The Edge of Platonism

DAVID HAWKINS

Imre Lakatos, *Proofs and refutations: the logic of mathematical discovery.* Cambridge University Press, 1976
Philip J. Davis and Reuben Hersh, *The mathematical experience.* Birkhäuser Boston, 1980
Philip Kitcher, *The nature of mathematical knowledge.* Oxford University Press, 1983

The first of these three works has been much discussed. An early version of it appeared some twenty years ago; the author died young, before the Cambridge edition was completed. It is a small essay, narrow in scope and wide in the range of its implications. The second is a whole collection of essays in and about mathematics by two mathematicians who are both well informed and reflective about the history and philosophy of their discipline. The book is rich in detail, and for illustration it samples well from the kingdoms, phyla and classes of what is all, somehow, mathematics. The third book is by a philosopher who wants to bring the discussion of the nature of mathematical knowledge into a more fruitful relation to its own history and to that of the sciences generally, when studied with serious concern for the understanding of conceptual and methodological change. Its argument is in a style which you will recognize if you have followed the same philosophical paths, not easy for the outsider.

I bring these three books together not for detailed review but for reference along the way. The way I wish to go is to open up, for discussion, some questions about the relevance of the philosophy of mathematics to mathematics teaching, and on early mathematics learning in particular. I shall not break fresh ground in the philosophy, though I'll try to crack a clod or two.

Philip Kitcher wants to take a philosophical position which has for a long time been the most scorned, the empiricism of John Stuart Mill. Convinced as he was that all knowledge comes by way of inductive generalization from sense experience, Mill was then also committed to the belief that the subject matter of mathematics is simply the world of nature: different from physics or biology not in kind, but only in degree of generality. The only alternative for such an empiricist is that of Formalism, to which Kitcher gives little attention. Davis and Hersh give it more, particularly in its logical positivist version. Their criticism is neat and effective, and could be a text for several sermons. For any strict formalist mathematical statements are empty, their truth value is only that of tautology. Mathematics is only a language for describing phenomena — of physics, for example. Formalism drops out everything except the formalized end-product, and therefore also drops out the working mathematician and all the great world of mathematical conjecture, argument and discovery.

At the opposite pole is the metaphysics called Platonism. The term is commonly used without much attention to the writings of Plato, who never became his own disciple and thus cannot properly be called a Platonist. But we are stuck with the isms; Plato did formulate this one, and his dialogues hover around it, in quite unforgettable ways. But be careful. He was a dramatist of ideas, and sometimes the dramatic line is not soberly to answer the questions raised by his cast of characters; it can be, rather, a kind of insobriety which only deepens the questions and blocks the easy answers. "Courage," says the old soldier, "is standing your ground." Before the dialogue is over courage has been successively redefined as "the knowledge of hope and fear." That seems a puzzle, but the dialogue leads to it. It is a mathematician's kind of answer. It seems far from the initial conjectures, but if you define courage that way it tightens all the arguments.

I mention this aspect of Plato because it puts Platonism in a more interesting light than any pat definition can manage. I think we have to recognize a strong family relation, for example, between this Platonic dialectic and that which dominates the dialogue of Lakatos. If you study his *Proofs and refutations* and then look carefully at some of the dialogues like *Gorgias* or *Hippias Minor* or parts of *The Republic* the parallel is clear. Despite all differences in subject matter there is an implicit structure, implied by the ways our ideas engage each other, in ethics or mathematics. By naive formulations, examples and proofs, counterexamples and refutations, reformulations et seq., we can hope to see these ideas as part of a larger and more coherent structure, transformed beyond our initial grasp of them, still familiar but, for the new context embedding them, strangely so. In the one case the virtue of courage gets embedded in a context of the modes of knowing, in the other the elements of polyhedra reappear as structures within a vector algebra. In both cases there is residual doubt as to whether the translation has not left something out. Davis and Hersh discuss Lakatos' analysis of the method of discovery and provide us with their own case-history illustrations. They set this account in the context of George Polya's rich and many-sided work on heuristics, where it should be.

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Plato shows his talent for mathematical heuristics in the *Meno,* with the story of the slave boy, and gives us a taste of the metaphysics. The teaching story is splendid, and the metaphysical story is told to block easy explanations of the slave-boy's perspicacity. [1] In some of the dialogues he treats it lightly, as a way station in the dialectic of proofs and refutations. Sometimes he seems to have convinced himself and takes it, without his customary irony, for truth. The whole material world, which we inhabit and learn about through sense experience, is patterned architecturally after a prior world of Ideas (Forms) which our souls were privy to before born in their earthly bodies. Bathed prior to birth in the waters of Lethe, of Forgetfulness — Plato's version of the birth trauma — the soul has now forgotten the Forms. It then knows not that it has known them from before. We do indeed learn from sense experience, but this learning is to be conceived in a way different from that of acquiring new empirical facts, it is more like the achievement of *re*cognition, of reminiscence.

Let me elaborate this, on my own. Against a background of routine experience, a ceaseless commerce, novelties appear and evoke some self-conscious searching, of the outer environment and of the inner store. The store is at least implicitly taxonomic. When not automatically responded to, the external experience leads to some file-searching, finding similarities and differences, match and mismatch. Things get filed, retrieved and refiled in similar or altered ways, and something like labels, categories, addresses get associated with its compartments. Sometimes inquiry is focused on what is externally problematic and we seek to characterize it more adequately. Retrieval then is mainly instrumental. In the search however we sometimes retrieve two or more partial patterns and find they link together, in previously unnoticed relation. Sometimes when things have been stored together their similarity is to be questioned and they are re-stored apart. Lumping and splitting. Ideas have valencies. Sometimes they bond together and sometimes stubbornly refuse.

If all this search, retrieval, application and restoration is initially instrumental to an inquiry which is externally oriented, there can also be a reversal of means and ends, context and content. Now it is the patterns retrieved which are problematic, and what is externally perceived has suggested, or been sought as providing, instances, examples to assist reflection. In the first case the file is used and, in the process, it spontaneously evolves. In the second case this evolution is intended and enjoyed, transiently at least, for its own sake, generating the same sort of tension and release characteristic of all inquiry. Subjected to this sort of inquiry the store can be more deliberately and more radically enriched, cross-referenced and sometimes reorganized.

Sometimes this latter kind of inquiry is provoked and assisted by a teacher, some more-or-less Socrates, supplying a stimulus and a kind of loop which the learner cannot as yet provide for himself. And when the bonding or separation of ideas reduces disorder — in a technical sense, redundancy — we enjoy some release of tension, some emotion of enhancement. However achieved, that is the kind of experience which Plato was intent upon, and which his drama or his metaphysics was intended to celebrate. I think him superbly one-sided, because he mostly neglected the other pole of inquiry, the piecing together of bits of factual information and the abstraction from it of new order to feed the store of ideas. For that emphasis among the classics you read Aristotle, who has been curiously neglected in the philosophy of mathematics. He holds as firmly as Plato to a doctrine of Forms, but he is an empiricist about it; the Forms belong to nature, and we learn them by abstraction from perceptual and practical experience.

My point, then, is that inquiry is always bipolar, stressed between the material environment and its furniture on one side, and on the other the storehouse of ideas, of prior conception and knowedge. In this matter I ought to have listed another writing besides Aristotle's: William James' *Psychology*. A hundred years ago philosophy and psychology had not yet parted, and James discusses a topic, a fact, about thinking which I believe is central to the issues reviewed here. He gets it from Kant, I think; but never mind, he gets it. In a chapter on Conception (12, v. I) he lays down what he called "the principle of Constancy in the mind's meanings". "The same matters can be thought of in successive portions of the mental stream, and some of these portions can know that they mean the same matters which the other portions meant." "The function by which we thus identify a numerically distinct and permanent subject of discourse is called CONCEPTION." "Every one of our conceptions is of something which our attention originally tore out of the continuum of felt experience and provisionally isolated so as to make of it an individual topic of discourse. Every one of them has a way... of suggesting other parts of the continuum from which it was torn, for conception to work upon in a similar way."

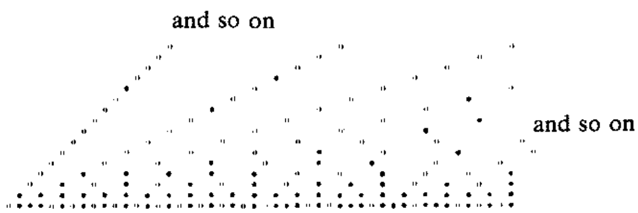
James' notion of conception is wider than our customary usage. The recognition of identity across time, the ability to come back to what we know or can recover as the *same* thing, in different contexts, applies both to the things of the world around us and also to the furniture of the mind, to ideas, "concepts" in our rather murky usage. I give one last quotation, in which James is now referring tò the latter domain: "Thus, amid the flux of opinions and physical things, the world of conceptions, of things intended to be thought about, stands stiff and immutable, like Plato's Realm of Ideas."

But why immutable? Surely ideas do change, with fresh experience and thought. No, says James, new ones appear, perhaps more adequate, and the old may be buried. But we still can retrieve the old and recognize it. We could not acknowledge the change unless that retrieval were possible. Like almost everyone else of his time and before, James was not a developmentalist; his subject is the adult human being. Fair enough. But the study of mental development gives us another track for examining his thesis. It was Piaget who first carefully observed (at least among the learned) and analyzed the game of peek-a-boo, the development of "object constancy" in infants, a prime example of James' principle. Transported back into the Jamesian context this suggests a wider developmental rubric, *concept* constancy. [2] It has to do with a shift from *using* ideas to *intending* them. The desk calculator uses the rules of arithmetic but does not intend them Children learn a string of nonsense syllables "yet, yi, sam, set, ... " which when well used through practice become ordinal counters and cardinal adjectives. Having learned to match these against discrete sets they can be said to use the numbers though not yet intending them. In traditional pedagogy one supposed marker of the shift from use to intention is that from the adjectival use to a nominal one: no longer three apples or marbles, but just "three". Indeed the logician Quine takes this shift as marking a new "ontological commitment". The properties of things are dependent on the things, instrumental to their recognition and control. But when we nominalize them — *the* number three — there is an existential quantifier involved, and, he proposed, some kind of ontological commitment that goes with it, Platonic or otherwise.

Whatever the commitment, there is a clear epistemological shift reflected in this verbal one, and it is precisely the shift from using to intending, the reversal of context and content I spoke of above. In this shift the Cheshire Cat's Grin can be intended, can be referred to, even though it cannot exist in nature without the cat. Shakespeare marks such a shift (in Troilus and Cressida): "... man ...cannot make boast to have that which he hath, nor feels not that which he owes, but by reflection; as when his virtues shining upon others heat them, and they retort that heat again to the first giver." Needless to say Ulysses was not talking about a child's arithmetic but about the performatory virtues which Achilles did not yet know he had; the key metaphor, of course, is that of the mirror, of reflection. In recognizing children's early reflective shift to the numbers, to make boast that they have them, I think we come back to James and to the edge of Platonism.

Years ago I noticed, as many have, a sort of behavioral marker of this shift, or at least of the last stage of it: concern about "the biggest number", and the final recognition that there isn't any; the first infinity. Practical counting doesn't get you there, a stop rule always works. I remember an image of my own (age 5 or 6), the numbers became fence posts stretching across the desert; but the string of them wouldn't end. You could start to count them, but the stop rule never worked. In her memoir (forthcoming) of work as a nursery school teacher Frances Hawkins tells a story of number play and exploration among a group of fours and fives. Questions and responses ran over the available range, from chairs, children, teachers — even the grains of sand in the sandbox. And here is one of her children, with a new kind of question: "When the numbers get to the end of the world, do they have to curl?" She comments on the danger of assuming that such momentary bursts of insight reveal a consolidation of understanding that is not there yet. They are often only leading-strokes, auguries of what may come. In Plato's allegory of the Cave the ability to make this sort of shift is called the art of turning about, from the shadow world of people and chairs and Archimedean sand-grains to the archetypal world of Ideas. Never mind the Platonic disparagement of nature; we get again the metaphor of reflection, in this case of turning about. In our normal outward preoccupations the domain of ideas is behind us.

Because of his empiricist commitments Philip Kitcher wants to get Platonism out of the way and goes to a good deal of trouble to do so. I agree with much of his affirmative position, but I think he errs in his understanding of Platonism, as "espousing the claim that mathematics owes its truth to some abstract *instantiation* of mathematical structure" (my italics), known by a faculty of intuition somehow analagous to visual perception as the source of empirical knowledge. He goes on then to say that this instantiation would be redundant, we have that already in the world around us. I can't vouch for all Platonists but Plato doesn't claim, and Quine doesn't claim, that the "ontological commitment" to universals involves *any* kind of instantiation; quite the contrary. To say there are *instances* of the numbers, or the shapes, or what-not universals, stored in some heavenly Bureau of Standards, is to suppose them still in the adjectival mode, and miss the whole point. Words like reflection and intuition imply some kind of seeing. My string of fence posts gave me a *monary code* or *image* of the number sequence; as an image it was portable and as a code it had the great virtue of instantiating each number it signified. But what I "saw" was not the archetype but the token of it. James I think uses the right kind of word: the universals of structure are not seen, not "intuited". They are and can only be signified, and *intended*. So when children think about counting the numbers, they are intending something which puts them in league with number theorists, who could now meet them as equals. For some older children, who can use multiplication as well as addition, there is the graph (which uses only the monary code) which goes as follows:



Professor Polya once showed me this graph, which is to be found in Leibniz. He said modestly he had never worked in elementary schools but he thought it would be good there. The rows mark the numbers additively, "counting by n's". The columns mark the divisors of m. Indeed it is good for elementary school children, and in Polya's words a "gateway into number theory". We have seen a good many children get hooked on it, from third grade to sixteenth. Numbers of divisors? Primes and round ones? Odd numbers of divisors? What happens if you put in a prior dot, for zero? Can you see any curves? If you carry it further they begin to stand out. A good programmer friend turned it sideways and we ran it out for thirty feet or so, with divisors on the printout up to $n^{1/2}$. Are all the curves you see parabolas?

When you work with children in and around such matters you want to be *both* a Platonist and at the same time a pretty rank empiricist. They already have the conceptual start, a potentially infinitary world But the fresh empiricism is vital. It has to do with the novel representation which gives you all those visual patterns. When you catch on to building that graph, you can do "factoring" in a twinkle. But it is the coordination of hand and eye which gives you all this, it does not come (not first) by "reflection". It gives rise to representations and transformations which both simplify and enlarge an area of inquiry. An odd number of divisors? A good many adults, working with a pegboard-golf-tee graph, generate the sequence 1, 4, 9, 16, . That is pure empiricism. Next, they notice the odd-number differences (though not the squareness!). This is pure empiricism. One got the idea from noticing that each such number (after 1) marked the tip of a parabola, thus "losing" a divisor. Immediately, then, the idea of a proof is in the air, and we have made the Platonic move; the empiricism has done its job and is in retreat. The didactic will say that the diagram is only for illustration, for example. But that's the backward view. For inquiry it is much more, the light shines forward: better words would be lustration, lumination.

All the concrete math materials have the virtue that they necessarily present what they represent, like the monary arithmetic code. Not just the nice manufactured stuff, the world is full of them. In their very diversity they invite analogies and suggest questionings. The *Erlebnis* which *Erkenntnis* presupposes is not less necessary here than elsewhere, but can accrue earlier, from all of any fortunate child's surroundings and activities since birth. Mathematics is notoriously a field where precocity can flourish, well ahead of high skill and insight in many other involvements. Nor does access to high-order abstraction have here to follow only in later years. Young children can sometimes intend ideas about which they lack, as yet, the means for reaching untuned adult ears.

So it is a bad consequence of uncritical Piagetianism to identify the developmental contrast between "concrete operational" and "formal operational" stages with the contrast between use and intention, manipulation and abstraction. Some such belief implies that because most children cannot early-on make much use of our abbreviated mathematical script and scribble, in place of their own more enactive and perceptual representation of ideas, that therefore they cannot think abstractly. Sometimes indeed their insight is only a leading-stroke, to be forgotten for a time. But we need no dogma which forbids us to recognize it in any of its beginnings.

Worse: we often fail to realize, ourselves, that much of our script has evolved from just such iconic representation as children can use. Children already skilled in groupings and regroupings, matching, ordering, iteration, and some measure intending such matters, are subjected typically to the logistic formalism as though *that* were the subject matter. Much research confirms what imaginative teachers have often understood, that schoolchildren so taught can quite fail to connect the standard script and rules with an already informed understanding. Our arithmetical scripture is only a second language.

I think this is no mere matter of pedagogy but goes straight to a philosophic issue. In arithmetic the tokens which first instantiate the numbers must, in all rigor, come first The numbers and the iterative successor operation can only be defined by what we call abstractive definition. It is only so that we can define the visual colors, for example, or identify the two senses of the helix, and more generally the primitives of any formal discourse. In a world which lacked collections of discrete elements (of reliable stability) arithmetic could not get started. Such collections and operations with them are brought literally into our discourse, with the help of demonstrative gestures or pronouns. We listen with Kant to the strokes of a bell or we use poker chips or Cuisenaire units. We can get back to zero with the null set, but Old Mother Hubbard's cupboard will do as well At this level set theory has no privileged status. Kitcher is good about all this, though for my taste with far too little effort to nail it down. He reminds us that there is a large variety of ways of "defining" the natural numbers within set theory. I would add that what they all have in common is that with this or that stipulation about what to notice, they all give us an instantiation, an ordered sequence of sets of typographic marks on paper, equivalent to children's marbles or Pythagorean pebbles. Any monary code will do.

Instantiation is only necessary, we must *intend* the univeral instantitated That is the point of Kant's unity of apperception or Brouwer's two-ing, //-ing And instantiation is left behind when we abstract *from* it the successor operation and all the rest. Conceptual definition takes over. We name the numbers (or better, address them) each by an operational code word which would, *if* carried out, instantiate (in the monary code) that number. So now it is clear, I think, just what is the "ontological commitment": the numbers are well-defined potentialities which we know about by abstraction from their realized instances and (to use an apt phrase of Kitcher) from the operations which nature allows us to perform. So that's my philosophic pitch: *Everybody's* understanding of arithmetic depends on the world's concrete math materials; not just children's

This need, I think, impose no limits of the sort which Brouwer advocated. Every number has a host of properties, shared with others but also distinguishing it uniquely from all the others. As in the old joke, every number is interesting. And this is another aspect of the ontological commitment. The properties of the numbers and the number system, once conceptually defined as a system of potentialities, are inexhaustible. Today we know that not all possible facts of number can be found by finite proof, even mathematical induction can fail. But it need not be the last resort; the very kinds of investigation which reveal its limitations may provide for some kinds of super-inductive argument. So I opt for mathematical realism, and this brings me back again to the edge of Platonism. There is nothing wrong with it unless we start to believe that there are attributes of numbers somehow of a higher logical order than that of their relations to each other within the system. The Pythagoreans and others later [see Davis and Hersh, p. 96 et seq.] got into that trouble in the elaboration of number mysticism and gematria. The physicists come near it with the "magic numbers", but the label is only a nice joke.

My second suggestion from Polya had to do with geometry. "Draw a cube," he said, "that is easy for children to learn" — plane parallel perspective. Can you draw a tent inside the cube? A house? When you are done you can erase the lines you don't need or that would be hidden if you couldn't see through. Add a bit of shading? What other shapes? Find the midpoints of each face and connect them round the belly and from these to those of top and bottom, six lines. Erase the cube's edges, and what have you got? Try going the other way to a shape with just four triangular faces. That is harder. Look at the other regular solids; can you get them as well, with just straightedge and compass? Age is a factor, for careful drawing; that is the only obvious prerequisite.

I won't elaborate, but Polya was right again. His topic links up with the real 3-D solids, homemade with card and glue, and with all kinds of 2-D patterns with tiles and drawings, figurate numbers, 2-D and 3.

Again instantiation is a first language. To work with children on such matters needs no special words, though some may well get used or invented along the way. The empirical initiative here needs no discussion. A fine teacher thought to breathe some life into the understanding of fractions. She challenged her ten-year-olds to divide the hexagon into congruent parts in as many ways as they could think of. So they quickly got halves, thirds, fourths, sixths, twelfths, . . , not all simple shapes by any means. Then hatching and coloring came in and fractions gave way to an extraordinary profusion of decorative design. Which fractions couldn't you get? Give a reason? Surely not an easy question. And where does the "couldn't" come from? Fifths? Thirtieths? You have immediately, perhaps, a kind of argument, the idea of a proof? Is this still empiricism? We all might take it so at first, and blindly try? Failure may be evidence, but not final. Where then to look? A bit of arithmetic, indecisive. Symmetry? Aha, maybe. And suddenly we are back to the edge of Platonism. And to Lakatos?

In all such work the numbers begin to play a part and it can sometimes be brought to center stage, geometry into number theory and then to measure, as with the Pythagoreans: figurate numbers, for example, and their sums led to areas and volumes. In a work like Euclid those tracks got all erased, but children can rediscover them. So now our teachers (and other adult friends) don't recognize the squareness of square numbers, and most advanced undergraduates can't sum them except by advanced methods. The coentanglement of arithmetic and geometry has a history far older and deeper than that of the reals, than Descartes and Fermat, than the nineteenth century synthesis. I, for one, do not see how we can give them any essentially different epistemological status.

I know that much of the kind of practice I've drawn my examples from is familiar to readers of this journal, and long practiced by them. Yet the practice is rare most everywhere, and children suffer because the raw empiricism of it is treated with fear or contempt, and its power to take children to what I have called the edge of Platonism goes unrecognized. I am afraid this often applies as well to those who teach teachers, despite the efforts of the happy few.

What can be added to those efforts, from the side of philosophy? Some general framework possessing philosophic dignity, of course. But also some instruction of philosophers and reflective mathematicians from the practice of teaching young children. To bring the present discussion to a close, I would like to return to emphasize the polarity, the tension of reflective thought, between the observation of what is problematic in the material environment and the recognition of what is problematic within the mindstore and its organization or clutter. What we call knowledge is a territory, wide or narrow, between two others, of ignorance, very different in kind. One is ignorance of nature and human affairs, the other of the relationships — initially implicit relationships — among the conceptions and categories which have evolved as empirical information has been many times stored and retrieved, addressed, returned and re-addressed, culturally stabilized through language and practice, but always reflecting, more or less, the generic orders of experience. We do not *observe* this kind of order, a literal intuition (seeing-in) is the wrong metaphor. Our eyes are in front, these structures behind. Hence "reflection," the mirror. "Intending" is better. Physical instantiation, imaging, word-patterning are what is perceived, but they "bring to mind" the structures of interest. We objectify them, externalize them, by "operations which nature allows us to perform".

I cannot soberly believe that one gets to any metaphysical Platonism by this way of thinking. I think it is a *kind* of empiricism but not that of John Stuart Mill. At least half of Kitcher's book is devoted to an argument (often nicely illustrated from the history) that mathematics differs from the empirical sciences precisely because (after early crude beginnings) it is self-referential, self-centered, and evasive of empirical test, evolving more out of itself than of nature. I at least find that argument cluttered and sometimes a bit muddy, yet I want in essence to agree with it. But I think the philosopher we should bow to for ancestral support is Aristotle, not Mill. He was as much a realist about the universals as Plato, but he described them and their orderings as the minds' distillation from the patterns of experience, and their status as potentialities within the order of the world. That is where I, at least, would stop, at the edge which Plato stepped over. *Amicus Plato, amicior veritas!*

References

[1] David Wheeler, Teaching for discovery. *Outlook* 14 (Winter 1974). Mountain View Publishing Co., 2929 Sixth Street, Boulder, CO 80302, U.S.A

[2] The Piagetian connection I owe to Andrew Garrison, in a private communication