involves the reprocessing of an activity. Once the activity is interiorized, it is available prior to being carried out. Thus it can be treated as given and acted upon or used to make sense of another activity. In this case, interiorized counting acts will eventually lead to the ability to curtail counting activity. However, more basically, the mental counting act (as opposed to re-presented figurative material or physical movements associated with counting) will eventually serve as material to be operated on further by the counting scheme. The interiorized records of counting are called *arithmetical unit items* (Glaserfeld, 1981; Steffe, 2010).

Here I will give an example from Steffe's (2010) research that exemplifies the conditions that engender the behavior that characterizes the initial construction of arithmetical unit items: a student named Jason is attempting to count the union of a hidden collection of 8 cookies and a hidden collection of 10 cookies. He counts as he makes 8 pointing actions, presumably pointing at re-presented figural unit items, and then monitors himself counting 10 more figural unit items while forming a spatial pattern with his pointing actions. In doing so, he apparently constructs a *numerical pattern* for 10. When counting, he does not simply use spatial patterns to re-present images of cookies, which would qualify as counting figural unit items. In the midst of counting on 10 more, he pauses, after having counted on by only 8, presumably to reflect on the results of his previous counting acts. He seems to be aware, not of having counted eight re-presented cookies, but of the pattern of his 8 previous counting acts themselves. He then uses these abstracted records of counting to determine that he stills need to count twice more, does so, and arrives at the answer of 18. In monitoring his counting actions, he is taking the *counting act* of figurative unit items as an object of reflection, which leads to *interiorization* of counting acts. It is possible for this interiorization of counting acts to occur initially only in action. In this case, the child would be aware of monitoring counting acts only *while* counting. In such a case, the records of the arithmetical unit items would decay so that the arithmetical units that were formed during counting would no longer be available for reflection after counting.

When the arithmetical units are formed during counting but decay, I would say that the child can construct one level of units but not use these units in assimilation. This is because, like Jason, the child makes temporary modifications in his counting scheme that allow him to start counting each individual count *once he is in the process of counting*; the child cannot plan or anticipate that he will be able to count his individual counting acts before he starts counting and form the goal of keeping track of his counting behavior. Similarly, an adult might experience a temporary modification when having a mathematical insight if the insight quickly fades away, and it becomes necessary to think through it again from the beginning to experience the insight once again. In the same way, the arithmetical unit items that the child constructs fade away (decay) before the child interiorizes them.

Once the child makes a permanent modification to his counting scheme, the child is *aware of counting acts* as countable items *before* counting commences. The child now has an anticipatory meaning for 8, for example, as referring to the cardinality of counting acts before counting commences. At this point the child is assimilating with (one level of) units in that the child can use units to make sense of a mathematical situation before operating on the situation. Pinpointing the precise time when students go from assimilating situations with figurative unit items and constructing arithmetical units in action to assimilating situations with arithmetical units is quite hard using behavioral indicators because, in either case, children are able to form countable items so that they do not have to count re-presented perceptual unit items in order to make sense of a number word. For example, Steffe gives examples of two students who count the union of two covered collections, but he attributes the use of figurative unit items to one and arithmetical unit items to the other based on subtle differences in behavior (Steffe & Cobb, 1988).

Coordinating two levels of units

Coordinating two levels of units requires that there is a new level of unit. This new kind of unit is called the *composite unit*, which is the result of unitizing a subsequence of a child's number sequence and abstracting out the cardinality as a property separate from the particular location on the number sequence that has been unitized: if *four* is a composite unit, then it describes counting "1, 2, 3, 4" or counting "83, 84, 85, 86." A composite unit of four also allows the student to count how many times he counts by four when counting from 1 to 20, but only in action. In this section, I outline the development of composite units and I describe the simplest coordination of two levels of units in which a student uses units of one to count the number of times a composite unit has been applied.

Just as monitoring is an important instigator of the reprocessing necessary to form arithmetical units of 1, monitoring counting acts instigates students' construction of composite units. As before, monitoring involves experientially or attentionally bounding a subsequence of counting acts in order to keep track of its cardinality. However, when constructing composite units, the subsequence as a whole would be interiorized in addition to each arithmetical unit item that comprises it.

The first step of interiorization of a subsequence of counting acts is internalization, allowing a child to "run through a counting activity and produce its results in thought without motor action and without given sensory material to act on" (Steffe, 2010, p. 35). This mental activity can then be condensed so that saying the final word in the counting sequence stands in for counting the entire subsequence, resulting in an *initial number sequence* (INS, see Steffe & Cobb, 1988). The INS is an initial number sequence in two senses. First, it is the result of the first (initial) numerical counting scheme. Second, the initial number sequence is characterized by the fact that each number word can stand in for the interiorized acts of counting from 1 to that number word (Olive, 2001), *i.e.,* an initial number sequence. For example, when figuring out how many cookies you get when you add three cookies to five cookies (no cookies are actually present), the student may count on from five, "5...6, 7, 8." When a child has not yet constructed an INS, they would still have to count from 1, so to figure out how many cookies you would have if you add three cookies to five cookies (when no cookies are actually present), the child might count, "1, 2, 3, 4, 5...6...7...8." In other words, it allows students to *count on* instead of *counting all.*

Despite this noticeable advance in the child's counting scheme, monitoring when counting on remains problematic for INS students because their initial number sequences are only used when counting from 1. When counting on (starting from a number other than1), INS students still rely on figurative material, including spatio-motor patterns, in order to keep track of when to stop counting, similarly to Jason's example above. They can, at most, experientially bound the subsequence of counting acts.

Eventually, in their attempts to attend to the cardinality of non-initial subsequences, the student will attentionally bound them, opening up the possibility for application of the unitizing operation to the subsequence as a whole, not just the individual counting acts that comprise it. This results in a composite unit, which reflects the records of attentionally bounding, or marking off, a subsequence of a given cardinality. This ability to produce stable composite units, means that a student remains aware that the subsequence associated with each number word is contained in, or marked off as part of, the next largest number word. This represents a reorganization of the student's INS into a *tacitly nested number sequence* (TNS, Steffe & Olive, 1988; see Table 1). The TNS is called *tacitly nested* because students are tacitly aware of the presence of one number sequence in another.

Another byproduct of reorganizing an INS into a TNS is that what used to be initial number sequences are now understood to be equivalent to non-initial subsequences with the same cardinality. Therefore, the child can utilize these initial number sequences to monitor counting-on activity. For a younger TNS student this would simply allow them to count on without figurative material. For older, advanced TNS students, this leads to more explicit *double counting,* For example, in response to the task, "How many pennies need to be added to a pile of 7 pennies to get a pile of 16 pennies?" the child might say, "7,…, 8 is 1, 9 is 2, 10 is 3, 11 is 4, 12 is 5, 13 is 6, 14 is 7, 15 is 8, 16 is 9…9!" As illustrated in Figure 1, when double counting, a TNS child is utilizing

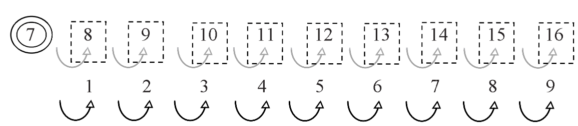

Figure 1. Double counting.

one number sequence to assimilate the units of another number sequence. Although there is an attentionally bounded composite of items being counted, neither the attentionally bounded subsequence, {8, 9, …, 16}, nor the assimilating subsequence, {1, 2, …, 9}, is necessarily unitized to form a composite unit of 9. Therefore the student is *coordinating* two sequences of units, but the units being coordinated are all of the same basic type; there is not a new kind of unit structure being formed, nor are there necessarily two different unit structures being coordinated. Therefore, the student is only coordinating with one level of units.

Furthermore, this counting on can be used in new ways. For example, an INS student could not independently solve a missing addend problem such as, "I have 7 marbles in this cup. Now I am adding some more. I now have 16 marbles in this cup. How many did I add?" because they have to form non-initial subsequences in activity. They cannot posit a hypothetical subsequence from 8 to 16 and form the goal of enumerating it. A TNS student can. Nonetheless, the TNS student, once again, only needs to attentionally bound the subsequences in activity in order to find the solution. Therefore, solving a missing addend task still involves coordinating number sequences, not two levels of units.

Finally, TNS students can also monitor their use of composite units. For example, given the expression, 7 × 4, TNS students could interpret this as a request to keep track of how many times they applied composite units of four: "4 [puts up a finger]; four more is (5, 6, 7,) 8 [puts up a second finger]; four more is (9, 10, 11,) 12 [puts up a third finger]; […the student continues in this way….] four more is (25, 26, 27,) 28." In particular, students would know they need to apply a composite unit of 4 more than once, would keep track of how many times it has been applied [5], and would stop once they had counted seven instances of 4. It is important to note for the later discussion of multiplicative reasoning that while such students are solving what we would call a multiplication task, a student probably interprets the question as asking what number you end at after seven instances of counting four times. Therefore, for the student, this is a counting problem, and the student is not aware of a multiplicative relationship between 4 and 28.

Nonetheless, the way the student is solving this problem represents an important shift in quantitative complexity in that the student is using arithmetical units of 1 to count the application of composite units and is thereby *coordinating these two levels of units in activity* (See Table 1). This is also the first case of what is often simply referred to as *units coordination,* distributing the units of one composite unit across the elements of another composite unit (*e.g.,* Steffe, 1992). Hackenberg and colleagues have developed a hierarchy of three multiplicative concepts (MC1, MC2, and MC3) that characterize a student's multiplicative thinking. MC1 is defined as students who can carry out units coordination in activity (*e.g.,* Hackenberg & Tillema, 2009). Therefore, the ability to coordinate two levels of units in activity is equivalent to the construction of the first multiplicative concept (MC1).

While the coordination of two levels of units in activity is an important milestone in the development of quantitative complexity, a TNS student will still experience significant constraints in dealing with added complexity because of the nature of the units of a TNS. These units still contain records of counting and are experienced by the student as *equivalent*, in that each represents one counting act, but they are not experienced as *identical*: the location of a unit in the number sequence is still very salient for TNS students, as indicated by the arrows in Figure 1.

Conclusion to Part 1

The stages of constructing and coordinating units represent a way of characterizing mathematical thinking that focuses on the nature of the units a student can use to make sense of a mathematical situation. Some of the major steps in this process that I have explored in Part 1 are the development of arithmetical units and composite units to reach the first two numerical stages of this framework. In Part 2, I discuss the re-interiorization of both of these constructs to form the iterable units of one and iterable composite units that correspond to the last two stages of this framework. So far I have shown that the composite unit opens the way for double counting and a coordination of two levels of units in activity. Iterable units will open the way for more complex constructions and coordinations, including multiplicative reasoning.

Notes

[1] Part 2 will be published in the next issue.
[2] Related research on fractions includes Hackenberg and Tillema (2009), Norton and Wilkins (2012) and Steffe and Olive (2010). Related research on negative numbers includes Ulrich (2012). Related research on algebraic notation includes Hackenberg and Lee (2015). Related research on combinatorial reasoning includes Tillema (2013).
[3] Both of these stem from Steffe's (*e.g.,* 1992) construct, *units coordination.* For the use of levels of units, see Boyce (2014). For the use of multiplicative concepts, see, for example, Hackenberg (2013), Hacken-

Number sequence	New types of units	Multiplicative concept	Arithmetic coordinations
Pre-numerical	Perceptual items Figurative items	None	None
INS	Arithmetical units	None	None
TNS	Composite units	MC1	Double counting: use one number sequence to assimilate another in order to count on or solve an additive comparison task.
			Coordination of two levels of units in activity: use one number sequence to explicitly count composite units in order to solve multiplicative comparison tasks.

Table 1. Correspondences between unit types and various frameworks presented in Part 1.

berg and Lee (2015) or Hackenberg and Tillema (2009).
[4] In fact, Steffe does not call them numerical until they are assimilating with what he calls numerical composites of these units, corresponding to the initial number sequence described in the next section (*e.g.,* Steffe, 2010).
[5] I do not mean to imply that a student would be aware of applying a composite unit but rather that the student is attending to the number of instances of counting by 4.

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To show the presence of the contradictions may be all that is required by a mathematician if he is to be justified in saying that the notion *x* is a mathematical impossibility—it may, that is, be a conclusive demonstration of its impossibility—but the force of calling it impossible involves more than simply labelling it as 'leading to contradictions'. The notion *x* involves one in contradictions and is *therefore* or *accordingly* an impossibility: it is impossible *on account of* the contradictions, impossible *qua* leading on into contradictions. If 'mathematically impossible' meant precisely the same as 'contradictory', the phrase 'contradictory and so mathematical impossible' would be tautologous—'contradictory and so contradictory'. But this will not do: to say only, 'This supposition leads one into contradictions or, to use another equivalent phrase, is impossible', is to rob the idea of mathematical impossibility of a crucial part of its force, for it fails to draw the proper moral—it leaves the supposition un-ruled out.

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