Meaning in Arithmetic from Four Different Perspectives

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Questions of meaning keep coming up in the teaching of arithmetic. Teachers continually say such things as, "What do you mean?" or, "I know what you mean" or, "Does he mean to say that?". Children's perceptions, actions, and speech are all described as either meaningful or meaningless. It is not surprising then that mathematics educators and psychologists have persisted in trying to find a satisfactory theory of meaning in arithmetic. No real agreement has yet been reached on a theory of meaning. Mathematics educators and psychologists do not even agree as to whether the term meaning should describe something inside a person or something outside a person. For some, meaning is an image; for others, meaning is a response; for still others, meaning is a set of circumstances in the outside world which causes children to do something.

B.F. Skinner [1957], responding to this confusion, has complained that the term "meaning" is vague and not objectively defined and that its use may be misleading. He feels it would be best to banish it from scientific discussion altogether. I am a non-word-banisher. My own opinion is that the term "meaning" serves important functions in arithmetic and that it cannot as yet be replaced by a more precise term. In this paper I will examine four important schools of thought which have addressed the problem of meaning in arithmetic. They are as follows:

1. Connectionist school
2. Structural school
3. Operational school
4. Constructivist school

At the end of the paper I will argue that the constructivist perspective is a potentially fruitful framework within which to recast the issues involved in the analysis of meaning in arithmetic.

The connectionist school

The connectionist learning theory has also been known as the associationist, rote, or drill theory of learning. This theory has been described as follows:

> Whenever there is a response of the learner to a stimulus, a "bond" is formed. This "bond" is conceived as involving changes in the nervous tissue... the strength and the permanence of a "bond" are largely a matter of repetition, of "exercise". That is, the "trace" left in the nervous organism by each response is strengthened by repetition, by drill. [Betz, 1936, p. 118]

This theory assumes that, through conditioning, specific responses are linked with specific stimuli—a position considerably distant from that of faculty psychology, which sought to improve a general faculty that would then serve the learner in a variety of situations far removed from the stimulus situation that provided the initial learning.

Thorndike [1922], the principal contributor to this school, established three primary laws of learning as a part of S-R bond theory: the law of exercise or repetition, the law of effect, and the law of readiness. According to the first, the more times a stimulus-induced response is elicited, the longer the learning (response) will be retained. The effect of this law is to put a heavy emphasis on many practice exercises for each stimulus-response bond that is to be established. The law of effect simply states that responses associated with satisfaction are strengthened and those associated with pain weakened. The law of readiness associates satisfaction or annoyance with action or inaction in the face of a bond's readiness to act or not act.

In *The psychology of arithmetic* [1922] and *The psychology of algebra* [1923], Thorndike stressed the importance of establishing many "bonds" by means of much practice. This psychological approach led to the fragmentation of arithmetic into many small facts and skills to be taught and tested separately. This theory even led to the avoidance of teaching closely-related facts close in time to one another for fear of establishing incorrect bonds.

All knowledge, even the most complex, was to be built of simple connections between stimuli and responses. Learning thus consisted of establishing and strengthening the needed associations. Thorndike [1922] asserted that:

> When analysis of the mental functions involved in arithmetical learning is made thorough it turns into a question, "what are the elementary bonds or connections that constitute these functions?" and when the problem of teaching arithmetic is regarded, as it should be in the light of present psychology, as a problem in the development of a hierarchy of intellectual habits, it becomes in large measure a problem of the choice of the bonds to be formed and of the discovery of the best order in which to form them and the best means of forming each in that order. [p. 70]

However, Thorndike emphasized that connectionists do not wish to make the learning of arithmetic a mere matter of acquiring thousands of disconnected habits. To him learning arithmetic consisted of learning a series of rules to apply. The successful child is the one who makes the proper connections at the proper times. Perception and memory are closely tied to this view of success. Applying a correct series of connections at the proper time implies the ability to perceive the structure of a problem, choose the appropriate series of connections, and produce all of the connections in series

The impact of this theory on the schools was such that arithmetic was taught by drill techniques. Each problem was analyzed into a great many isolated units or "elements of knowledge" and skill. The student was then drilled on the mechanical mastery of these units with little regard for understanding. The prevailing viewpoint was that arithmetic was a tool subject consisting of a series of computational skills. The rote learning of skills was paramount, with rate and accuracy the criteria for measuring learning. This approach was described by Brownell [1935] as follows:

> Arithmetic consists of a vast host of unrelated facts and relatively independent skills. The pupil acquires the facts by repeating them over and over again until he is able to recall them immediately and correctly. He develops the skills by going through the processes in question until he can perform the required operations automatically and accurately. The teacher need give little time to instructing the pupil in the meaning of what he is learning. [p. 2]

The structural school

William Brownell, the most articulate spokesman for the structural school, says that arithmetic becomes meaningful when the child sees the structure of the subject. By "structure" he means the internal organization, the logic of the subject. Thus Brownell [1945] explains:

> Meaning is to be sought in the structure, the organization, the inner relationship of the subject itself. [p. 481]

The "meaning" theory emphasizes understanding the structure of the subject. It proposes a view of meaning that is related to ideas inherent in the subject matter. This approach seems to be closely related to Gestalt psychology, although Brownell does not appeal directly to it. Weaver and Suydam [1972] note the similarities between Brownell's emphasis on teaching and meanings and Bruner's [1960] emphasis on teaching the fundamental structure of the subject.

In the Tenth Yearbook of the NCTM, Brownell [1947] provides a detailed explanation of the notion of meaningful arithmetic:

> "Meaningful" arithmetic, in contrast to "meaningless" arithmetic, refers to instruction which is deliberately planned to teach arithmetical meanings and to make arithmetic sensible to children through its mathematical relationships. . The meanings of arithmetic can be roughly grouped under a number of categories. I am suggesting four.
>
> 1. One group consists of a large list of basic concepts. Here, for example, are the meanings of the whole numbers, of common fractions, of decimal fractions, or percent, and most persons would say, of ratio and proportion .
>
> 2. A second group of arithmetical meanings includes understanding of the fundamental operations. Children must know when to add, when to subtract, when to multiply and when to divide. They must possess this knowledge and they must also know what happens to the numbers used when a given operation is employed. . . .
>
> 3. A third group of meanings is composed of the more important principles, relationships, and generalizations of arithmetic, of which the following are typical: when 0 is added to a number, the value of that number is unchanged. The product of two abstract factors remain the same regardless of which factor is used as multiplier. The numerator and denominator of a fraction may be divided by the same number without changing the value of the fraction.
>
> 4. A fourth group of meanings relates to the understanding of our decimal number system and its use in relationalizing our computational procedures and our algorithms. [p. 257]

While Brownell stressed the importance of teaching the structure of arithmetic, he did allow that there were appropriate uses of both drill and incidental learning in teaching that subject. His writings suggest that not only the structure of the subject, ". . . a closely knit system of understandable ideas, principles, and process . . ," but also the use of concrete materials and practical applications have a place in "meaning" theory. [McKillip and Davis, 1980] Brownell believes that within the structural school there is a place for drill when ideas and processes, already understood, are to be practiced to increase proficiency, to be fixed for retention, or to be rehabilitated after disuse. However he rejects the view of arithmetic as a heterogeneous mass of unrelated elements to be trained through repetition. Brownell [1935] states his position as follows:

> The "meaning" theory conceives of arithmetic as a closely knit system of understandable ideas, principles, and processes. According to this theory, the test of learning is not mere mechanical facility in "figuring". The true test is an intelligent grasp upon number relations and the ability to deal with arithmetical situations with proper comprehension of their mathematical as well as their practical significance. [p. 19]

In reference to the learning of arithmetic, Brownell makes four major criticisms of the connectionist theory. These are the prohibitive task of memorizing multitudes of bonds, the invalidity of product measures as accurate measures of process, the invalidity of the analysis of adults' performance of arithmetic, and the negative effect of the connectionist theory may have on the later learning of mathematics.

However when it comes to the explanation of children's difficulties in arithmetic, Brownell would have agreed with

Thorndike that it is to be found in the disparity between the mental processes and capacities required to learn arithmetic on the one hand, and the mental processes and capacities that young children have at their command on the other hand. But for Thorndike, the argument that children do not possess the necessary capacity for inferential thinking is not sufficient criticism of an instructional method. His problem is to specify instructional procedures that are sufficient to bring about the generalization, not to account for children's generalizations.

In contrast, Brownell believes that investigating children's mental abilities is an essential aspect of curriculum and instruction. He contends that the remedy for children's difficulties in arithmetic

> appears to lie in an adjustment of the curriculum and the methods of instruction which, recognizing fully the gap between pupil's abilities and powers and those required for the mastery of numbers, will adapt instruction in number to the mental capacities of the pupils. [Brownell, 1928, p. 195]

The operational school

The origin of the operational school can be traced to Percy Bridgman's operational analysis of the fundamental concepts of modern physics [Bridgman, 1927]. Van Engen [1949], the principal contributor to this school, says:

> In any meaningful situation there are always three elements. (1) There is an event, an object, or an action. In general terms, a referent. (2) There is a symbol for the referent. (3) There is an individual to interpret the symbol as somehow referring to the referent. [p. 323]

A symbol was taken as referring to something outside itself. This something may be anything, even another symbol, subject only to the condition that in the end it leads to a meaningful act or a mental image.

In an arithmetic situation, for instance, the phrase "1/3 of an orange" is the symbol. The referent is the one-third of an orange, and the interpretation, if known to the child, is the act of cutting an orange into three pieces, either in actuality or in imagination. The one-third orange is regarded as being the result of an operation that exists apart from interpretation. In other words, van Engen believes that the meaning of a symbol such as "1/3" is "an intention to act and the act need not in itself take place". However, if the child is challenged to demonstrate the meaning of the symbol, then the action takes place. [Steffe and Cobb, 1984]

Further, in the case of arithmetic, the referents are initially either actual or mentally visualized overt physical acts. Van Engen calls these referents operations. He states his position as follows:

> Arithmetic is predominantly concerned with the study of operations and not with the study of groups (or collections). In fact, it is predominantly interested in the operations which can be performed on groups [van Engen, 1949, p. 328]

Thus physical acts or operations are symbolized by a spoken word and are the primary instruments of knowledge.

Van Engen calls those arithmetical meanings that involve associating a symbol with an operation: semantic meanings. He constrasts these with the syntactical meanings of arithmetic which involve the formation of relationships between operations. Van Engen asserts that

> "Syntactical meanings" are those very important meanings which are established after the child has acquired a semantic base for these meanings of higher order. [p. 398]

Van Engen views semantic meanings as interpretations of operational definitions which are taken to be universal and therefore identical for all children. Especially in arithmetic, the main concepts seem transparent and can be expected to become self-evident provided they are properly explained.

Van Engen argues that Brownell has taken the semantic meanings of arithmetic for granted and has focused solely on the syntactical meanings. He believes that the structural school fails to recognize at least two kinds of meaning in arithmetic and to establish their place in the instructional program.

Van Engen's operational analysis of meaning constitutes an advance when compared with the analysis of the two earler meaning theorists. First, van Engen stresses that the individual actively gives meaning to symbols. Second, he does not seek the roots of arithmetical meaning in the structure of the subject matter. Instead, he views physical or sensory-motor activity as its source.

The constructivist school

From the constructivist perspective, the meaning of a symbol like "1/3" consists of the child's interpretation of the symbol based on the schemes that are available to the child. The primary task of the development of meaning in arithmetic is viewed as the construction of such schemes (or mental activity).

The fundamental building block in the construction of knowledge is, for Piaget, the scheme. "All action that is repeated or generalized through application to new objects engenders by this very fact a "scheme"." [Piaget, 1980, p. 24] Functionally, a scheme coordinates either sensory-motor, internalized, or interiorized actions. It can be viewed as a basic sequence of events that consist of three parts.

> An initial part that serves as trigger or occasion. In schemes of action, this roughly corresponds to what behaviourists would call "stimulus", i.e., a sensory-motor pattern. The second part, that follows upon it, is an action ("response") or an operation (conceptual ... activity). The third part is what I call the result or sequel of the activity (and here again, there is a rough and only superficial correspondence to what behaviourists would call "reinforcement"). [von Glasersfeld, 1980, p. 81]

This structural analysis of scheme which places the emphasis on "situations" that serve to trigger the scheme should not be interpreted as if those situations are external to the child The crucial difference between the constructivist and behaviourist views of the stimulus or trigger has been expressed by Piaget [1964]:

> A stimulus is really a stimulus only when it is assimilated into a structure and it is this structure which sets off the response Consequently, it is not an exaggeration to say that the response is there first, or if you wish, at the beginning there is the structure. [p 15]

The constructivists believe that conceptual knowledge cannot be transferred ready-made from one person to another, but must be built up by every child solely on the basis of his own experience. For Piaget, "to know is to assimilate reality into systems of transformation". [Piaget, 1970, p. 15] The primary instruments of assimilation, and thus, of knowledge, are systems of transformations—schemes. Schemes constitute mental activity that children take as being material for reflection and abstraction.

Piaget [1980] discussed three kinds of reflecting abstraction: pseudo-empirical, reflective and reflected abstraction They are distinguished by the degree to which the abstracted sequence of actions is disassociated from figural content Reflective abstraction is an indispensable component in the constructivist meaning theory of arithmetic because all new knowledge presupposes some types of an abstraction. [Piaget, 1980] If there is no theory of abstraction, it is difficult to account for the development of symbolic activity which involves the intention to act In fact, one of the primary shortcomings of the three previous theories of meaning is their inability, because of the absence of a theory of abstraction, to account for the progress that children do make in arithmetic

According to Piagetian theory, the construction of knowledge involves two types of action structure that give rise to two types of knowledge: physical and logico-mathematical. [Furth, 1969] Physical knowledge is derived from activities related to the children's abstractions from objects themselves In general, physical knowledge relates to the figurative aspects of knowing because the child attempts to represent "reality" as it appears without transforming it On the other hand, logico-mathematical knowledge is abstracted from activities and operations themselves. It emerges from the child's reflection on his own coordinating activity which yields an expanded internal structure of logico-mathematical relations Thus knowledge based on physical experience alone is knowledge of static states of affairs, while knowledge emerging from logico-mathematical experience is knowledge of the transformation of states and quite another matter. Smock [1981] provides a more detailed explanation:

> Physical experiences, then, provide for the construction of invariants (figurative process) relevant to the properties of objects (i e states) through exchanges (sensory mechanisms) with objects. For example, one may touch something and it is hard, cold, hot, soft, supple, etc., or one may see something—an object is red, a diamond cutting glass, the shape of a banana, etc The course of logico-mathematical experience, however, is assumed to be abstraction (operative knowledge) from the coordination of actions vis-a-vis representations of "objects", i e tranformations of the "states" associated with series of discrete physical experiences [p. 60-61]

Figurative and operational processes represent two types of functional structures necessary to account for knowledge acquisition In general psychological terms, the distinction is between the selection, storage, and retrieval versus the coordination and transformation of information. [Inhelder et al, 1966] Figurations are defined as those action schemes that apprehend, extract and reproduce aspects of a prior structured physical and social environment. Such actions schemes included components of perception, language, imagery, and memory On the other hand, operations are those action schemes that construct "logical" transformations of "states". An operative scheme, then, includes both conceptual operations which coordinate interiorized actions, purely conceptual acts whose results are abstract, structural forms that do not depend on sensory-motor or figural features of experience, and figurative schemes as subschemes.

The processes of construction of figurative and logico-mathematical knowledge are functionally different from the symbolic processes by which the child represents actions. An object or event within the child-object interaction that recalls some knowledge to the child about another object or event is a "signifier". There are three types of signifier The first type is called an "index", where an index representation involves a direct relationship between an object and a representation of that object: e g. a bark is an index of a dog's presence. The second type of signifier is a symbol. It is differentiated from an object but retains a degree of similarity to an object: e.g a child represents a house with seven LEGO blocks Finally the third type of signifier is a sign which is differentiated from the signified in a conventional and often arbitrary way: e g. linguistic or other socially agreed upon representations are considered signs [Smock, 1981] It is important to note that Piaget uses the terms "sign" and "symbol" differently from most other authors

The constructivist characterizes meaning as a signifier-signified relationship [de Saussure, 1959] established by the child where both the signifier and the signified are themselves constructed by the child. Thus the child creates meaning by establishing a relationship between items isolated within the stream of experience However, it is important to note that the child does not carry around in his head a ready-made hierarchy of signifier-signified relationships. Instead, the child carries around the means to construct or assemble signifieds to the degree of specificity required in a particular situation Von Glasersfeld puts it this way [1983a, p. 207]:

> Whenever a piece of language is interpreted, the process involves building up a conceptual structure Though the structure may be a novel composition, the elements of which it is composed are always derived from the interpreter's own experience.

Von Glasersfeld continues:

> Whenever we say "S interprets X", we bring to mind

a specific situation which, I would argue, is always composed of the following elements: (1) an active subject (S), the interpreter; (2) an object (X) which is experienced by S; (3) a specific activity (interpreting) carried out by S; (4) the activity's result (Y), which is not part of S's immediate experience of X but is linked to X by some relation known to S. [p. 208]

Thus from the constructivist perspective, words do not have an external content or meaning of their own. Each child abstracts the meaning of words from his own experiences. For the purpose of communication, it is not necessary for children to share the same representations associated with a given word; it is sufficient that the speaker's representations are "compatible" with the listener's [von Glasersfeld, 1983b]

Discussion

From the constructivist perspective, the structural school is a significant advance over the connectionist school. A constructivist would accept Brownell's [1935] three aphorisms:

> 1. According to the "meaning" theory, the ultimate purpose of arithmetic instruction is the development of the ability to think in quantitative situations [p. 28]
> 2. In order to achieve this purpose, .. this theory makes meaning, the fact the children shall see sense in what they learn, the central issue in arithmetic instruction [p. 19]
> 3. True arithmetic learning is seen to be a matter of growth which needs to be carefully checked... and guided at every stage [p. 19] [Thus]....the true measure of status and of development is therefore to be found in the level of the thought processes employed. [p. 29]

Differences between these two schools become apparent when the relationship between meaning and behaviour is considered. Brownell [1935], for instance, claimed that "...meanings are dependent upon reactions." [p. 11] This is in contrast to the perspective of constructivism where an attempt is made to differentiate between meanings and reactions. This involves analyzing a child's solutions to a wide variety of tasks. For instance, Steffe, von Glasersfeld, Richards, and Cobb [1983] considered children's solutions to missing addend and subtraction tasks as well as to addition tasks when they inferred the meaning of addition for children who counted.

While analysing children's counting, Brownell [1935] argued that numbers are constructed by the individual [p. 10], but there is no indication that he questioned the ontological nature of an abstract numerical "one". Hence, in contrast to other numbers, "one" would seem to have an ontological existence. The constructivist, however, makes an attempt to question implicit assumptions about the ontological nature of mathematical objects and structures. Thus in the case of counting, for instance, the things the child counts are considered to be his own construction. Bridgman [1959] puts it this way:

> The thing we count was not there before we counted it, but we create it as we go along. It is the acts of creation that we count. [p. 103]

From the constructivist perspective, the operational school's analysis of meaning in arithmetic constitutes a definite advance when compared with the analyses of earlier meaning theorists. [Steffe and Cobb, 1984] The operational school stresses that the individual actively gives meaning to symbols; it does not seek the roots of arithmetical meaning either in the simple connections between stimuli and responses or in the structure of the subject matter. The operational school is broadly compatible with the constructivist school. Both trace the origins of arithmetical knowledge to sensory-motor activity. The primary difference between these two schools is an epistemological one and concerns the existence of the referents and the symbols.

According to the operational school, reality for the child is assumed to be outside his mental constructions. This is in contrast to the perspective of constructivism where there is an explicit rejection of the notion that knowledge must reflect an absolute or "ontological" reality. Von Glasersfeld puts it this way:

> The reality of knowledge is in one respect radically different from the reality sought by metaphysical realists: viable knowledge fits into the ontic world but makes no claim whatever that it represents that world iconically. [1982, p. 620]
> The epistemological view which I find to be the most compatible with Piaget's work is basically an instrumentalist one in which "knowledge" does not mean knowledge of an experience-independent world. From that perspective, cognitive structures, i.e., action schemes, concepts, rules, theories, and laws, are evaluated primarily by the criterion of success, and success must ultimately be understood in terms of the organism's efforts to gain, maintain, and extend its internal equilibrium in the face of perturbations. [p. 619]

Thus the constructivist believes that knowledge cannot be transferred ready-made from one child to another, but must be built up by every child solely on the basis of his own experience.

In summary, the predominant notion of early meaning theorists that meaning was to be found in simple connections between stimuli and responses, in structural relationships, and in operational definitions from the adult's point of view led to classic works that emphasized standard algorithmic procedures dictated by adult conventions. [Thorndike, 1922; Brownell, 1945; van Engen & Gibb, 1956] In contrast, the constructivist school looks at the child's arithmetical activity in order to infer the meaning of arithmetical words, numerals, and procedures. The constructivists acknowledge the gap between formal arithmetical procedures and children's methods, and in general they view arithmetical knowledge as the coordinated schemes of actions and operations that a child has constructed at a particular point in time. According to Steffe [1985], this view provides a psychological interpretation for the astute statement made by Thom [1973] that:

> The real problem which confronts mathematics teaching is ..the problem of the development of "meaning", of the "existence" of mathematical objects [p. 202]

The constructivist would interpret the "existence" of mathematical objects as psychological existence—existence as concepts in the context of schemes. [Steffe, 1985] I personally believe that the approach of focusing on children's schemes will open up a novel understanding of arithmetical knowledge; the result of such an approach has been at least partially captured in a book called *Children's counting types* by Steffe, von Glasersfeld, Richards, and Cobb.

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